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***Research paper***

# Inverse problem of electrocardiography

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## Abstract

Inverse problem in Electrocardiography (ECG) is the mathematical formulation of the electrical activity of the heart surface from the measured body surface potential. This paper presents a state of art review of the inverse problem in ECG and the recent development in the solution of the mathematical model.

***Keywords***: *ECG; Electric Potential; Epicardium; Ill-Posedness; Inverse Problem.*

## 1. Introduction

Heart disease is the major cause of death in United States, United Kingdom, Canada and Australia [2]. It is the main organ of a human body which is responsible to stay alive. Any abnormalities in the heart may lead to heart attack. To increase the survival rate in heart attack it is very important to detect early the typical sign of abnormalities. An electrocardiogram (ECG) is a machine used for finding anomalies in the heart. It uses electrodes on the body surface and reconstructs the electric potential on the heart from the measured body surface potential. Using ECG a well trained doctor read the body surface signal and can detect and analyze certain cardiac arrhythmia. ECG is crucial for people who have high blood pressure or heart disease in order to prevent heart attack.

Inverse problem in ECG finds the electric potential on the heart surface from the electric potential on the body surface. It is a non invasive process. Solving the inverse problem in ECG requires mathematically solving the forward and inverse problem of the electric potential which is the main process in correctly diagnosing heart disease. Many mathematical and computational techniques are applied to the forward and inverse problem of ECG. The inverse problem is ill-posed problem in the sense of Hadamard whereas forward problem is well posed. Ill-posed means small error in the input data gives large error in the inverse solution.

ECG signal measured the electric potential on the body surfaces. ECG suffers from low resolution and inadequate sensitivity and specificity [8]. ECG inverse problem or Electrocardiographic Imaging (ECGI) is the process of simulating the electrical activity of the heart which helps the doctor in understanding the relationship between the ECG signal and the electrical activity of the heart.

The objective of forward and inverse problem in ECG is to accurately determine the anomalies on the heart surface. For this purpose experiment has to be performed on equivalent source similar to torso. The forward problem calculates the electric potential on the body surface due to the presence of selected equivalent source inside the conductor. Mathematically solving an inverse problem does not give a unique mathematical solution. The main difficulty faced in solving the inverse problem is its ill-posedness. Much research work has been done in finding a clinically acceptable solution to the inverse problem of ECG by using different computational and mathematical techniques. This paper presents a review of the mathematical model of the inverse problem in ECG and the recent development in the solution of the mathematical model.

## 2. ECG inverse problem

ECG Inverse problem is the construction of the electric potential on the heart from the body surface potential. The inverse problem is solved based on the ECG information on the body surface and volume conductor that surround the heart. Formulation of an ECG inverse problem requires quantitative description of the sources, the geometry of the volume conductor and the equations associated with the geometry of the volume conductor and the heart. To formulate the mathematical model a set of model equation is selected that mathematically articulate the biophysical process of the electrical potential [29], [35] on the torso volume. The model equations are then discretized using mathematical techniques which is in the form of numerical system and then solved to approximate the true solution of the inverse problem.

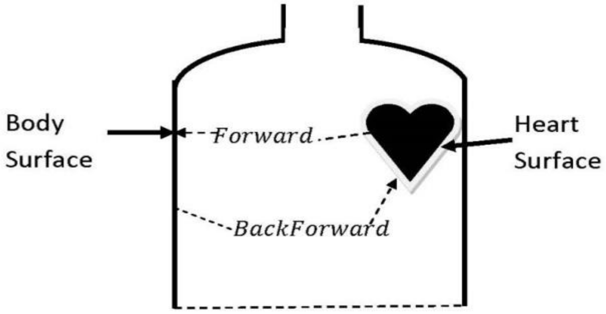

**Fig. 1:** Simulation Pipeline of an Inverse Problem.

The simulation strategy of an inverse problem is followed by first solving the forward problem is which obtain the electric potential on the body surface from the electric potential on the epicardium, and using the transfer matrix of the forward solution the electric potential on the heart surface is reconstructed which solved the inverse problem. Figure 1 shows the simulation pipeline of an inverse problem.

Inverse problem in electrocardiography is the mathematical formulation of the electrical activity of the heart surface from the measured body surface potential. Inverse problem is solved using the solution of the forward problem which mathematically determines the potential on the body surface using the measured potential on the heart surface or the epicardium.

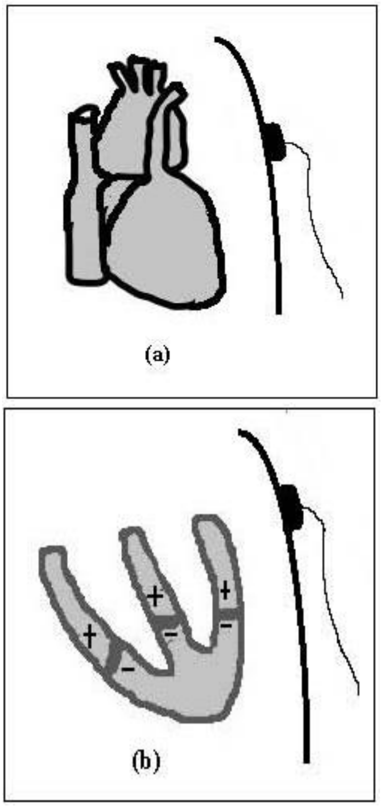

**Fig. 2:** Source Model (A) Epicardial Potential and (B) Activation Time.

The inverse problem has two different source models as shown in figure 2. The first source model is potential based approaches which reconstruct the epicardial potential, electrograms and isochrones from the measured body surface electrocardiographic data. The source model has two parts the forward problem and the inverse problem which is in a form of partial differential equations with cauchy boundary conditions. Figure 3 illustrates the different boundary condition. From the figure H denote the heart, B represent the open domain of the human body which has two boundary $\partial H$ represent the inner boundary which is the heart surface and $\partial B$ represent the outer boundary which is the body surface. The forward or direct problem is the determination of the electric potential from the heart potential.

The forward problem is well posed i.e. the problem gives a unique solution. A number of different source formulations of the mathematical model are possible. In the literature it has been found that two source formulations are frequently used by researchers for finding inverse ECG Problem. The first one represents the entire cardiac activity by the potential distribution on a closed surface that enclosed the heart. This formulation gives a unique inverse solution but it is not a true representation of the underlying mechanism of the heart. If the enclosing surface is the epicardial surface of the heart that the electric potential on the body surface can be measured and validate the solution [13]. The mathematical form of this kind of formulation in terms of the forward or direct problem is,

$$\nabla.(\alpha\nabla u) = 0 , u(x,y) \in B \quad (1)$$

$$n.\nabla u = 0, u(x,y) \in \partial B \quad (2)$$

$$u_H = g, u(x,y) \in \partial \quad (3)$$

Where, n denote the outward unit normal on the body surface, $\alpha$ is the conductivity tensor and g is the known potential on the heart surface. The solution of equation (1) (2) and (3) is the electric potential $u(x,y)$on the body.

The inverse problem assume that we have a given electric potential on the body surface $\partial B$ from the result of the direct problem and determine the electric potential on the heart surface $\partial H$. The mathematical formulation is,

$$\nabla.(\alpha\nabla u) = 0 , u(x,y) \in B \quad (4)$$

$$\nabla u.n = 0, u(x,y) \in \partial B \quad (5)$$

$$u_B = d, u(x,y) \in \partial H \quad (6)$$

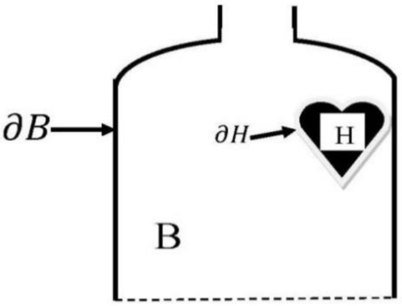

**Fig. 3:** Geometry of Torso.

The goal of inverse problem is to calculate u that satisfies equation (4) (5) and (6). n represents the outward unit normal on the body surface $\partial B$ and d is the measured potential on the body surface. An inverse problem is ill-posed according to Hadamard. Ill-posed means that a unique solution does not exist as a small variation on the measured body potential will lead to large changes in the computed solution. This mathematical formulation has been used in [1], [13].

The second source model is activation time approach which is a bidomain model or a uniform dipole layer (UDL). The mathematical form of a bidomain model in terms of the forward problem is given by [33],

$$A_m\left(C_m v' + I_{ion}(v,w)\right) - \nabla(\sigma_i\nabla v) = \nabla(\sigma_i\nabla u_e) + I_{stim}, \text{ in H} \quad (7)$$

$$-\nabla\left((\sigma_i + \sigma_e)\nabla u_e\right) = \nabla(\sigma_i\nabla v), \text{ in H} \quad (8)$$

$$w' + g(v,w) = 0, \text{ in H} \quad (9)$$

$$\sigma_i\nabla v.n = -\sigma_i\nabla u_e.n \text{ on } \partial H \quad (10)$$

Equation (7), (8), (9) and (10) is the equation in the heart domain H. In equation (7) v and $u_e$ stands for the transmembrane potential and extracellular potential. The constants $A_m$ and $C_m$ represents membrane surface area per unit volume and capacitance of the membrane, respectively. $\sigma_i$and $\sigma_e$ are intracellular and extra cellular conductivity tensor. $I_{ion}$ and $I_{stim}$ are the stimulation and ionic currents. w is a vector characterizing the state of the cell membrane. The time derivative of v and w is denoted by $v'$ and $w'$.

In the torso domain B the electric potential is described by,

$$\nabla(\sigma_T\nabla u_T) = 0 \text{ in B} \quad (11)$$

$$\sigma_T\nabla u_T.n = 0, \text{ on } \partial B \quad (12)$$

Where, $u_T$ represent electric potential on the torso domain and $\sigma_T$ represent the conductivity tensor of torso.

Electric potential in the extracellular myocardial region and the torso region is given by,

$$u_e = u_T, \text{ on } \partial H \tag{13}$$

$$n.\left((\sigma_i + \sigma_e)\nabla u_e\right) = n.\,(\sigma_T \nabla u_T) \text{ on } \partial B \tag{14}$$

Equation (7) to (14) represents the cardiac electrical activity from the heart to the body surface. The inverse problem is given by,

$$\nabla.\,(\sigma_T \nabla u_T) = 0\ \text{, in B} \tag{15}$$

$$\nabla u.\,n = 0 \text{ and } u_T = d, \text{ on } \partial B \tag{16}$$

$$u_T = u_e, \text{ on } \partial H \tag{17}$$

Equation (15) to (17) represents the equation in the body surface domain.

## 3. Solving inverse problem

Solving inverse problem in ECG is the mathematical problem of solving the forward and inverse problem. Forward problem is a well posed problem i.e., it gives a unique solution. Solving the inverse problem reconstruct the electric potential on the epicardium. Due to its ill-posedness nature the electric potential on the epicardium are not sufficiently accurate. This ill-posedness in the inverse problem arises due to noise and measurement error in the forward problem. Identification of the error is a complication in the inverse problem, which has to be handled with great care. Regularization techniques are used to reduce the error in the inverse solution. To obtain an improved solution of the inverse problem many regularization techniques are developed and analyze. Some of the regularization techniques used so far in inverse problem are: Tikhnov regularizations [22], [17], Truncated Singular Value Decomposition (TSVD) [34], Total Variation [15], and Total Variation with Laplacian [15]. Tikhnov regularization technique is the most used regularization technique [22], [17].

Tikhnov regularization requires a prior knowledge about the solution which acts as a constraint of the problem. It also requires regularization parameter which determines the level of constrained to be applied. Determining the regularization parameter is also an important research area. In [22] Tikhnov regularization is used to impose constraint on the reconstructed potential or derivatives. Accuracy of this method depends on the prior knowledge of the solution and the regularization parameter. Many methods so far have been developed to identify an accurate regularization parameter which is affected by the noise in data. Some of the method developed to accurately identify regularization parameter are composite residual and smoothing operator (CRESO) [9], L-curve, and zero crossing [9].

In [17] the authors applied 13 regularization techniques in ECG inverse problem. The regularization techniques are, (i) tikhnov regularizations: zero order, and second order. (ii) iterative techniques: truncated singular value decomposition(zero order, first order and second order) [23], conjugate gradient [36] (zero order, first order and second order), v-method [14], and MINRES method [7], (iii) non-quadratic techniques : total variation, and total variation with Laplacian. Comparison in the solution of the different regularization techniques is done in the source model of a single dipole to triplet of dipoles and it is found that the performance of non-quadratic technique and first order and second order Tikhnov regularization gives better performance as compared to the other regularization techniques.

Many contributions has been made by many researchers in finding a clinically acceptable solution for the inverse problem which will allow us to detect, quantify and localized cardiac electrical activity from body surface measurement. To find such solution we have to first solve the associated forward problem. Some of the mathematical methods used in finding solution to inverse problem in ECG are: Numerical method such as Finite Elemant Method (FEM) [4], Boundary Element method BEM [4], bidomain method [10] and domain decomposition method [25].

Finite Elemant Method (FEM) has been used by [29], [3] for finding solution to the inverse problem of ECG. In [29] numerical technique such as FEM is used in forward and inverse problem. The objective of this paper is to develop discretization and refinement strategies involving hybrid finite elements to minimize approximation errors. The strategy is applied on both a simplified and realistic 2-D torso model. Much improvement has also been done so far on finite element method. In [13] the authors improve finite element method for solving both the forward and inverse problem of ECG.

Boundary Element method is applied to the potential based method [28]. This method requires meshing the heart and torso surface and optimizes the mesh which is done manually. A new BEM called h-adaptive is developed [24] for finding the solution to the forward and inverse problem. The simulation result shows that the new method is more accurate and efficient than the traditional BEM. BEM are commonly used in computation of ECGI [26], [24]. In [26] the author used Boundary Element Method (BEM) using the point collocation weighting and the constant or linear basis function to reconstruct the epicardial potential. It is found that the best optimal technique depends on the electrode potential and chosen errors. In [24] the author used Adaptive BEM [36] and compare to BEM.

Domain Decomposition method for finding solution to the inverse problem is proposed by Zemzemi [33]. This method is used for finding the activation time on epicardium and endocardium. The method is applied in equation (7); the forward problem is solved using finite element method. Domain decomposition is applied in the ill-posed inverse problem by splitting it into two well posed problems. The method is tested on the synthetic data generated from the forward problem. Simulation is performed on the volume between three concentric spheres. Experimental result shows that the activation times on the heart are constructed with good accuracy.

In [1] the author used an approach which is called an admissible solution set that reformulate the inverse problem, which allow the use of multiple constraints. The solution of the problem is calculated based on iterative convex optimization and the ellipsoid algorithm. In [32] the author solves the inverse problem using an iterative Kozlov- Maz'ya-Fomin method on the synthetic data generated by the bidomain method of [33].

In [22] ECGI image is used to study the normal and abnormal cardiac electrophysiological activity during activation and repolarization during canine experiment. In this paper the author proposed a generalized minimal residual method (GMRes) to smoothen the ill posed inverse problem of ECG. GMRes is an iterative technique applied in experimental data during activation/repolarization of normal and infracted hearts. This technique is compared to Tikhonov regularistion and experimental result shows that the accuracy of both the methods is the same. But the author found that GMRes recovered localized potential features which are not there in the solution of the Tikhnov regularization method. The authors conclude that both the method can be used if a priori information about a patient condition is not known.

Successful employment of Mathematical techniques in solving the Forward and inverse problem of ECG required the quantification and minimization of approximate error in discretization process [29]. Many researchers have used many different methods for computing this epicardial potential such as finite element method, boundary element method, domain decomposition method, bidomain method, admissible solution set and GMRes method. Numerical techniques such as BEM and FEM require mesh generation of the heart and torso surface which is time consuming. In [28] the author used a meshless method called Method of Fundamental Solution (MFS) in ECGI. The method performance is then compared with (BEM). Experiment result shows that both the

methods have similar accuracy. And the following advantages of MFS over BEM have been obtained: Elimination of meshing and manual mesh optimization processes, thereby enhancing automation and speeding the ECGI procedure, elimination of mesh-induced artifacts and complex singular integrals that must be carefully computed in BEM and simpler implementation [28].
Improvement in the inverse solution is associated with the input noise. Input nose has to be reduced up to the certain level such that the noise obscured the improvement. This noise may be due to discretization of the body surface, heart surface, other organs and volume conductor. To obtained an improved solution to the inverse problem volume refinement and discretize refinement is necessary which also improves computational efficiency.

## 4. Conclusion

Goal of ECG inverse problem is the reconstruction of the electrical activity of the heart from the body surface using an inverse solution of the forward problem. Solving the inverse ECG problem is finding a numerical technique that gives the accurate solution to the electrocardiographic forward and inverse problems when using realistic geometries. Descretization of the geometries using numerical techniques such as BEM and FEM required mesh generation of the organs and torso which is time consuming while refining and descretizing the mesh structure to reduce nose due to descretization. Introduction of the meshless technique MFS is an important development in inverse problem of ECG which speed up the ECGI procedure. This meshless method can further be extended by improving the ill-posedness and accuracy of the inverse problem of ECG.

## References

[1] Ahmad GF, Brookes DH. and Macleod RS. An Admissible Solution Approach to Inverse Electrocardiography. *Annals of Biomedical Engineering*, Vol. 26, (1998) 278-292. https://doi.org/10.1114/1.56.

[2] Chambolle A, Caselles V, Novaga M, Cremers D and Pock T. An introduction to total Variation for Image Analysis. 2009.<hal-00437581>

[3] Colli-Franzone P, Guerri L, Tentoni S et al.. A mathematical procedure for solving the inverse potential problem of Electrocardiography-Analysis of the time space Accuracy from In Vitro Experimental Data, *Mathematical Biosciences* 77 (1985), 353-396, 1985.

[4] Costabel M and Darmstadt TH. Principles of Boundary Element Methods. https://perso.univ-rennes1.fr/martin.costabel/publis/Co_PrinciplesBEM.pdf.

[5] Denisov AM, Zakharov EV, Kalinin AV and Kalinin VV. Numerical Methods for Some Inverse Problems of Heart Electrophysiology. *Differential Equations* 2009, Vol. 45 No. 7. 1034-1043. https://doi.org/10.1134/S0012266109070106.

[6] Gantumur T. Adaptive boundary element methods with convergence rates. *Math.NA*, Dec. 2012.

[7] Gerald CF and Wheatley PO. *Applied Numerical Analysis*. Pearson 7th edition.

[8] Ghosh S and Rudy Y. Accuracy of Quadratic versus Linear Interpolation in Noninvasive Electrocardiographic Imaging (ECGI), *Ann. Biomedical Eng.*, 2005. https://doi.org/10.1007/s10439-005-5537-x.

[9] Hansen PC. The truncated SVD as a method for regularization, *NA*-86-36, (1986).

[10] Horacek BM, Wang L, Dawoud F, Xu J and Sapp JL. Noninvasive electrocardiographic imaging of chronic myocardial infarct scar. *Journal of Electrocardiology* 48 (2015), 952-958. https://doi.org/10.1016/j.jelectrocard.2015.08.035.

[11] Hullerum M. The inverse Problem of Electrocardiography: Explanation based on a simple Example. August 2012.

[12] Intini A, Goldstein RN, Jia P et. al. Electrocardiographic imaging (ECGI), a novel diagnostic modality used for mapping of focal left ventricular tachycardia in a young athlete. *Heart Rhythm* 2005 Nov; 2(11) 250-1252. https://doi.org/10.1016/j.hrthm.2005.08.019.

[13] Johnson CR and MacLeod RS. Adaptive Local Regularization Methods for the Inverse ECG Problem. *Prog Biophys Mol Biol*, 69(2-3), (1998), 405-23. https://doi.org/10.1016/S0079-6107(98)00017-0.

[14] Liesen J, Rozloznik M and Strakos Z. Least Squares Residuals and Minimal Residual Methods. *SIAM J. SCI.Comput*. Vol. 23, No. 5, 1503-1525. https://doi.org/10.1137/S1064827500377988.

[15] Macfarlane PW, Oosterom AV, Pahlm O, Kligfield P, Janse M and Camm J. Comprehensive Electrocardiography, *Springer*.

[16] Milan Horacek B and John. C. Clements. The inverse problem of Electrocardiography: A Solution in Terms of Single- and Double-Layer Sources on the Epicardial surface. *Mathematical Biosciences* 144, 1997, 119-154. https://doi.org/10.1016/S0025-5564(97)00024-2.

[17] Milanic M, Jazbinsek V, Macleod RS. Assessment of Regularization Techniques for electrocardiographic Imaging. *Journal of Electrocardiology* 47 (2014). https://doi.org/10.1016/j.jelectrocard.2013.10.004.

[18] Mocanu D, Morega M, Eisenberg SR, Morega AM. Nonsmooth Regularization in Electrocardiographic Imaging. Rev. Round. Sci. Tech.-Electotechn. et Energ., 48, I,p, *Bucarest*, 2003.

[19] Pullan AJ, Cheng LK, Nash MP, Ghodrati A, MacLeod R and Brooks D. The Inverse Problem of Electrocardiography. *Comprehensive Electrocardiology* 299-344.

[20] Ramanathan C and Rudy Y. Electrocardiographic Imaging: II. Effect of Torso Inhomogeneties on Noninvasive Reconstruction of Epicardial Potentials, Electrograms and Isochrones. *Journal of Cardiovascular Electrophysiology*, Volume 12, No. 2, February 2001. https://doi.org/10.1046/j.1540-8167.2001.00241.x.

[21] Ramanathan C, Jia P, Ghanem R and Calvetti D and Rudy Y. Noninvasive Electrocardiographic Imaging (ECGI): Application of the Generalized Minimal Residual (GMRes) Method. *Annals of Biomed Eng* Vol. 31, pp. 981–994, 2003. https://doi.org/10.1114/1.1588655.

[22] Ramanathan C, Jia P, Ghanem R and Calvetti D and Rudy Y. Noninvasive Electrocardiographic Imaging (ECGI)”. *Annals of Biomed Eng* 2007 December 24.

[23] Shewchuk JR. An Introduction to the Conjugate Gradient Method without the Agonizing Pain, Edition 1(1/ 4), (1994). https://www.cs.cmu.edu/~quake-papers/painless-conjugate-gradient.pdf.

[24] Shou G, Xia L, Jiang M, Liu F and Crozier S. Forward and Inverse Solutions of Electrocardiography Problem Using an Adaptive BEM Method. *In proceeding “FIMH 2007”*, LNCS 4466 (2007), 290-299.

[25] Smidth BF, Domain Decomposition Methods for Partial Differential Equations. *Parallel Numeric Algorithms, Springer* (1997).

[26] Stenroos M and Haueisen J. Boundary Element Computations in the Forward and Inverse Problems of Electrocardiography: Comparison of Collocation and Galerkin Weightings. *IEEE transaction of Biomedical Engineering*, vol. 55 Issue 9. https://doi.org/10.1109/TBME.2008.923913.

[27] Verfürth P. Reconstruction of the Epicardial Potential from Body Surface Potential Maps. 2011.

[28] Wang Y and Rudy Y. Application of the method of fundamental Solutions to Potential based Inverse Electrocardiography. *Ann Biomed Eng*. 2006 August: 34(8). 1272-1288. https://doi.org/10.1007/s10439-006-9131-7.

[29] Wang D, Kirby RM and Johnson CR. Resolution Strategies for the Finite-Element-Based solution of the ECG Inverse problem. *IEEE Transactions on Biomedical Engineering*, Vol. 57, No. 2, Feb. 2010.

[30] Wang L. Computational reduction for noninvasive transmural electrophysiological imaging. *Computers in Biology and Medicine* 43 (2013), 184-199. https://doi.org/10.1016/j.compbiomed.2012.12.003.

[31] Zemzemi N. An Iterative Method for Solving the Inverse Problem in Electrocardiography in Normal and Fibrillation Conditions: A simulation study. *Proceeding of “Electrography 2014: 41st International Congress on Electrocardiography”.*

[32] Zemzemi N, Bourenane H, Cochet H. An Iterative Method for Solving the Inverse Problem in Electrocardiography Imaging: From Body Surface to Heart Potential. *Computing in Cardiology* 2014, 717-720.

[33] Zemzemi N. A Domain Decomposition Approach in the Electrocardiography Inverse Problem. Domain decomposition methods in Science and Engineering XXII, *Lecture Notes in Computational Science and Engineering*. 641-647.

[34] http://www.mit.edu/~9.520/scribe-notes/cl7.pdf (tikhonov cl17).

[35] https://www.medicalnewstoday.com/articles/237191.php.

[1] https://www.tutorialspoint.com/sdlc/sdlc_v_model.htm.