

# Driving Two Numerical Methods to Evaluate the Triple Integrals R(MSM),R(SMM) and Compare Between Them

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## Abstract

The main objective of present work is the conclusion of two new numerical methods to evaluate the triple integrals with continuous integrands and their partial derivatives are continuous too, the first method through using Mid-point rule on the interior dimension X, Simpson's rule on the middle dimension Y and Mid-point rule on the exterior dimension Z, which denoted by the symbol MSM. The second method by using Mid-point rule on both two dimensions of interior X and middle dimension Y and Simpson's rule on the exterior dimension Z which denoted by SMM, where the number of divisions on the three dimensions are equals. We have concluded two theorems with their proves to find their rules and the correction terms that we found it and to improve the results we used Romberg acceleration which denoted by R(MSM),R(SMM) where we got high accuracy in the results by little sub-intervals relatively and short time.

**Keywords:** Mid-point, dimension.

## 1. Introduction

The triple integrals have an importance in finding the volumes, centroids, and the moment of inertia of volumes, for example, the volume which lies inside  $x^2 + y^2 = 4x$ , above the plane  $z = 0$  and below the paraboloid  $x^2 + y^2 = 4z$ , the volume inside the cylinder  $\rho = 4\cos(\theta)$ , bounded from the upper by the sphere  $\rho^2 + z^2 = 16$  and from below by the plane  $z = 0$ . Another example about the using the triple integrals is the calculation of the centroid for the volume below  $z^2 = xy$  and above the triangle  $y = x, y = 0, x = 4$ . It also used to calculate the moment of inertia for the volume inside  $x^2 + y^2 = 9$ , above the plane  $z = 0$  and below the plane  $x + z = 4$ . Its importance arises by finding the masses which have a various density as a piece of fine wire or thin plate of metal, Frank Ayres [6].

Some researchers study the triple integrals filed, one of them Dheyaa [2], in 2009, using a composite numerical methods  $RMRM(RS)$ ,  $RMRM(RM)$ ,  $RMRS(RM)$  and  $RMRS(RS)$ , these methods result from Romberg acceleration with Mid-point rule ( $RM$ ) on the exterior dimension Z,  $RS(RS)$ ,  $RS(RM)$ ,  $RM(RM)$ ,  $RM(RS)$  on the middle dimension Y and the interior dimension X, it has been concluded that the composite method  $RMRM(RM)$   $RMRM(RM)$  is the best when calculating triple integrals with continuous integrands in terms of accuracy, number of sub-interval used and time.

In 2010, Eghaar [5] suggest a numerical method to calculating triple integrals by using Romberg acceleration on the resulted values from implementation Mid-point rule on the three dimensions X, Y and Z when the number of sub-intervals are equal on the three dimensions which denoted by  $RMMM$  and she obtained good results in terms of accuracy and sub-intervals of few minutes relatively. In 2013, Mohamed et al.[1]introduced a numerical method  $RSSS$  to calculating triple integrals with continuous integrands by using Romberg acceleration with Simpson's rule on the three dimensions X, Y and Z with the same number of sub-intervals and got good results in terms of accuracy and speed of approaching the approximate values to the exact values of integrals by a few sub-intervals relatively. Also in 2014, Shuber [3] offer six numerical methods (RMTS, RMST, RSTM, RSMT, RTMS, and RTSM) to evaluate the triple integrals with continuous integrands by using Romberg acceleration with Newton-Cotes rules (Trapezoidal, Mid-point and Simpson) on the three dimensions X, Y and Z with the same number of sub-intervals and got good results. In 2015, Aljassas [10] introduced a numerical method  $RM(RMM)$  to calculating triple integrals with continuous integrands by using Romberg acceleration with Mid-point rule on the three dimensions when the number of divisions on the interior dimension is equal to the number of divisions on the middle dimension, but both of them are deferent from the number of divisions on the exterior dimension and she got a high accuracy in the results in a little sub-intervals relatively and a short time. Also in 2015, Sarada et al. [9] use the generalized Gaussian Quadrature to evaluate triple integral and got a good results.

In this research we presented two theories with their proves to derive two new numerical methods to evaluate the triple integrals with continuous integrands and their correction terms, this two methods product from applied two rules of Newton-Cotes (Mid-

point and Simpson) the first method by using Mid-point rule on the interior dimension X, Simpson's rule on the middle dimension Y and Mid-point rule on the exterior dimension Z, the second method by using Mid-point rule on both two dimensions of interior X and middle Y, while Simpson's rule used on the exterior dimension Z which denoted by symbols MSM and SMM respectively with the same number of divisions on the three dimensions. Then we improved the results by using Romberg acceleration which denoted by R(MSM) and R(SMM), later we compare between them.

### 2. Newton-Cotes Formulas

The Newton-Coates formulas are the most important methods of numerical integration, we review two rules of them Mid-point and Simpson and their correction terms if the function of integration continuous. Let the integral  $J$  defined by the formula

$$J = \int_0^p g(x) dx = \beta(k) + E_\beta(k) + R_\beta \tag{1}$$

Fox [7], where  $\beta(k)$  is the numerical rule to evaluate the integral  $J$ ,  $\beta$  denoted to a type of rule,  $E_\beta(k)$  is the correction terms for  $\beta(k)$  and  $R_\beta$  is the remainder which is related to truncation from  $E_\beta(k)$  after using a several terms of  $E_\beta(k)$ . The general formulas of Mid-point rule  $M(k)$  and Simpson's rule  $S(k)$  are:

$$M(k) = k \sum_{i=1}^u g(o + (i - 0.5)) \tag{2}$$

$$S(k) = \frac{k}{3} \left[ g(o) + g(p) + 2 \sum_{i=1}^{\frac{u}{2}-1} g(x_{2i}) + 4 \sum_{i=1}^{\frac{u}{2}} g(x_{2i+1}) \right] \tag{3}$$

where  $k = (p - o)/u$ .

The correction terms are large importance in improving the value of integration and accelerating the numerical value of integration to exact value. Fox [8] and Eghaar [5] are working to find the correction terms of each rule of Newton-Cotes, whereas the correction terms of Mid-point rule  $M(k)$  is:

$$E_M(k) = \frac{1}{6} k^2 (g'_u - g'_o) - \frac{7}{360} k^4 (g^{(3)}_u - g^{(3)}_o) + \frac{31}{15120} k^6 (g^{(5)}_u - g^{(5)}_o) - L \tag{4}$$

Fox [7] prove that the correction terms of Simpson's rule  $S(k)$  is:

$$E_S(k) = -\frac{1}{180} k^4 (g^{(3)}_u - g^{(3)}_o) + \frac{1}{1512} k^6 (g^{(5)}_u - g^{(5)}_o) - L \tag{5}$$

By using the mean value theorem for the formulas (4) and (5) we get:

$$E_S(k) = \frac{-(p-o)}{180} k^4 g^{(4)}(\mu_1) + \frac{(p-o)}{1512} k^6 g^{(6)}(\mu_2) + \dots \tag{6}$$

$$E_M(k) = \frac{(p-o)}{6} k^2 g^{(2)}(\eta_1) - \frac{7(p-o)}{360} k^4 g^{(4)}(\eta_2) + \frac{31(p-o)}{15120} k^6 g^{(6)}(\eta_3) - \dots \tag{7}$$

where  $\mu_i, \eta_i \in (o, p) \forall i = 1, 2, 3, \dots$ , Eghaar [5].

### 3. Romberg Accelerating

Suppose that  $\beta_1$  and  $\beta_2$  are two approximations values of the integral  $J = \int_0^p g(x) dx$  for two different values of  $k$  say  $k_1, k_2$  by

using any rule of Newton-Cotes formulas then:

$$J = \int_0^p g(x) dx = \beta_1 + \sum_{j=\delta}^{\infty} A_j k_1^j \tag{8}$$

$$J = \int_0^p g(x) dx = \beta_2 + \sum_{j=\delta}^{\infty} A_j k_2^j \tag{9}$$

where  $\sum_{j=\delta}^{\infty} A_j k_1^j$  is the correction terms. By solve the equations (8)

and (9) for  $J$  we get:

$$J = \int_0^p g(x) dx = \frac{k_1^\delta \beta_2 - \beta_1}{k_1^\delta - k_2^\delta} + \sum_{j=\delta+1}^{\infty} A_j \frac{k_1^\delta k_2^j - k_2^\delta k_1^j}{k_1^\delta - k_2^\delta} \tag{10}$$

Now let  $k_2 = 2 k_1$  then:  $J = \frac{2^\delta \beta_2 - \beta_1}{2^\delta - 1} + \sum_{j=\delta+1}^{\infty} \frac{2^\delta - 2^j}{2^\delta - 1} A_j k_1^j$ .

Ralston [4]

So the formula of Romberg Accelerating we depended on it is

$$J = \frac{2^\delta \beta_2 - \beta_1}{2^\delta - 1} \tag{11}$$

Where  $\delta$  is a power of  $k$  in the correction terms.

### 4. The Two Numerical Methods MSM, SMM and Their Correction Terms

#### Numerical method MSM

**Theorem 1:** If  $g(x, y, z)$  continuous and differentiable function at any point of the region  $D = [o, p] \times [q, r] \times [s, t]$  then the approximated value of integral  $J = \iiint_D g(x, y, z) dx dy dz$  can be determined

by:

$$MSM = \frac{k^3}{3} \sum_{i=1}^u \sum_{j=1}^v \sum_{l=1}^w \left[ g(x_i, y_j, z_l) + g(x_i, r, z_l) + 4 \sum_{j=1}^{\frac{v}{2}-1} g(x_i, y_{(2j-1)}, z_l) + 2 \sum_{j=1}^{\frac{v}{2}} g(x_i, y_{(2j)}, z_l) \right]$$

where  $y_{(2j)} = q + (2j)k$ ,  $j = 1, 2, \dots, \frac{u}{2} - 1$ ,

$y_{(2j-1)} = q + (2j-1)k$ ,  $j = 1, 2, \dots, \frac{u}{2}$ ,

$x_i = o + (i - 0.5)k$ ,  $i = 1, 2, \dots, u$ ,

$z_1 = s + (1 - 0.5)k$ ,  $1 = 1, 2, \dots, u$ .

And the formula of correction terms is

$$J - MSM(k) = \varpi_1 k^2 + \varpi_2 k^4 + \varpi_3 k^6 + L$$

where  $\varpi_1, \varpi_2, \varpi_3, K$  are constants.

**Proof:** We can rewrite the integral  $J$  in the following form:

$$J = \iiint_{s, q, o}^{t, r, p} g(x, y, z) dx dy dz = MSM(k) + \alpha_{MSM}(k) \tag{12}$$

Where  $MSM(k)$  is the approximation value of integral by using Mid-point rule on the interior dimension X, Simpson's rule on the middle dimension Y and Mid-point rule on the exterior dimension Z,  $\alpha_{MSM}(k)$  is the correction terms which could be added to the values of  $MSM(k)$ , and

$$k = (p - o)/u = (r - q)/u = (t - s)/u.$$

Calculate the approximated value of integral  $J$  by using Mid-point rule on the interior dimension X, Simpson's rule on the

middle dimension Y and Mid-point rule on the exterior dimension Z we get:

$$\begin{aligned} \int_s^t \int_o^p \int_q^r g(x, y, z) dx dy dz &= \frac{k^3}{3} \sum_{l=1}^u \sum_{i=1}^u \left[ g(x_i, q, z_1) + g(x_i, r, z_1) + 4 \sum_{j=1}^{\frac{u}{2}} g(x_i, y_{(2j-1)}, z_1) + 2 \sum_{j=1}^{\frac{u}{2}-1} g(x_i, y_{(2j)}, z_1) \right] \\ &+ \int_s^t \left[ \left( \frac{p-o}{6} k^2 g_{x^2}(\lambda_1, y, z) - \frac{7(p-o)}{360} k^4 g_{x^4}(\lambda_2, y, z) + \frac{31(p-o)}{15120} k^6 g_{x^6}(\lambda_3, y, z) - L \right) dy dz \right. \\ &+ k \int_s^t \sum_{i=1}^u \left[ -\frac{(r-q)k^4}{180} g_{y^4}(x_i, \phi_{1i}, z) + \frac{(r-q)k^6}{1512} g_{y^6}(x_i, \phi_{2i}, z) - L \right] dz \\ &+ k^2 \sum_{i=1}^u \left( \frac{t-s}{6} k^2 g_{z^2}(x_i, q, \xi_{1i}) - \frac{7(t-s)}{360} k^4 g_{z^4}(x_i, q, \xi_{2i}) + \frac{31(t-s)}{15120} k^6 g_{z^6}(x_i, q, \xi_{3i}) - L \right) \\ &+ k^2 \sum_{i=1}^u \left( \frac{t-s}{6} k^2 g_{z^2}(x_i, r, \xi_{1i}) - \frac{7(t-s)}{360} k^4 g_{z^4}(x_i, r, \xi_{2i}) + \frac{31(t-s)}{15120} k^6 g_{z^6}(x_i, r, \xi_{3i}) - L \right) \\ &+ 4k^2 \sum_{l=1}^{\frac{u}{2}} \sum_{i=1}^u \left( \frac{t-s}{6} k^2 g_{z^2}(x_i, y_{(2j-1)}, \xi_{1ij}) - \frac{7(t-s)}{360} k^4 g_{z^4}(x_i, y_{(2j-1)}, \xi_{2ij}) + \frac{31(t-s)}{15120} k^6 g_{z^6}(x_i, y_{(2j-1)}, \xi_{3ij}) - L \right) \\ &+ 2k^2 \sum_{l=1}^{\frac{u}{2}} \sum_{i=1}^u \left( \frac{t-s}{6} k^2 g_{z^2}(x_i, y_{(2j)}, \xi_{1ij}) - \frac{7(t-s)}{360} k^4 g_{z^4}(x_i, y_{(2j)}, \xi_{2ij}) + \frac{31(t-s)}{15120} k^6 g_{z^6}(x_i, y_{(2j)}, \xi_{3ij}) - L \right) \end{aligned}$$

Since  $g_{x^2}, g_{x^4}, \dots, g_{y^2}, g_{y^4}, \dots, g_{z^2}, g_{z^4}, \dots$  are continuous functions at any point in the region, then the correction terms becomes:

$$\begin{aligned} \alpha_{MSM}(k) &= (t-s)(r-q)(p-o) \left( \frac{k^2}{6} g_{x^2}(\zeta_1, \tau_1, \nu_1) - \frac{7k^4}{360} g_{x^4}(\zeta_2, \tau_2, \nu_2) + \frac{31k^6}{15120} g_{x^6}(\zeta_3, \tau_3, \nu_3) - L \right) \\ &+ (t-s)(r-q)(p-o) \left( -\frac{k^4}{180} g_{x^4}(\overline{\zeta_1}, \overline{\tau_1}, \overline{\nu_1}) + \frac{k^6}{1512} g_{x^6}(\overline{\zeta_2}, \overline{\tau_2}, \overline{\nu_2}) - L \right) + (t-s)(r-q)(p-o) \left( \frac{k^2}{6} g_{z^2}(\overline{\overline{\zeta_1}}, \overline{\overline{\tau_1}}, \overline{\overline{\nu_1}}) \right. \\ &\left. - \frac{7k^4}{360} g_{z^4}(\overline{\overline{\zeta_2}}, \overline{\overline{\tau_2}}, \overline{\overline{\nu_2}}) - \frac{31k^6}{15120} g_{z^6}(\overline{\overline{\zeta_3}}, \overline{\overline{\tau_3}}, \overline{\overline{\nu_3}}) - L \right) \\ \alpha_{MSM}(k) &= (t-s)(r-q)(p-o) \frac{k^2}{6} \left( g_{x^2}(\zeta_1, \tau_1, \nu_1) + g_{z^2}(\overline{\overline{\zeta_1}}, \overline{\overline{\tau_1}}, \overline{\overline{\nu_1}}) \right) + (t-s)(r-q)(p-o) \frac{-k^4}{180} \left( \frac{7}{2} g_{x^4}(\zeta_2, \tau_2, \nu_2) \right. \\ &\left. + g_{x^4}(\overline{\zeta_1}, \overline{\tau_1}, \overline{\nu_1}) - \frac{7}{2} g_{z^4}(\overline{\overline{\zeta_2}}, \overline{\overline{\tau_2}}, \overline{\overline{\nu_2}}) \right) + (t-s)(r-q)(p-o) \frac{k^6}{1512} \left( \frac{31}{10} g_{x^6}(\zeta_3, \tau_3, \nu_3) + g_{y^6}(\overline{\zeta_2}, \overline{\tau_2}, \overline{\nu_2}) \right. \\ &\left. - \frac{31}{10} g_{z^6}(\overline{\overline{\zeta_3}}, \overline{\overline{\tau_3}}, \overline{\overline{\nu_3}}) \right) + L \end{aligned}$$

where  $(\overline{\overline{\zeta_c}}, \overline{\overline{\tau_c}}, \overline{\overline{\nu_c}}), (\overline{\zeta_c}, \overline{\tau_c}, \overline{\nu_c}), (\zeta_c, \tau_c, \nu_c) \in D$ ,

Since  $g(x, y, z)$  is a continuous and differentiable function at any point in the region  $D$ , then we can write the correction terms as a formula:

$$J - MSM(h) = \omega_1 k^2 + \omega_2 k^4 + \omega_3 k^6 + L \dots (13)$$

where  $\omega_1, \omega_2, \omega_3, K$  are constants which depended on the partial derivatives.

### Numerical method SMM

**Theorem 1:** If  $g(x, y, z)$  continuous and differentiable function at any point of the region  $D = [o, p] \times [q, r] \times [s, t]$  then the approximated value of integral  $J = \int \int \int_D g(x, y, z) dx dy dz$  can be determined by:

$$SMM = \frac{k^3}{3} \sum_{j=1}^u \sum_{i=1}^u \left[ g(x_i, y_j, s) + g(x_i, y_j, t) + 4 \sum_{l=1}^{\frac{u}{2}} g(x_i, y_j, z_{(2l-1)}) + 2 \sum_{l=1}^{\frac{u}{2}-1} g(x_i, y_j, z_{(2l)}) \right]$$

where  $z_{(2l-1)} = s + (2l - 1)k$ ,  $l = 1, 2, \dots, \frac{u}{2}$ ,  $z_{(2l)} = s + (2l)k$ ,  $l = 1, 2, \dots, \frac{u}{2} - 1$ ,

$$y_j = q + (j - 0.5)k, \quad j = 1, 2, \dots, u, \quad x_i = o + (i - 0.5)k, \quad i = 1, 2, \dots, u.$$

And the formula of correction terms is  $J - SMM(k) = \psi_1 k^2 + \psi_2 k^4 + \psi_3 k^6 + L$

where  $\psi_1, \psi_2, \psi_3, K$  are constants.

**Proof:** We can rewrite the integral  $J$  in the following form:

$$J = \int_s^t \int_q^r \int_o^p g(x, y, z) dx dy dz = SMM(k) + \alpha_{SMM}(k) \dots (14)$$

where  $SMM(k)$  is the approximation value of integral by using Mid-point rule on both two dimensions of interior X and middle

$$\begin{aligned} \int_s^t \int_q^r \int_o^p g(x, y, z) dx dy dz &= \frac{k^3}{3} \sum_{j=1}^u \sum_{i=1}^u \left[ g(x_i, y_j, s) + g(x_i, y_j, t) + 4 \sum_{l=1}^{\frac{u}{2}} g(x_i, y_j, z_{(2l-1)}) + 2 \sum_{l=1}^{\frac{u}{2}-1} g(x_i, y_j, z_{(2l)}) \right] \\ &+ \int_s^t \int_q^r \left( \frac{(p-o)}{6} k^2 g_{x^2}(\vartheta_1, y, z) - \frac{7(p-o)}{360} k^4 g_{x^4}(\vartheta_2, y, z) + \frac{31(p-o)}{15120} k^6 g_{x^6}(\vartheta_3, y, z) - L \right) dy dz \\ &+ k \int_s^t \left( \sum_{i=1}^u \left( \frac{(r-q)}{6} k^2 g_{y^2}(x_i, \sigma_{1i}, z) - \frac{7(r-q)}{360} k^4 g_{y^4}(x_i, \sigma_{2i}, z) + \frac{31(r-q)}{15120} k^6 g_{y^6}(x_i, \sigma_{3i}, z) - L \right) \right) dz \\ &+ k^2 \sum_{j=1}^u \sum_{i=1}^u \left[ -\frac{(t-s)k^4}{180} g_{y^4}(x_i, y_j, \varepsilon_{1ij}) + \frac{(t-s)k^6}{1512} g_{y^6}(x_i, y_s, \varepsilon_{2ij}) - L \right] \end{aligned}$$

Since  $g_{x^2}, g_{x^4}, \dots, g_{y^2}, g_{y^4}, \dots, g_{z^2}, g_{z^4}, \dots$  are continuous functions at any point in the region, then the correction terms become:

$$\begin{aligned} \alpha_{SMM}(k) &= (t-s)(r-q)(p-o) \left( \frac{k^2}{6} g_{x^2}(\varphi_1, \nu_1, \theta_1) - \frac{7h^4}{360} g_{x^4}(\varphi_2, \nu_2, \theta_2) + \frac{31h^6}{15120} g_{x^6}(\varphi_3, \nu_3, \theta_3) - L \right) \\ &+ (t-s)(r-q)(p-o) \left( \frac{k^2}{6} g_{y^2}(\hat{\varphi}_1, \hat{\nu}_1, \hat{\theta}_1) - \frac{7h^4}{360} g_{y^4}(\hat{\varphi}_2, \hat{\nu}_2, \hat{\theta}_2) + \frac{31h^6}{15120} g_{y^6}(\hat{\varphi}_3, \hat{\nu}_3, \hat{\theta}_3) - L \right) \\ &+ (t-s)(r-q)(p-o) \left( \frac{-k^4}{180} g_{x^4}(\bar{\varphi}_1, \bar{\nu}_1, \bar{\theta}_1) + \frac{k^6}{1512} g_{x^4}(\bar{\varphi}_2, \bar{\nu}_2, \bar{\theta}_3) - L \right) \end{aligned}$$

$$\begin{aligned} \alpha_{MSM}(k) &= (t-s)(r-q)(p-o) \frac{k^2}{6} \left( g_{x^2}(\varphi_1, \nu_1, \theta_1) + g_{y^2}(\hat{\varphi}_1, \hat{\nu}_1, \hat{\theta}_1) \right) + (t-s)(r-q)(p-o) \frac{-k^4}{180} \left( \frac{7}{2} g_{x^4}(\varphi_2, \nu_2, \theta_2) \right. \\ &+ g_{y^4}(\hat{\varphi}_2, \hat{\nu}_2, \hat{\theta}_2) - \left. \frac{7}{2} g_{z^4}(\bar{\varphi}_1, \bar{\nu}_1, \bar{\theta}_1) \right) + (t-s)(r-q)(p-o) \frac{k^6}{1512} \left( \frac{31}{10} g_{x^6}(\varphi_3, \nu_3, \theta_3) + g_{y^6}(\hat{\varphi}_3, \hat{\nu}_3, \hat{\theta}_3) \right. \\ &\left. - \frac{31}{10} g_{z^6}(\bar{\varphi}_2, \bar{\nu}_2, \bar{\theta}_3) - L \right) \end{aligned}$$

where  $(\bar{\varphi}_b, \bar{\nu}_b, \bar{\theta}_b), (\hat{\varphi}_b, \hat{\nu}_b, \hat{\theta}_b), (\varphi_b, \nu_b, \theta_b) \in D, \forall b = 1, 2, 3, \dots$

where  $\psi_1, \psi_2, \psi_3, K$  are constants which depended on the partial derivatives.

Since  $g(x, y, z)$  is a continuous and differentiable function at any point in the region  $D$ , then we can write the correction terms as a formula:

$$J - SMM(k) = \psi_1 k^2 + \psi_2 k^4 + \psi_3 k^6 + L \quad (15)$$

### 5. Calculate the Triple Integrals By Using Two Rules R(MSM), R(SMM)

To calculate the triple integrals numerically by using the method R(MSM) or R(SMM), first compute the approximated value of integral  $J$  when  $u = 2$  by using the rule  $MSM$  which is:

$$\int_s^t \int_q^r \int_o^p g(x, y, z) dx dy dz = \frac{k^3}{3} \sum_{l=1}^2 \sum_{i=1}^2 [g(x_i, q, z_1) + g(x_i, r, z_1) + 4g(x_i, y_1, z_1)]$$

Fixed this value in our table, then put  $u = 4$  to finding another approximated value :

$$\int_0^t \int_0^r \int_0^p g(x, y, z) dx dy dz = \frac{k^3}{3} \sum_{l=1}^4 \sum_{i=1}^4 \left[ g(x_i, q, z_1) + g(x_i, r, z_1) + 4 \sum_{j=1}^2 g(x_i, y_{(2j-1)}, z_1) + 2g(x_i, y_2, z_1) \right]$$

Fixed this value in our table and doubling  $u$  each time. Next, we use Romberg acceleration with  $\delta = 2$  to find new approximated values and fixed them in the table then use Romberg acceleration with  $\delta = 4$  and so on until we get a value with the exact precision we want.

### 6. Examples and Results

**Example 1:** The integral  $\int_0^1 \int_0^1 \int_0^1 (0.2y+1.4z)e^{(x+y+z)} dx dy dz$

which analytical value equal to 4.723987907220 (rounded to twelve decimal places) with continuous integrand for each point  $(x, y, z) \in [0, 1] \times [0, 1] \times [0, 1]$ . Therefore, the formula of correction terms for this integral is similar to the two formulas (13),(15). Applying two methods R(MSM) and R(SMM), we obtained the results listed in tables (1) and (2). We conclude the following :

1. From table (1): when  $u = 32$  the approximated value is correct for three decimal places by using MSM rule while it is equal to analytical value with applying Romberg acceleration.
2. From table (2): when  $u = 64$  the approximated value is correct for three decimal places by using SMM rule while it is equal to analytical value with applying Romberg acceleration.  
Thus the method R(MSM) is the best in accuracy and the speed of approach.

**Example 2:** The integral  $\int_1^2 \int_1^2 \int_1^2 \sqrt{z} \ln(x+y+z) dx dy dz$  which analytical value equal to 1.83346759814308 with continuous integrand for each point  $(x, y, z) \in [1, 2] \times [1, 2] \times [1, 2]$ .

Applying two methods R(MSM) and R(SMM), we obtained the results listed in tables (3),(4) and we conclude the following :

1. From table (3): when  $u = 32$  the approximated value is correct for five decimal places by using MSM rule while it is equal to analytical value with applying Romberg acceleration.
2. From table (4): when  $u = 64$  the approximated value is correct for five decimal places by using SMM rule while it is equal to analytical value with applying Romberg acceleration.  
Thus the method R(MSM) is the best in the speed of approach.

**Example 3:** The integral  $\int_3^4 \int_2^3 \int_1^2 \frac{(1+y+z)}{\sqrt{(2x+y+3z)^3}} dx dy dz$  which analytical value equal to 0.10982704325114 with continuous integrand for each point  $(x, y, z) \in [1, 2] \times [2, 3] \times [3, 4]$ .

When calculated in both methods R(MSM) and R(SMM), we obtained the results listed in tables (5) and (6). We conclude the following :

1. From table (5): when  $u = 16$  the approximated value is correct for five decimal places by using MSM rule

while it is equal to analytical value with applying Romberg acceleration. [11]

2. From table (6): when  $u = 32$  the approximated value is correct for three decimal places by using SMM rule while it is equal to analytical value with applying Romberg acceleration.  
Thus the method R(MSM) is the best in the speed of approach.

**Example 4:** The integral  $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{x \sin(y) + \cos(z)}{x^4 + y^4 + z^4} dx dy dz$  which analytical value is unknown, and it have a continuous integrand for each

$$\text{point } (x, y, z) \in \left[ \frac{\pi}{2}, \frac{\pi}{4} \right] \times \left[ \frac{\pi}{2}, \frac{\pi}{4} \right] \times \left[ \frac{\pi}{2}, \frac{\pi}{4} \right].$$

We can't use the fundamental calculation theorems to evaluate this integral, therefore, we can replace it by the approximation methods R(MSM) and R(SMM) where we obtain the results listed in tables (7) and (8) respectively. Where we note that the values converge vertically towards the value of 0.2779460489911 as well as matching the values of integration in the last two rows when  $u = 64$  and  $u = 128$  by using the two methods R(MSM) and R(SMM). Therefore, it is possible to say that the value of integration is correct for thirteen decimal places when applying these two methods

**Example 5:** The integral  $\int_1^2 \int_1^2 \int_1^2 \frac{z \ln(x)e^y}{x^2 + y^2 + z^2} dx dy dz$  which analytical value is unknown, and it have a continuous integrand for each point  $(x, y, z) \in [1, 2] \times [1, 2] \times [1, 2]$ .

We can't use the fundamental calculation theorems to evaluate this integral, therefore, we can replace it by the approximation methods R(MSM) and R(SMM) where we obtain the results listed in tables (9) and (10) respectively. Where we note that the values converge vertically towards the value of 0.357703307442 as well as matching the values of integration in the last two rows when  $u = 64$  and  $u = 128$  by using the two methods R(MSM) and R(SMM). Therefore, it is possible to say that the value of integration is correct for thirteen decimal places when applying these two methods.

### 7. Discussion and Conclusion

The tables show the calculation of the approximate values of the triple integrations with continuous inversions in the two composite methods  $R(SMM)$ ,  $R(MSM)$  when the number of divisions on the three dimensions are equal and when applying Romberg acceleration gives them correct values (for several decimal places) compared to the actual values of integrations. We obtained a precision ranging from 13-12 in a decimal order and at from  $u = 64$  to  $u = 128$  for the method  $R(MSM)$  and ranging from  $u = 32$  to  $u = 128$  for the method  $R(SMM)$ , show table(11), charts (1) and (2).[12]

We concluded that the two methods  $R(SMM)$ ,  $R(MSM)$  can be used to calculate the values of the triple integrations, it values can't be found by the fundamental calculation theorems, where we observed approach of the approximate values towards a specific value as the number of divisions increased and at last the values are matched for different divisions when using correction terms, thus it can be act the value of integration, which is significantly evident in the integrations of fourth and fifth examples.

$$\int_0^1 \int_0^1 \int_0^1 (0.2y + 1.4z)e^{(x+y+z)} dx dy dz \quad R(MSM) \text{ by } u$$

Table 1:

u	MSM	R(MSM)
2	4.485297639273	
4	4.662885747387	4.722081783424
8	4.708619930815	4.723983516972
16	4.720140085959	4.723987904838
32	4.723025586921	4.723987907220

$$\int_0^1 \int_0^1 \int_0^1 (0.2y + 1.4z)e^{(x+y+z)} dx dy dz \quad R(SMM) \text{ by } u$$

Table 2:

u	SMM	R(SMM)
2	4.616969797898	
4	4.694916476577	4.720898702803
8	4.716567358244	4.723976693200
16	4.722123098644	4.723987900106
32	4.723521098596	4.723987907219
64	4.723871167127	4.723987907220

$$\int_1^2 \int_1^2 \int_1^2 \sqrt{z} \ln(x+y+z) dx dy dz \quad R(MSM) \text{ by } u$$

Table 3:

u	MSM	R(MSM)
2	1.835030834789	
4	1.833862754204	1.833473394009
8	1.833566674952	1.833467621059
16	1.833492385618	1.833467598192
32	1.833473796159	1.833467598143

$$\int_1^2 \int_1^2 \int_1^2 \sqrt{z} \ln(x+y+z) dx dy dz \quad R(SMM) \text{ by } u$$

Table 4:

u	SMM	R(SMM)
2	1.834687126496	
4	1.833784229163	1.833483263385
8	1.833547666537	1.833467848925
16	1.833487676292	1.833467600137
32	1.833472621571	1.833467598150
64	1.833468854244	1.833467598143

$$\int_3^4 \int_3^2 \int_3^2 \frac{(1+y+z)}{\sqrt{(2x+y+3z)^3}} dx dy dz \quad R(MSM) \text{ by } u$$

Table 5:

u	MSM	R(MSM)
2	0.1096983922776	
4	0.1097947734515	0.1098269005094
8	0.1098189690545	0.1098270431719
16	0.1098250242794	0.1098270432511

$$\int_3^4 \int_3^2 \int_3^2 \frac{(1+y+z)}{\sqrt{(2x+y+3z)^3}} dx dy dz \quad R(SMM) \text{ by } u$$

Table 6:

u	SMM	R(SMM)
2	0.1097736018997	
4	0.1098134878914	0.1098267832219
8	0.1098236417603	0.1098270425941
16	0.1098261920801	0.1098270432505
32	0.1098268304084	0.1098270432511

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \sin(y) + \cos(z)}{x^4 + y^4 + z^4} dx dy dz \quad R(MSM) \text{ by } u$$

Table 7:

u	MSM	R(MSM)
2	0.2798012279673	
4	0.2783661753968	0.2778878245400

8	0.2780484349711	0.2779461679596
16	0.2779714885614	0.2779460592776
32	0.2779523992019	0.2779460489425
64	0.2779476359406	0.2779460489911
128	0.2779464456908	0.2779460489911

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \sin(y) + \cos(z)}{x^4 + y^4 + z^4} dx dy dz \quad R(SMM) \text{ by } u$$

Table 8:

u	SMM	R(SMM)
2	0.2818993084029	
4	0.2789518137940	0.2779693155911
8	0.2781973868623	0.2779443509268
16	0.2780088712720	0.2779460676676
32	0.2779617537385	0.2779460489590
64	0.2779499751256	0.2779460489911
128	0.2779470305214	0.2779460489911

$$\int_1^2 \int_1^2 \int_1^2 \frac{z \ln(x) e^y}{x^2 + y^2 + z^2} dx dy dz \quad R(MSM) \text{ by } u$$

Table 9:

u	MSM	R(MSM)
2	0.367651030716	
4	0.360222898238	0.357746854078
8	0.358335236085	0.357703292787
16	0.357861417184	0.357703308609
32	0.357742842865	0.357703307443
64	0.357713191798	0.357703307442
128	0.357705778562	0.357703307442

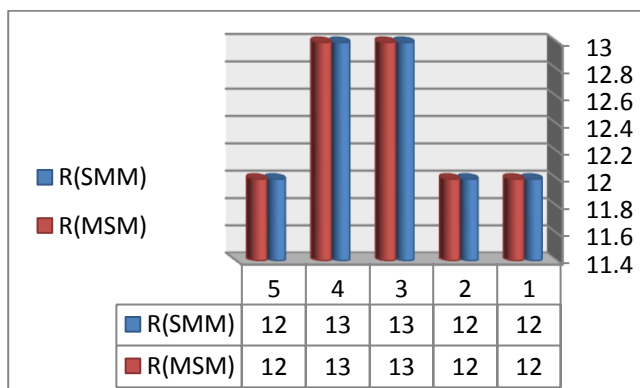
$$\int_1^2 \int_1^2 \int_1^2 \frac{z \ln(x) e^y}{x^2 + y^2 + z^2} dx dy dz \quad R(SMM) \text{ by } u$$

Table 10:

u	SMM	R(SMM)
2	0.370449247119	
4	0.360906877273	0.357726087325
8	0.358505371394	0.357703454908
16	0.357903898707	0.357703308076
32	0.357753459998	0.357703307443
64	0.357715845878	0.357703307442
128	0.357706442070	0.357703307442

Table 11: The Number of Correct Hierarchies to the Number of Divisions

No.	The integral	$\frac{R(SMM)}{u}$	$\frac{R(MSM)}{u}$
1	$\int_0^1 \int_0^1 \int_0^1 (0.2y + 1.4z)e^{(x+y+z)} dx dy dz$	$\frac{12}{64}$	$\frac{12}{32}$
2	$\int_1^2 \int_1^2 \int_1^2 \sqrt{z} \ln(x+y+z) dx dy dz$	$\frac{12}{64}$	$\frac{12}{32}$
3	$\int_3^4 \int_3^2 \int_3^2 \frac{(1+y+z)}{\sqrt{(2x+y+3z)^3}} dx dy dz$	$\frac{13}{32}$	$\frac{13}{16}$
4	$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \sin(y) + \cos(z)}{x^4 + y^4 + z^4} dx dy dz$	$\frac{13}{128}$	$\frac{13}{128}$
5	$\int_1^2 \int_1^2 \int_1^2 \frac{z \ln(x) e^y}{x^2 + y^2 + z^2} dx dy dz$	$\frac{12}{128}$	$\frac{12}{128}$



R(SMM) and R(MSM)Chart(1) showing the accuracy of the two methods

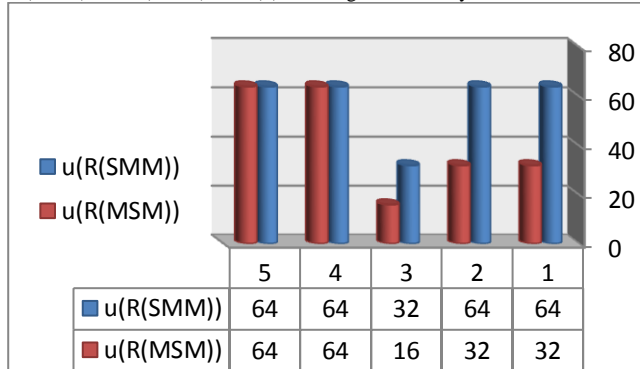


Chart 2: Showing the speed of approach of the two methods R(SMM) and R(MSM)

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