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Research paper

Parikh factor matrices for finite words of rectangular Hilbert space filling curve

S. Jeya Bharathi[1], K. Thiagarajan[2], K. Navaneetham[3*]

[1]*Associate Professor, Department of Mathematics, Thiagarajar College of Engineering, Madurai.*
[2]*Associate Professor, Department of Mathematics, PSNA College of Engg. & Tech., Dindigul.*
[3]*Assistant Professor, Department of Mathematics, PSNA College of Engg. & Tech., Dindigul.*
Research Scholar, Ph.D-CB, Bharathiar University, Coimbatore.
**Corresponding author E-mail:navaneekandhan@gmail.com*

Abstract

Ordered Factor Patterns in a word over an ordered alphabet aredefined. Also, Parikh StrictlyAscending FactorMatrix and Parikh StrictlyDescending Factor Matrix of a given word are introduced. The relation of these matrices with Ordered Factor Patterns is discussed. Moreover, the ParikhStrictly Ascending FactorMatrices and the Parikh Strictly Descending FactorMatricesfor finitewords of Rectangular Hilbert Space Filling Curve are determined.

Keywords: *Ordered patterns, rises, descents, parikh matrix, factors, rectangular space filling curve.*

1. Introduction

Space Filling Curves are useful in applications where a traversal of a multidimensional grid is needed. Sample applications are image halftoning,data organization, data compression and color quantization. The goal of the research presented in this paper is to extend the concept of Space Filling Curves(SFC) on square frame to Space Filling Curves on rectangular frame. Just as SFCs are convoluted linesthat fill a square, these SFCs are carefully elaborated to fill a rectangle.

In [4], the Parikh matrix of a word *u* is introduced. The basic numerical quantity investigated in this paper is $|w|_u$,the number of occurrences of a word u as a scattered subword of a word w. This matrix contains information on the number of occurrences of some sub words of *u*.The Authors [1] investigated the injectivity of the Parikh matrix mapping mainly on the binary alphabet. Huldah Samuel [3] discussed about some results and properties of Generalized Parikh Matrix for finite words. The author has introduced the notion of Hilbert words in [5]. Combinatorics on words has been analyzed in [2]. Counting occurrences of some patterns was done in [6].The author has discussed in [7] about occurrences of some patterns, subsequences and sub words in sigma-sequence. Counting ordered patterns in words generated by morphisms was done in [8]. The authors in [9] described Rectangular Hilbert Curve through 7-power free infinite word.

Geometric Generation and representation through a grammar for Rectangular Hilbert Space Filling Curve (RHSFC) is done in section II. Finite words for finite iterations of this RHSFC are formed in the third section. Ordered Factor Patterns in the finite words were analyzed in the next section. Finally, Parikh Strictly Ascending Factor Matrix and Parikh Strictly Descending Factor Matrix arepresented and they are established to the finite words of RHSFC.

2. Rectangular hilbert space filling curve

Geometric Generation of RHSFC[9]

A Space Filling Curve maps a 1-dimensional space onto a higher-dimensional space e.g., the unit interval onto the unit square. A geometric generation principle for the extension of Hilbert curve construction to fill a rectangle is suggested. Let us assume that the unitinterval Ican be mapped continuously onto the rectangle R $\left[0,\frac{2}{3}\right] x [0,1]$.

Initial mapping

If I is partitioned into six congruent subintervals then it should be possible to partition R into six congruent sub squares, such thateach subinterval will be mapped continuously onto one of the sub squares.

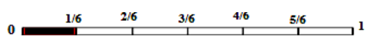

Fig. 1(a): Initial iteration

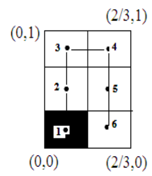

Fig. 1(b): Initial iteration

Iteration mapping

If the subintervals are partitioned into nine congruent subintervals then it should be possible to partition the sub squares into nine congruent sub squares, such that each subinterval will be mapped continuously onto one of the sub squares. This reasoning can be repeated by again partitioning each subinterval into nine congruent subintervals and doing the same for the respective sub squares. When repeating this procedure make sure that the sub squares are arranged in such a way that adjacent sub squares correspond to adjacent subintervals. Like this the overall continuity of the mapping is preserved. If an interval corresponds to a square, then its subintervals must correspond to the sub squares of that square. This inclusion relationship assures that a mapping of the n^{th} iteration preserves the mapping of the $(n\text{-}1)^{th}$ iteration.

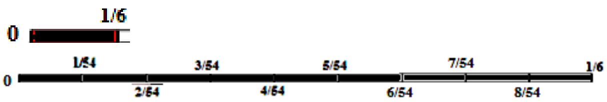

Fig. 2(a): First iteration

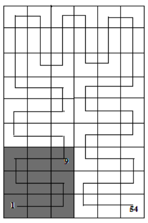

Fig. 2(b): First Iteration

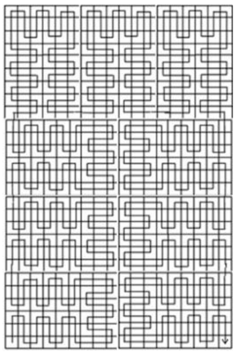

Fig. 3: Second Iteration

Representation of RHSFC through a grammar [9]

In the construction of the Rectangular Hilbert curve, four templates are used in every iteration. The template H is the initial graph or it can be called as the generator graph. The template A is obtained from H by rotating through an angle 90°anticlockwise and by reversing the direction. The template B is obtained from H by rotating anticlockwise through an angle 180°. The template C is obtained from H by rotating through an angle 270°anticlockwise and by reversing the direction. Their formationsare illustrated in the Fig. 4(a). These templates are substituted in every iteration step into a first iteration of the RHSFC.

These fixed substitution procedure can be described by a grammar G = (V,T,P,S) where V = $\{H, A, B, C\}$,

T = $\{\uparrow, \downarrow, \rightarrow, \leftarrow\}$, S = $\{H\}$ and P is defined by

H $\rightarrow$ [C $\uparrow$ C $\uparrow$ C $\uparrow$ H $\rightarrow$ H $\rightarrow$ H $\downarrow$ A $\downarrow$ A $\downarrow$ A]

A $\rightarrow$ [B $\leftarrow$ B $\leftarrow$ B $\leftarrow$ A $\downarrow$ A $\downarrow$ A $\rightarrow$ H $\rightarrow$ H $\rightarrow$ H]

C $\rightarrow$ [H $\rightarrow$ H $\rightarrow$ H $\rightarrow$ C $\uparrow$ C $\uparrow$ C $\leftarrow$ B $\leftarrow$ B $\leftarrow$ B]

B $\rightarrow$ [A $\downarrow$ A $\downarrow$ A $\downarrow$ B $\leftarrow$ B $\leftarrow$ B $\uparrow$ C $\uparrow$ C $\uparrow$ C]

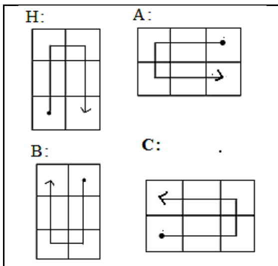

Fig. 4(a): Templates

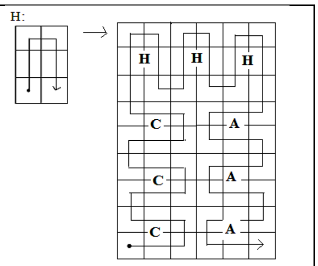

Fig. 4(b): Substitution for H

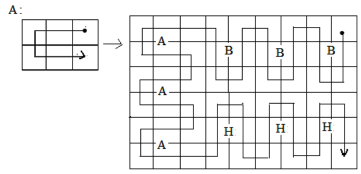

Fig. 4(c): Substitution for A

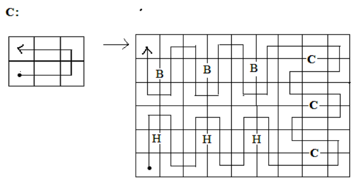

Fig. 4(e): Substitution forC

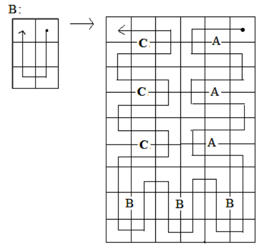

Fig. 4(d): Substitution for B

3. RHSFC and Finite Word

Finite word S_n for RHSFC[9]

The construction of the Rectangular Hilbert Space Filling Curve is obtained by drawing, without removing the pen from the surface of the paper to fill the rectangle of size $3^n \times 2^n$, an infinite succession of unit lines and double unit lines left, right, up or down. Thus the finite iterations of this curve can be represented by an finite words over the alphabet $\sum = \{u, d, r, l, \bar{u}, \bar{d}, \bar{r}, \bar{l}\}$ where u stands for unit line up, d stands for unit line down, r stands for unit line right, ℓ stands for unit line left, $\bar{u}$ stands for double units line up, $\bar{d}$ stands for double units line down, $\bar{r}$ stands for double units line right, $\bar{\ell}$ stands for double units line left.

Let us explain the algorithm of RHSFC. The general idea is to divide, at step n, the rectangle into $6(9)^{n-1}$ equal sub squares each of them containing an equal length part of the curve (except the first and the last ones which contain a part of length ½). The curve so obtained is then depicted by a word of length $(4(9)^{n-1}-1)$ which we call the n^{th} Rectangular Hilbert finite word S_n.

Therefore $S_1 = \bar{u}\ r\ \bar{d}$

Similarly $S_2 = \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}$

u

$\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}$

d

$\bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}$

$S_3 = \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}$

$\ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}$

$\ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}\ \ell\ \bar{u}$

$\ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d$

$\bar{\ell}\ d\ \bar{r}$

r

$\bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d$

$\bar{\ell}\ d\ \bar{r}$

r

$\bar{r}$ u $\bar{\ell}$ u $\bar{r}$ u $\bar{\ell}$ u $\bar{r}$ u $\bar{\ell}$ u $\bar{u}$ r $\bar{d}$ r $\bar{u}$ r $\bar{d}$ r $\bar{u}$ r $\bar{d}$ d $\bar{\ell}$ d $\bar{r}$
d $\bar{\ell}$ d $\bar{r}$ d $\bar{\ell}$ d $\bar{r}$
d
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$\bar{d}$ r $\bar{u}$ r $\bar{d}$ r $\bar{u}$ r $\bar{d}$

Now let us define on $\sum$ three literal morphisms f, t_ℓ and t_r by

f(u) = d, f(d) = u, f(r) = r, f(ℓ) = ℓ ,
f($\bar{u}$)= $\bar{d}$,f($\bar{d}$) = $\bar{u}$,f($\bar{r}$) = $\bar{r}$,f($\bar{\ell}$) = $\bar{\ell}$
t_ℓ (u) = ℓ , t_ℓ (d) = r, t_ℓ (r) = u, t_ℓ (ℓ) = d
t_ℓ ($\bar{u}$) = $\bar{\ell}$, t_ℓ ($\bar{d}$) = $\bar{r}$, t_ℓ ($\bar{r}$) = $\bar{u}$, t_ℓ ($\bar{\ell}$) = $\bar{d}$
t_r (u) = r, t_r (d) = ℓ t_r (r) = d, t_r (ℓ) = u,
t_r ($\bar{u}$) = $\bar{r}$, t_r ($\bar{d}$) = $\bar{\ell}$, t_r ($\bar{r}$) = $\bar{d}$, t_r ($\bar{\ell}$) = $\bar{u}$.

These three morphisms represent a vertical flip, a quarter turn left rotation, and a quarter turn right rotation of H respectively.
Using these morphisms, the recurrence relation for finite word S_n can be given by

$$S_{n+1} = \rho(S_n)u\rho(S_n)u\rho(S_n)uS_nrS_nrS_nd\lambda(S_n)d\lambda(S_n)d\lambda(S_n) \quad \textbf{(1)}$$

where $\rho = t_\ell \circ f$ and $\lambda = t_r \circ f$

Theorem 3.1: For any n ≥ 1,

1. $|S_n|$ = $4(9)^{n-1} - 1$
2. $|S_n|_{\bar{u}} = |S_n|_{\bar{d}}$ and $|S_n|_{\bar{r}} = |S_n|_{\bar{\ell}} = |S_n|_u = |S_n|_d$
3. $|S_{n+1}|_{\bar{u}} = 3|S_n|_{\bar{u}} + 3|S_n|_{\bar{r}} + 3|S_n|_{\bar{\ell}}$
4. $|S_{n+1}|_{\bar{d}} = 3|S_n|_{\bar{d}} + 3|S_n|_{\bar{\ell}} + 3|S_n|_{\bar{r}}$
5. $|S_{n+1}|_{\bar{r}} = 3|S_n|_{\bar{r}} + 3|S_n|_{\bar{u}} + 3|S_n|_{\bar{d}}$
6. $|S_{n+1}|_{\bar{\ell}} = 3|S_n|_{\bar{\ell}} + 3|S_n|_{\bar{d}} + 3|S_n|_{\bar{u}}$
7. $|S_{n+1}|_r = 3|S_n|_r + 3|S_n|_u + 3|S_n|_d + 2$
8. $|S_{n+1}|_u = 3|S_n|_u + 3|S_n|_r + 3|S_n|_\ell + 3$
9. $|S_{n+1}|_d = 3|S_n|_d + 3|S_n|_\ell + 3|S_n|_r + 3$
10. $|S_{n+1}|_\ell = 3|S_n|_\ell + 3|S_n|_d + 3|S_n|_u$
11. $|S_n|_{\bar{u}} = |S_n|_{\bar{d}} = \begin{cases} \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad even \\ \dfrac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad odd \end{cases}$
12. $|S_n|_{\bar{r}} = |S_n|_{\bar{\ell}} = |S_n|_u = |S_n|_d = \begin{cases} \dfrac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad even \\ \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \end{cases}$

$|S_n|_u$ = 0 means that there is zero occurrence of u in S_n.

13. $|S_n|_r = \begin{cases} \dfrac{3^n(3^{n-2}+1)}{2} - 1, & if \quad n \quad is \quad odd \\ \dfrac{3^{n-1}(3^{n-1}+1)}{2} - 1, & if \quad n \quad is \quad even \end{cases}$
14. $|S_n|_\ell = \begin{cases} \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \\ \dfrac{3^n(3^{n-2}-1)}{2}, & if \quad n \quad is \quad even \end{cases}$

Proof

The recurrence relation for $|S_n|$ is given by $|S_{n+1}|$ = $9|S_n|$ +8, $|S_1|$ = 3. Solving this equation with the initial condition, the equality (1) is obtained. The equalities from (2) to (10) can be obtained from the definition of ρ , λ and the formation **(1)** of S_n. Other equalities can be proved by using induction on n.

4. Ordered factor patterns

Ordered patterns [6]

Basic concepts

* Let A be a totally ordered alphabet with n elements and let N be the ordered alphabet whose letters are the first n positive integers. An ordered pattern is any word over
$\{N \cup \{\#\}\}$

Example

Let A = { a,b,c,d,e,f} with a < b < c < d < e < f.

- The word v = eafdbc contains one occurrence of the ordered pattern 2#31, namely the subsequence efd.
- In v, the ordered pattern 2#3#1 occurs in three occurences:efd, efb,efc.
- The ordered pattern 231 does not occur in v.

Rises and descents

- The number of Rises R(v) in a word v over Σ is the number of occurrences of the ordered pattern 12.
- The number of Descents D(v) in a word v over Σ is the number of occurrences of the ordered pattern 21.

Definition: ordered factor patterns

* Let A be a totally ordered alphabet with n elements and let N be the ordered alphabet whose letters are the first npositive integers.

- An Ordered Factor Pattern (OFP) is any word over N and it is denoted by OFP(12), OFP(342) etc.

Example

Let A = { a,b,c,d,e,f} with a < b < c < d < e < f.

- The word v = efdbca contains two occurrences of the ordered factor patternsOFP(231) namely the factors efd and bca.
- It does not contain the factors abc or bcd or cde or def. Hence it does not contain the ordered factor pattern OFP(123).

Definition: factor rises

The number of Factor RisesFR(v) in a word v over Σ is the number of occurrences of OFP(12) in v.

Definition: Factor Descents

The number of Factor Descents FD(v) in a word v over Σ is the number of occurrences of OFP(21) in v.

Factor rises and factor descents in S_n

Let us order the letters of S_n as

$\bar{u} < u < r < \bar{r} < \bar{d} < d < \ell < \bar{\ell}$

The number of Factor Rises $FR(S_n)$ is the number of occurrences of OFP(12). (i.e.) the number of occurrences of the factors $\bar{u}u$, ur, $r\bar{r}$, $\bar{r}\bar{d}$, $\bar{d}d$, $d\ell$ and $\ell\bar{\ell}$.

Occurrences of the factors $\bar{u}u$, ur, $r\bar{r}$, $\bar{r}\bar{d}$, $\bar{d}d$, $d\ell$ and $\ell\bar{\ell}$ in S_n

Let G be a grammar G = (V,T,P,S) where

V = $\{A, B, C, D\}$, T = $\{\bar{u}, u, r, \bar{r}, d, \bar{d}, \ell, \bar{\ell}\}$, S = $\{A\}$ and

P is defined by

A $\rightarrow$ BuBuBuArArAdDdDdD

B $\rightarrow$ ArArArBuBuBℓCℓCℓC

C $\rightarrow$ DdDdDdCℓCℓCuBuBuB

D $\rightarrow$ CℓCℓCℓDdDdDrArArA

A $\rightarrow$ $\bar{u}$ r $\bar{d}$, B $\rightarrow$ $\bar{r}$ u $\bar{\ell}$, C $\rightarrow$ $\bar{d}$ ℓ $\bar{u}$, D $\rightarrow$ $\bar{\ell}$ d $\bar{r}$

Then G generates the finite words S_n

Asan example, the derivation steps for S_2 are given below.

A $\Rightarrow$ BuBuBuArArAdDdDdD

(Substitution for a non-terminal should be done for all occurrences of that non-terminal in the string of the stage.)

$\Rightarrow$ $\bar{r}$ u $\bar{\ell}$ u $\bar{r}$ u $\bar{\ell}$ u $\bar{r}$ u $\bar{\ell}$ u $\bar{u}$ r $\bar{d}$ r $\bar{u}$ r $\bar{d}$ r $\bar{u}$ r $\bar{d}$ d $\bar{\ell}$ d $\bar{r}$ d $\bar{\ell}$ d $\bar{r}$ d $\bar{\ell}$ d $\bar{r}$

=S_2

Now let A_2 = BuBuBuArArAdDdDdD

(i.e.) the string of non-terminals and terminals obtained at the just previous stage of last step in the derivation of S_2.

Similarly A_3, A_4, …. and B_i, C_i, D_i for i = 1,2,3,… can be assumed.

Clearly $|S_n|_{\bar{u}u} = |A_n|_{Cu}$

(1) $|A_{n+1}|_{Cu} = |B_n|_{Cu} + |A_n|_{Cu} + |D_n|_{Cu} + \begin{cases} 3, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$

(2) $|B_{n+1}|_{Cu} = |A_n|_{Cu} + |B_n|_{Cu} + |C_n|_{Cu} + \begin{cases} 2, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$

(3) $|C_{n+1}|_{Cu} = |D_n|_{Cu} + |C_n|_{Cu} + |B_n|_{Cu} + \begin{cases} 2, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$

(4) $|D_{n+1}|_{Cu} = |C_n|_{Cu} + |D_n|_{Cu} + |A_n|_{Cu}$

Using the above recurrence relations from(1) to (4) the number of occurrences of the factor $\bar{u}u$ can be calculated.

Table 1: Number of occurrences of $\bar{u}$ u

n	$\|S_n\|_{u\bar{u}} = \|A_n\|_{Cu}$	$\|B_n\|_{Cu}$	$\|C_n\|_{Cu}$	$\|D_n\|_{Cu}$
1	0	0	0	0
2	0	0	1	0
3	3	5	5	3
4	33	39	39	33
5	318	335	335	315
6	2904	2964	2955	2904
7	26319	26471	26471	26289
8	237237	237783	237693	237237

Similarly, the number of occurrences of $r\bar{r}$, $\bar{d}d$ and $\ell\bar{\ell}$ can be found and they are given in the tables from 2 to 4 respectively.

Clearly $|S_n|_{ur} = |S_n|_{\bar{r}\bar{d}} = |S_n|_{d\ell} = 0$

Table 2: Number of Occurrences of $r\bar{r}$

n	$\|S_n\|_{r\bar{r}} = \|A_n\|_{bB}$	$\|B_n\|_{bB}$	$\|C_n\|_{bB}$	$\|D_n\|_{bB}$
1	0	0	0	0
2	0	1	0	0
3	5	5	3	3
4	39	39	33	33
5	335	335	315	318
6	2964	2955	2904	2904
7	26471	26471	26289	26319
8	237783	237693	237237	237237

Table 3: Number of Occurrences of $\bar{d}d$

n	$\|S_n\|_{\bar{d}d} = \|A_n\|_{Ac}$	$\|B_n\|_{Ac}$	$\|C_n\|_{Ac}$	$\|D_n\|_{Ac}$
1	0	0	0	0
2	1	0	0	0
3	5	3	3	5
4	39	33	33	39
5	335	315	318	335
6	2955	2904	2904	2964
7	26471	26289	26319	26471
8	237693	237237	237237	237783

Table 4: Number of Occurrences of $\ell\bar{\ell}$

n	$\|S_n\|_{\ell\bar{\ell}} = \|A_n\|_{mD}$	$\|B_n\|_{mD}$	$\|C_n\|_{mD}$	$\|D_n\|_{mD}$
1	0	0	0	0
2	0	0	0	1
3	3	3	5	5
4	33	33	39	39
5	315	318	335	335
6	2904	2904	2964	2955
7	26289	26319	26471	26471
8	237237	237237	237783	237693

Table 5: Number of Occurrences of FR(Sn)

n	**Factor Rises in S_n**
1	1
2	16
3	144
4	1303
5	11727
6	105550
7	949950
8	8549557

Similarly Factor DescentsinS$_n$can be calculated.

5. Parikh factor matrix

Parikh strictly ascending factor matrix

Consider the (strictly) ordered alphabet

$a_1 < a_2 < a_3 < .... < a_n$

The Parikh Strictly Ascending Factor Matrix(PSAFM)gives ordered factor patterns in strictly ascending way and isdefined for a word w as follows.

PSAFM(w) =

$$\begin{pmatrix} |w|_{a_1} & |w|_{a_1a_2} & |w|_{a_1a_2a_3} & |w|_{a_1a_2a_3a_4} & \cdots & |w|_{a_1a_2\ldots a_n} \\ 0 & |w|_{a_2} & |w|_{a_{21}a_3} & |w|_{a_2a_3a_4} & \cdots & |w|_{a_2a_3\ldots a_n} \\ 0 & 0 & |w|_{a_3} & |w|_{a_3a_4} & \cdots & |w|_{a_3a_4\ldots a_n} \\ 0 & 0 & 0 & \ldots\ldots & \cdots & |w|_{a_1a_2a_3} \\ 0 & 0 & 0 & 0 & |w|_{a_{n-1}} & |w|_{a_{n-1}a_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & |w|_{a_n} \end{pmatrix}$$

where $|w|_{a_ia_{i+1}\ldots a_j}$ (i≤ j)represents the number of occurrences of the factor $a_ia_{i+1}\ldots a_j$.

The sum of first leading diagonal elements of PSAFM gives the number of occurrences of all letters in w.

The sum of second leading diagonal elements ofPSAFM gives the number of occurrences of Factor Rises FR(w) in w.

(i.e.) FR(w) = $\sum_{j=1}^{n-1} |w|_{a_ja_{j+1}}$

The sum of third leading diagonal elements PSAFM gives the number of occurrences of OFP(123) in w.

(i.e.) $|w|_{OFP(123)} = \sum_{j=1}^{n-2} |w|_{a_ja_{j+1}a_{j+2}}$

Lastly, $|w|_{OFP(123..n)} = |w|_{a_1a_2\ldots a_n}$

Parikh Ascending Factor Matrix for S_n

As the letters of S_n are ordered as $\bar{u} < u < r < \bar{r} < \bar{d} < d < \ell < \bar{\ell}$, the PSAFM for S_n can be given by

PSAFM(S_n) =

$$\begin{pmatrix} |S_n|_{\bar u} & |S_n|_{\bar u u} & |S_n|_{\bar u ur} & |S_n|_{\bar u u r\bar r} & |S_n|_{\bar u u r\bar r\bar d} & |S_n|_{\bar u u r\bar r\bar d d} & |S_n|_{\bar u u r\bar r\bar d d\ell} & |S_n|_{\bar u u r\bar r\bar d d\ell\bar\ell} \\ 0 & |S_n|_{u} & |S_n|_{ur} & |S_n|_{u r\bar r} & |S_n|_{u r\bar r\bar d} & |S_n|_{u r\bar r\bar d d} & |S_n|_{u r\bar r\bar d d\ell} & |S_n|_{u r\bar r\bar d d\ell\bar\ell} \\ 0 & 0 & |S_n|_{r} & |S_n|_{r\bar r} & |S_n|_{r\bar r\bar d} & |S_n|_{r\bar r\bar d d} & |S_n|_{r\bar r\bar d d\ell} & |S_n|_{r\bar r\bar d d\ell\bar\ell} \\ 0 & 0 & 0 & |S_n|_{\bar r} & |S_n|_{\bar r\bar d} & |S_n|_{\bar r\bar d d} & |S_n|_{r\bar r\bar d d\ell} & |S_n|_{r\bar r\bar d d\ell\bar\ell} \\ 0 & 0 & 0 & 0 & |S_n|_{\bar d} & |S_n|_{\bar d d\ell} & |S_n|_{\bar d d\ell} & |S_n|_{\bar d d\ell\bar\ell} \\ 0 & 0 & 0 & 0 & 0 & |S_n|_{d} & |S_n|_{d\ell} & |S_n|_{d\ell\bar\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & |S_n|_{\ell} & |S_n|_{\ell\bar\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & |S_n|_{\bar\ell} \end{pmatrix}$$

$$= \begin{pmatrix} |S_n|_{\bar u} & |S_n|_{\bar u u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |S_n|_{u} & |S_n|_{ur} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & |S_n|_{r} & |S_n|_{r\bar r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |S_n|_{\bar r} & |S_n|_{\bar r\bar d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & |S_n|_{\bar d} & |S_n|_{\bar d d\ell} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & |S_n|_{d} & |S_n|_{d\ell} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & |S_n|_{\ell} & |S_n|_{\ell\bar\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & |S_n|_{\bar\ell} \end{pmatrix}$$

Using Theorem 3.1 and the Tables from 1 to 4, PSAFM for S_n can be formed.

$$PSAFM(S_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$PSAFM(S_2) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$PSAFM(S_3) = \begin{pmatrix} 45 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 53 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 45 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 36 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

By extending this process, PSAFM(S_n) can be acquired for any n.

Parikh strictly descending factor matrix

Consider the ordered alphabet $a_1 < a_2 < a_3 < \ldots. < a_n$

The Parikh Strictly Descending Factor Matrix (PSDFM) of a word gives ordered factor patterns in strictly descending way and is defined for a word was follows.

PSDFM(w)

$$\begin{pmatrix} |w|_{a_n} & |w|_{a_na_{n-1}} & |w|_{a_na_{n-1}a_{n-2}} & |w|_{a_na_{n-1}a_{n-2}a_{n-3}} & \cdots & |w|_{a_na_{n-1}\ldots a_1} \\ 0 & |w|_{a_{n-1}} & |w|_{a_{n-1}a_{n-2}} & |w|_{a_{n-1}a_{n-2}a_{n-3}} & \cdots & |w|_{a_{n-1}\ldots a_1} \\ 0 & 0 & |w|_{a_{n-2}} & |w|_{a_{n-2}a_{n-3}} & \cdots & |w|_{a_{n-2}a_{n-3}\ldots a_1} \\ 0 & 0 & 0 & \ldots\ldots & \cdots & |w|_{a_3a_2a_1} \\ 0 & 0 & 0 & 0 & |w|_{a_2} & |w|_{a_2a_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & |w|_{a_1} \end{pmatrix}$$

where $|w|_{a_ia_{i-1}\ldots a_k}$ (i ≥ k) represents number of occurrences of the factor $a_ia_{i-1}\ldots a_k$.

The sum of first leading diagonal elements of PSDFM gives the number of occurrences of all letters in w.

The sum of second leading diagonal elements of PSDFM gives the number of occurrences of FactorDescents in w.

(i.e.) FD(w) = $\sum_{j=1}^{n-1} |w|_{a_{j+1}a_j}$

The sum of third leading diagonal elements of PSDFM gives the number of occurrences of the ordered factor pattern OFP(321) in w.

(i.e.) $|w|_{OFP(321)} = \sum_{j=1}^{n-2} |w|_{a_{j+2}a_{j+1}a_j}$

Lastly, $|w|_{OFP(n(n-1)..21)} = |w|_{a_na_{n-1}\ldots a_1}$

As the way done for PSAFM, the PSDFM for the finite words S_n can be determined for any n.

6. Conclusion

Ordered Factor Patterns, defined in this article, are the patterns of an ordered sub factors of a word w. The Parikh Strictly Ascending Factormatrix and Parikh Strictly Descending Factor Matrix of a word wwaspresented. These matrices contain information on the number of occurrences of some ordered sub factors of w.

7. Further Research

Analytical properties and Algebraic structures of the Ordered Factor Patterns have to be analyzed.

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