International Journal of Engineering & Technology, 7 (3.31) (2017) 102-105

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Some Coupled Fixed Point Theorems in Modular Metric Space Using Caristi Type Contraction

G.Adilakshmi [1], G.N.V.Kishore [1, 2, *]

Department of Mathematics, K L E F, Vaddeswaram, Guntur, Andhra Pradesh, India - 522 502
[2] Department of Engineering Mathematics, S. R. K. R. Engineering College, Bhimavaram, Andhra Pradesh, India-534 204
** Corresponding Author Email: kishore.apr2@gmail.com*

Abstract

In this paper, we obtained a unique common coupled fixed point theorem using Caristi type contraction in modular metric spaces. Also furnished an example to support our main results.

Keywords: *Modular metric spaces , W - compatible maps, coupled fixed point, Caristi type contraction.*

1. Introduction

Fixed point theory is one of the very popular tools in various fields. Since Banach introduced this theory in 1922 [1], it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is modification of $\varepsilon -$ variational principle of Ekeland [6]. It is crucial in nonlinear analysis, in particular, optimization, variational inequalities, differential equations and control theory. The notion of modular space was introduced by Nakano and was intensively developed by Koshi, Shimogaki, Yamamuru [16] and others. A lot of mathematicians are interested in fixed point of modular space. In 2008, Chistyakov introduced the notion of modular metric space generated by $F -$ modular and developed the theory of this space. In this section, we will give some basic concepts and definitions about modular metric spaces.

Definition 1.1 [4] Let X be a nonempty set, a function $W: (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if it satisfying, for all $x, y, z \in X$ the following conditions holds

(i) $W_\lambda(x,y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
(ii) $W_\lambda(x,y) = W_\lambda(y,x)$ for all $\lambda > 0$;
(iii) $W_\lambda + \mu(x,y) \leq W_\lambda(x,z) + W_\mu(z,y)$ for all $\lambda, \mu > 0$.
If instead of (i), we have only the condition
(i) $W_\lambda(X,X) = 0$ for all $\lambda > 0$, then W is said to be a (metric) pseudo modular on X. The main property of a metric modular [9] W on a set X is the following: given $x, y \in X$, the function $0 < \lambda \mapsto W_\lambda(x,y) \in [0,\infty]$ is non increasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i) and (ii) imply
$W_\lambda(x,y) \leftrightarrow W_{\lambda-\mu}(x,x) + W_\mu(x,y) = W_\mu(x,y)$. It follows that at each point $\lambda > 0$ the right limit $W_{\lambda+0}(x,y) = \lim_{\mu\to\lambda+0} W_\mu(x,y)$ and the left limit
$W_{\lambda-0}(x,y) = \lim_{\varepsilon\to+0} W_{\lambda-\varepsilon}(x,y)$ exist in $[0,\infty]$ and the following two inequalities hold: $W_{\lambda+0}(x,y) \leftrightarrow w_\lambda(x,y) \leftrightarrow W_{\lambda-0}(x,y)$.

Theorem 1.2 [7] Let X_W be a complete modular metric space and T a contraction on X_w. Then, the sequence $(Tx)_n \in N$ converges to the unique fixed point of T in X_w for any initial $x \in X_w$.
Now we give some definitions, which are useful for our main results.

Definition 1.3 [4] Let X_w be a modular metric space. Then following definitions exits:
(i) The sequence $(x_n)_n \in N$ in X_w is said to be convergent to $x \in X_w$ if $W_1(x_n, x) \to 0$, as $n \to \infty$
(ii) The sequence $(x_n)_n \in N$ in X_w is said to be Cauchy if $w_1(x_m, x_n) \to 0$, as $m, n \to \infty$
(iii) A subset C of X_w is said to be closed if the limit of a convergent sequence of C always belong to C.
(iv) A subset C of X_w is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit is in C.

Definition 1.4 [4] An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F: X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.
Definition 1.5 [4] An element $(x,y) \in X \times X$ is called
(i) a coupled coincident point of mappings $F: X \times X \to X$ and $f: X \to X$ if $fx = F(x,y)$ and $fy = F(y,x)$.
(ii) a common coupled fixed point of mappings $F: X \times X \to X$ and $f: X \to X$ if $x = fx = F(x,y)$ and $y = fy = F(y,x)$.

Definition 1.6 [4] The mappings $F: X \times X \to X$ and $f: X \to X$ are called w - compatible if $f(F(x,y)) = F(fx, fy)$ and $f(F(y,x)) = F(fy, fx)$ whenever $fx = F(x,y)$ and $fy = F(y,x)$.

Introduction to Caristi fixed point theorems.
In 1976, Caristi proved the following famous fixed point theorem.
Theorem 1.7 [7] Let (X,d) be complete metric space and $f: X \to R$ be lower semi continuous function and bounded below function. A mapping $T: X \to X$ is said to be Caristi type map on X dominated by f if T satisfies $d(x,Tx) \leq f(x) - f(Tx)$ for each $x \in X$. Then T has a fixed point.

It is well-known that the Caristi's fixed point theorem is one of the most valuable generalization of the Banach contraction principle.

Definition 1.8 Let (X, d) be metric space. An extended real valued function $f: X \to (-\infty, +\infty]$ is called a lower semi continuous function at $w \in X$ if for any sequence in $\{x_n\}$ in X with $x_n \to w$ as $n \to \infty$, we have $f(w) \leq \liminf_{n\to\infty} f(x_n)$. The function f is called a lower semi continuous functions on X is f is lower semi continuous at every point of X.

2.Main Results

Theorem 2.1 Let X be a modular metric space and let $S, T: X \times X \to X$ and $f, g: X \to X$. Define $W: (0, \infty) \times X \times X \to [0, \infty]$ as

(2.1.1) $W_\lambda(S(x,y), T(u,v)) \leq (fx, gu) - \psi(S(x,y), T(u,v)) + \phi(fy, gv) - \phi(S(y,x), T(v,u))$

where $\psi, \phi: X \times X \to [0, \infty)$ are lower semi continuous functions

(2.1.2) $S(X \times X) \subseteq g(X)$ and $T(X \times X) \subseteq f(X)$

(2.1.3) (S, f) and (T, g) are w-compatible

(2.1.4) either $f(X)$ or $g(X)$ is complete. Then S, T, f, g have unique common coupled fixed point.

Proof: Let x_0, y_0 be arbitrary points in X. From (2.1.2), there exist sequences $\{x_{2n}\}$, $\{y_{2n}\}$,$\{z_{2n}\}$ and $\{w_{2n}\}$ in X such that

$$z_{2n} = gx_{2n+1} = S(x_{2n}, y_{2n}),$$
$$w_{2n} = gy_{2n+1} = S(y_{2n}, x_{2n}),$$
$$z_{2n+1} = fx_{2n+2} = T(x_{2n+1}, y_{2n+1}),$$
$$w_{2n+1} = fy_{2n+2} = T(y_{2n+1}, x_{2n+1}), \quad n = 0,1,2,\cdots$$

Choose $\lambda = 1$.

If $z_{2n} = z_{2n+1}$ and $w_{2n} = w_{2n+1}$ then the results is proved.

Suppose $z_{2n} \neq z_{2n+1}$ and $w_{2n} \neq w_{2n+1}$

Now

$$\begin{aligned} 0 &< W_1(z_{2n}, z_{2n+1}) \\ &= W_1(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1})) \\ &\leq \psi(fx_{2n}, gx_{2n-1}) - \psi(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1})) \\ &\quad + \phi(fy_{2n}, gy_{2n+1}) - \phi(S(y_{2n}, x_{2n}), T(y_{2n+1}, x_{2n+1})) \\ &\leq \psi(z_{2n-1}, z_{2n}) - \psi(z_{2n}, z_{2n+1}) + \phi(w_{2n-1}, w_{2n}) \\ &\quad - \phi(w_{2n}, w_{2n+1}). \end{aligned}$$

Therefore

$$\psi(z_{2n}, z_{2n+1}) + \phi(w_{2n}, w_{2n+1}) < \psi(z_{2n-1}, z_{2n}) + \phi(w_{2n-1}, w_{2n}).$$

This shows that the sequences $\{\psi(z_{2n}, z_{2n+1})\}$ and $\{\phi(w_{2n}, w_{2n+1})\}$ are a non-increasing sequence of non-negative real numbers. So it must converges to $k_1, k_2 \geq 0$ respectively.

Suppose $k_1 > 0$ or $k_2 > 0$.

Letting $n \to \infty$ in the above relation, we get a contradiction.

Therefore

$$\lim_{n\to\infty} \psi(z_{2n}, z_{2n+1}) = \lim_{n\to\infty} \phi(w_{2n}, w_{2n+1}) = 0.$$

Now consider

$$\begin{aligned} &\sum_{2n=1}^{2m} W_1(z_{2n}, z_{2n+1}) \\ &= W_1(z_1, z_2) + W_1(z_2, z_3) + \cdots + W_1(z_{2m}, z_{2m+1}) \\ &\leq \psi(z_0, z_1) - \psi(z_1, z_2) + \phi(w_0, w_1) - \phi(w_1, w_2) \\ &\quad + \psi(z_1, z_2) - \psi(z_2, z_3) + \phi(w_1, w_2) - \phi(w_2, w_3) + \cdots \\ &\quad + \psi(z_{2m-1}, z_{2m}) - \psi(z_{2m}, z_{2m+1}) \\ &\quad + \phi(w_{2m-1}, w_{2m}) - \phi(w_{2m}, w_{2m+1}) \\ &\leq \psi(z_0, z_1) + \phi(w_0, w_1) - \psi(z_{2m}, z_{2m+1}) - \phi(w_{2m}, w_{2m+1}) \\ &\leq \psi(z_0, z_1) + \phi(w_0, w_1). \end{aligned}$$

This shows that $\sum_{2n=1}^{2m} W_1(z_{2n}, z_{2n+1})$ is a convergent series.

Now for any positive integers n and m with $m > n$ as $n \to \infty$

$$W_1(z_{2n}, z_{2m}) \leq \sum_{i=2n}^{2m} W_1(z_i, z_{i+1}).$$

This shows $\{z_{2n}\}$ is a Cauchy sequence in X.

Similarly, we can prove that $\{w_{2n}\}$ is a Cauchy sequence in X.

Therefore $\lim_{n,m\to\infty} W_1(z_{2n}, z_{2m}) = 0$ and

$\lim_{n,m\to\infty} W_1(w_{2n}, w_{2m}) = 0$.

Since $f(X)$ is a complete subspace of (X, W) so $\{z_{2n+1}\} \subseteq f(X)$ and $\{w_{2n+1}\} \subseteq f(X)$ are converges in the complete metric space $(f(X), p)$.

Therefore

$\lim_{n\to\infty} W_1(z_{2n+1}, u) = 0$ and

$\lim_{n\to\infty} W_1(w_{2n+1}, v) = 0$, where $u, v \in f(X)$.

Since $f: X \to X$ and $u, v \in f(X)$ so there exists $s, t \in X$ such that $fs = u$ and $ft = v$.

Since $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences and $z_{2n+1} \to u$ and $w_{2n+1} \to v$, it follows that $z_{2n} \to u$ and $w_{2n} \to v$.

As $n \to \infty$

$$\begin{aligned} W_1(S(s,t), u) &= W_{\frac{1}{2}}(S(s,t), z_{2n+1}) + W_{\frac{1}{2}}(z_{2n+1}, u) \\ &= W_{\frac{1}{2}}(S(s,t), T(x_{2n+1}, y_{2n+1})) + W_{\frac{1}{2}}(z_{2n+1}, u) \\ &\leq \psi(fs, gx_{2n+1}) - \psi(S(s,t), T(x_{2n+1}, y_{2n+1})) \\ &\quad + \phi(ft, gy_{2n+1}) - \phi(S(t,s), T(y_{2n+1}, x_{2n+1})) \\ &\quad + W_{\frac{1}{2}}(z_{2n+1}, u) \\ &\leq \psi(u,u) - \psi(S(s,t), u) + \phi(v,v) - \phi(S(t,s), v) \\ &\quad + W_{\frac{1}{2}}(u,u) \\ &\leq \psi(u,u) + \phi(v,v) \\ &= 0. \end{aligned}$$

Therefore $W_1(S(s,t), u) = 0 \Rightarrow S(s,t) = u$ and $u = fs$ so we have $S(s,t) = u = fs$.

Similarly we can prove that $S(t,s) = v = fv$.

Since (S, f) are w-compatible mappings so we have $S(u,v) = fu$ and $S(v,u) = fv$.

Now to prove that $fu = u$ and $fv = v$.

$$\begin{aligned} W_1(fu, z_{2n}) &= W_1(S(u,v), T(x_{2n}, y_{2n})) \\ &\leq \psi(fu, gx_{2n}) - \psi(S(u,v), T(x_{2n}, y_{2n})) \\ &\quad + \phi(fv, gy_{2n}) - \phi(S(v,u), T(y_{2n}, x_{2n})) \\ &\leq \psi(fu, z_{2n-1}) - \psi(fu, z_{2n}) + \phi(fv, w_{2n-1}) \\ &\quad - \phi(fv, w_{2n}). \end{aligned}$$

Letting $n \to \infty$, we have that

$$W_1(fu, u) \to \psi(fu, u) - \psi(fu, u) + \phi(fv, v) - \phi(fv, v) = 0.$$

Therefore $fu = u$.

Similarly we can prove $fv = v$. Therefore $S(u,v) = fu = u$ and $S(v,u) = fv = v$.

Since $S(X \times X) \subseteq g(X)$, so there exist $a, b \in X$ such that $S(u,v) = ga$ and $S(v,u) = gb$.

Therefore $u = S(u,v) = ga$ and $v = S(u,v) = gb$.

$$\begin{aligned} W_1(u, T(a,b)) &= W_1(S(u,v), T(a,b)) \\ &\leq \psi(u,u) - \psi(u, T(a,b)) \\ &\quad + \phi(v,v) - \phi(v, T(b,a)) \\ &\leq \psi(u,u) + \phi(v,v) \\ &= 0. \end{aligned}$$

Therefore $W_1(u, T(a,b)) = 0$.

Therefore $u = T(a,b)$.

Similarly we can prove that $v = T(b,a)$.

Since (T, g) are w - compatible mappings so we have $T(u,v) = gu$ and $T(v,u) = gv$.

Now to prove that $gu = u$ and $gv = v$.

$$\begin{aligned} W_1(u, gu) &= W_1(S(u,v), T(u,v)) \\ &\leq \psi(u, gu) - \psi(u, gu) \\ &\quad + \phi(v, gv) - \phi(v, gv) \\ &= 0. \end{aligned}$$

Therefore $W_1(u, gu) = 0$ which implies $u = gu$.

Similarly we can prove that $v = gv$.
So we have $u = gu = T(u, v)$ and $v = gv = T(v, u)$.
This shows that (u, v) is the common coupled fixed point of the functions S, T, f and g.
Now to prove the uniqueness of (u, v).
Suppose (u^*, v^*) be an another coupled fixed point of S, T, f and g.

$$fu = u = S(u, v), \quad u^* = S(u^*, v^*) = fu^*,$$
$$fv = v = S(v, u), \quad v^* = S(v^*, u^*) = fv^*,$$
$$gu = u = T(u, v), \quad u^* = T(u^*, v^*) = gu^*,$$
$$gv = v = T(v, u), \quad v^* = T(v^*, u^*) = gv^*.$$

Now

$$\begin{aligned} W_1(u, u^*) &= W_1(S(u, v), T(u^*, v^*)) \\ &\le \psi(fu, gu^*) - \psi(S(u, v), T(u^*, v^*)) \\ &\quad + \phi(fv, gv^*) - \phi(S(v, u), T(v^*, u^*)) \\ &\le \psi(u, u^*) - \psi(u, u^*) + \phi(v, v^*) - \phi(v, v^*) \\ &= 0. \end{aligned}$$

Therefore $u = u^*$.
Similarly we can prove that $v = v^*$.
Hence $(u, v) = (u^*, v^*)$. Therefore (u, v) is the unique common fixed point of S, T, f and g.
Hence the results is proved.

Example 2.2 Let $X = [0,1]$ and define $W: (0, \infty) \times X \times X \to R^+ \cup \{0\}$ as $W(x, y) = |x - y|$ and define $S, T: X \times X \to X$ as

$$S(x, y) = \begin{cases} \frac{1}{2}(x - y) & \text{if } x \ge y \\ 0 & \text{if } x < y \end{cases},$$

$$T(x, y) = \begin{cases} \frac{1}{4}(x - y) & \text{if } x \ge y \\ 0 & \text{if } x < y \end{cases}$$

and define $f, g: X \to X$ as $gx = \frac{1}{2}x$, $fx = x$. Define a lower semi continuous functions $\psi, \phi: X \times X \to [0, \infty)$ as
$\psi(a, b) = a + b$, $\phi(c, d) = c + \frac{d}{2}$.
Then clearly S, T, f, g satisfies all the conditions of Theorem 2.1 except the condition (2.1.1).
Then S, T, f and g have a unique common coupled fixed point if they satisfies the condition (2.1.1) of Theorem 2.1. Now for any $(x, y), (u, v) \in X \times X$ there are four cases arises.
(i)$x < y, u < v$, (ii)$x < y, u \ge v$,
(iii)$x \ge y, u < v$, (iv)$x \ge y, u \ge v$.
We need to verify the condition (i) of above theorem.
Case(i): suppose $x < y, u < v$

$$\text{L.H.S} = W_\lambda(S(x, y), T(u, v)) = W_\lambda(0,0) = 0.$$

$$\begin{aligned} \text{R.H.S} &= \psi(x, \tfrac{1}{2}u) - \psi(0,0) + \phi(y, \tfrac{1}{2}v) - \phi(\tfrac{1}{2}(y - x), \tfrac{1}{4}(v - u)) \\ &= x + \tfrac{1}{2}u - 0 + y + 2(\tfrac{1}{2}v) - (\tfrac{1}{2}(y - x)) - 2(\tfrac{1}{4}(v - u)) \\ &= x + \tfrac{1}{2}u + y + v - \tfrac{1}{2}y + \tfrac{1}{2}x - \tfrac{1}{2}v + \tfrac{1}{2}u \\ &= \tfrac{3}{2}x + u + \tfrac{1}{2}y + \tfrac{1}{2}v \\ &> 0. \end{aligned}$$

Therefore for all $x, y, u, v \in (0,1]$ we have L.H.S < R.H.S .

Case(ii): Suppose $x < y, u \ge v$

$$\begin{aligned} \text{L.H.S} &= W_\lambda(S(x, y), T(u, v)) \\ &= W_\lambda\left(0, \frac{1}{4}(u - v)\right) \\ &= \left|0 - \frac{1}{4}(u - v)\right| \\ &= \tfrac{1}{4}(u - v). \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= \psi(x, \tfrac{1}{2}u) - \psi(0, \tfrac{1}{4}(u - v)) + \phi(y, \tfrac{1}{2}v) - \phi(\tfrac{1}{2}(y - x), 0) \\ &= x + \tfrac{1}{2}u - 0 - \tfrac{1}{4}(u - v) + y + 2(\tfrac{1}{2}v) - (\tfrac{1}{2}(y - x)) - 0 \\ &= x + \tfrac{1}{2}u - \tfrac{1}{4}u + \tfrac{1}{4}v + y + v - \tfrac{1}{2}y + \tfrac{1}{2}x \\ &= \tfrac{3}{2}x + \tfrac{1}{4}u + \tfrac{1}{2}y + \tfrac{5}{4}v. \end{aligned}$$

Clearly $\frac{1}{4}(u - v) < \frac{3}{2}x + \frac{1}{4}u + \frac{1}{2}y + \frac{5}{4}v$, for all $x, y, u, v \in (0,1]$.
Therefore we have L.H.S < R.H.S
Case(iii): suppose $x \ge y, u < v$

$$\begin{aligned} \text{L.H.S} &= W_\lambda(S(x, y), T(u, v)) \\ &= W_\lambda\left(\tfrac{1}{2}(x - y) - 0\right) \\ &= \left|\frac{1}{2}(x - y) - 0\right| \\ &= \tfrac{1}{2}(x - y). \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= \psi(x, \tfrac{1}{2}u) - \psi(\tfrac{1}{2}(x - y), 0) + \phi(y, \tfrac{1}{2}v) - \phi(0, \tfrac{1}{4}(v - u)) \\ &= x + \tfrac{1}{2}u - \tfrac{1}{2}(x - y) - 0 + y + 2(\tfrac{1}{2}v) - 2(\tfrac{1}{4}(v - u)) - 0 \\ &= x + \tfrac{1}{2}u - \tfrac{1}{2}x + \tfrac{1}{2}y + y + v - \tfrac{1}{2}v + \tfrac{1}{2}u \\ &= \tfrac{1}{2}x + u + \tfrac{3}{2}y + \tfrac{1}{2}v. \end{aligned}$$

Clearly $\frac{1}{2}(x - y) < \frac{1}{2}x + u + \frac{3}{2}y + \frac{1}{2}v$, for all $x, y, u, v \in (0,1]$.
Therefore we have L.H.S < R.H.S
Case(iv): suppose $x \ge y, u \ge v$

$$\begin{aligned} \text{L.H.S} &= W_\lambda(S(x, y), T(u, v)) \\ &= W_\lambda\left(\tfrac{1}{2}(x - y) - \tfrac{1}{4}(u - v)\right) \\ &= |\tfrac{1}{2}(x - y) - \tfrac{1}{4}(u - v)|. \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= \psi(x, \tfrac{1}{2}u) - \psi(\tfrac{1}{2}(x - y), \tfrac{1}{4}(u - v)) + \phi(y, \tfrac{1}{2}v) - \phi(0,0) \\ &= x + \tfrac{1}{2}u - \tfrac{1}{2}(x - y) - \tfrac{1}{4}(v - u) + y + 2(\tfrac{1}{2}v) - 0 \\ &= x + \tfrac{1}{2}u - \tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{4}u + \tfrac{1}{4}v + y + v \\ &= \tfrac{1}{2}x + \tfrac{1}{4}u + \tfrac{3}{2}y + \tfrac{5}{4}v. \end{aligned}$$

Clearly $|\frac{1}{2}(x - y) - \frac{1}{4}(u - v)| < \frac{1}{2}x + \frac{1}{4}u + \frac{3}{2}y + \frac{5}{4}$, for all $x, y, u, v \in (0,1]$.
Therefore we have L.H.S < R.H.S
Clearly from case (i), case (ii), case (iii) and case (iv) S, T, f and g have a unique common coupled fixed point.

Corollary 1: Let (X, W) be a modular metric space and let $T: X \times X \to X$ and $f: X \to X$ be mappings satisfies

$$(1.1) W_\lambda(T(x, y), T(u, v)) \le \psi(fx, fu) - \psi(T(x, y), T(u, v)) + \phi(fy, fv) - \phi(T(y, x), T(v, u))$$

where $\psi, \phi: X \times X \to [0, \infty)$ are lower semi continuous functions and
(1.2) $T(X \times X) \subseteq f(X)$
(1.3) (T, f) are w-compatible
(1.4) suppose $f(X)$ is complete. Then T, f have unique common coupled fixed point.

3. Conclusion

In this article, the new Caristi type contraction in a modular metric space has been introduced and a fixed point theorem for four maps has been established with help of the new contraction. Our results is unified in the literature. The results discussed in this paper are mainly concerned with the fixed points in Modular metric space. The study of uniqueness of the fixed points in the current context would be an interesting topic for future study.

4. Acknowledgement

All authors are equally contributed in developing of this article. The authors are grateful to the learned referees for their careful reading of our manuscript.

References

[1] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales", Fund. Math, 3 , 1922, 133-181.

[2] V.V. Chistyakov, "Modular metric spaces generated by F-modulars" Folia Math, 14, 2008, 3-25.

[3] V.V. Chistyakov, "Modular metric spaces I. basic conceps" Nonlinear Anal, 72, 2010, 1-14.

[4] Bhaskar TG, Lakshmikantham V "Fixed point theorems in partially ordered metric spaces and applications", Nonlinear Analysis, 65, 2006, 1379 - 1393.

[5] Duran Turkoglu and Emine Kilinc, "Some fixed point results for caristi type mappings in Modular metric spaces with an application", Internation Journal of Analysis and applications, ISSN 2291-8639, Volume 12, Number 1, 2016, 15-21.

[6] I. Ekeland, "On the variational principle", Journal of Mathematical Analysis and Applications, 47(2), 1974, 324-353.

[7] J.Caristi, "Fixed point theorems for mappings satisfying inwardness conditions" Transactions of the American Mathematical Society, vol. 215, 1976, pp. 241- 251.

[8] Karapınar E, Erhan IM, "Fixed point theorems for operators on partial metric spaces" Applied Mathematics Letters, 24 (11), 2011, 1900 - 1904.

[9] Karapınar E, "Weak ϕ -contraction on partial contraction and existence of fixed points in partially ordered sets" Mathematica Aeterna, 1(4), 2011, 237 - 244.

[10] Matthews SG, "Partial metric topology. Research Report 212", Dept. of Computer Science, University of Warwick, 1992.

[11] M.R.Alfuraidan and M.A.Khamsi, "Caristi fixed point theorem in Metric space with a Graph", Hindawi Publishing corporation Abstract and Applied Analysis, Article ID 303484, 2014.

[12] ONeill SJ, "Two topologies are better than one", Tech. report. University of Warwick, Coventry 1995, UK, http://www.dcs.warwick.ac.uk/reports/283.html, (1995).

[13] Rao KPR, Kishore GNV, Nguyen Van Luong, "A unique common coupled fixed point theorem for four maps under $\psi - \phi$ contractive condition in partial metric spaces" CUBO A Mathematical Journal Vol.14, No03, 2012, 115 - 127.

[14] Rao KPR, Kishore GNV, Raju VCC, "A coupled fixed point theorem for two pairs of w - compatible maps using altering distance function in partial metric space", Journal of Advanced Research in Pure Mathematics, 4(4), 2012, 96 - 114.

[15] T.G.Bhaskar and V.Lakshmikantham, "Fixed point theorems in partially ordered cone metric spaces and applications" Nonlinear Analysis Theory, Methods and Applications, 65(7), 2006, 825-832.

[16] S. Koshi, T. Shimogaki, "On F-norms of quasi-modular space", J. Fac. Sci. Hokkaido Univ, 72, 1961, 1-14.