International Journal of Engineering & Technology, 7 (3) (2018) 1150-1153

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET
doi: 10.14419/ijet.v7i3.12603
Research paper

Monophonic Wirelength in Graph Embedding

Sindhuja G. Michael [1] *, K. Uma Samundesvari [2]

[1] Department of Mathematics, Ponjesly College of Engineering, Nagercoil-629 003, Kanyakumari, Tamilnadu, India
[2] Department of Mathematics, Noorul Islam Centre for Higher Education,Thuckalay-629 180, Kanyakumari, Tamilnadu, India
**Corresponding author E-mail:sindhujajefic@gmail.com*

Abstract

In this paper, we define the monophonic embedding of graph G into another graph H and this paper presents a monophonic algorithm to find the monophonic wirelength of circulant networks G[n, ±S], where S ⊆ {1,2,3,⋯,n/2} into the family of Cycle C_n, n≥ 4. The mono-phonic embedding of a graph G into a graph H is an embedding denoted by $f_m$is a bijective map from the vertex set of G into the vertex set of H and f_m is a one-one mapping from the edge set (x, y) of G into $P_m(H)$ where $P_m(H)$ is the set of monophonic paths between $f_m(x)$ and $f_m(y)$ for every $f_m(x)$, $f_m(y) \in H$. The monophonic wirelength of f_m of G into H is the sum of distances of monophonic paths between two vertices $f_m(x)$ and $f_m(y)$ in H such that (x, y) ∈ E(G). In addition, the eccentricity, radius and diameter of an embedding of G into H are defined. The average wirelength of an embedding is defined and the bounds of average wirelength of some embeddings have been found.

Keywords:*Circulant Networks; Congestion; Cycles; Embedding; Wirelength.*

1. Introduction

For vertices u and v in a connected graph G, The distance d(u, v) is the length of the shortest u-v path in G. A chord of a path u_0, u_1, u_h is an edge u_iu_j, with j ≥ i + [2]. A u - v path is called a monophonic path if it is a chordless path. For two vertices u and v in a connected graph G, the monophonic distance d_m(u, v) is the length of the longest u - v monophonic path in G. An u - v monophonic path of length d_m(u, v) is called an u - v monophonic as stated in [1, 2].

By an embedding f : G → H and a monophonic embedding f_m: GH, it is meant that the graphs G(V, E) and H(V, E) are finite, simple and connectedwith n vertices. Given a host graph H, which represents the network into which other networks are to be embedded, and a guest graph G, which represents the network to be embedded, the problem is to find a mapping from V(G) to V(H) such that each edge of G can be mapped to a path in Has given in [3-7].

An embedding f of G into H is defined as follows:

1) f is a bijective map from V(G) to V(H).
2) f is an one-to-one mapping from E(G) to P_f(f(u), f(v)) where P_f(f(u), f(v)) is a path in H between f(u) and f(v) for (u,v) ∈ E(G) as defined in [8-10].

An embedding f_m: G → H is called a monophonic embedding if f_m maps each vertex of G into a vertex of H and each edge (x, y) of G is mapped to a monophonic path between f_m(x) and f_m(y) in H. The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. The wirelength of an embedding f of G into H is given by,

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H\big(f(u),\ f(v)\big) = \sum_{e\in E(H)} EC_f(G,\ H(e))$$

The wirelength problem of a graph G into H is to find an embedding of G into H that induces the minimum wirelength WL(G, H) as defined in [6],[11], [12].

2. Preliminaries

In this section, we have given definition, example, Lemma and Theorem which are needed in the sequel.

Definition 2.1:*f_m: G→H be an monophonic embedding. The monophonic wirelength MWL(G, H) of f_m is given by,* $\mathrm{MWL}_{f_m}(G,H) = \sum_{(x,y)\in E(G)} d_m(f_m(x),\ f_m(y))$

Definition 2.2: *Let f_m :G → H be a monophonic embedding. The monophonic edge congestion off_m of G into H is the maximum number of edges of the graph G that are embedded on an edge e ∈ H and is given by, $MEC_{f_m}(G,\ H) = \max MEC_{f_m}(G,\ H(e))$. The monophonic wirelength problem of a graph G into H is the problem of finding a monophonic embedding f_m :G → H that produces the monophonic wirelength MWL(G, H).*

Definition 2.3:*The eccentricity of a vertex of graph G into H of an embedding f is given by $e_f(v)$ = max{d(f(u), f(v)) / (u, v) ∈E(G) and P_f(f(u), f(v)) is a path in H for every (u, v) ∈E(G) and for any vertex u in G}.*

Definition 2.4:*The radius of f : G→H is given by r_f(G,H) = min{e_f(v)/v ∈ V(G)} and the diameter of f : G→H is given by dm_f(G, H) = max{e_f(v)/v ∈V(G)}.*

Definition 2.5:*The monophonic eccentricity of a vertex of graph G into H of an embedding f_m is given by* $e_{f_m}(v) =$

Copyright © 2018Sindhuja G. Michael, K. Uma Samundesvari. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

$\max\{d_m(f_m(u), f_m(v))/ (u,v) \in E(G)$ *and* $P_{f_m}(f_m(u), f_m(v))$ *is a path in H for every (u, v) ∈E(G) and for any vertex u in G}.*

Definition 2.6: *The monophonic radius of f_m: G→H is given by* $r_{f_m}(G,H) = min\{e_{f_m}(v)/\ v \in V(G)\}$ *by and the diameter of f_m: G→H is given by* $dm_{f_m}(G,H) = max\{e_{f_m}(v)/\ v \in V(G)\}$.

Lemma 2.7: (*Congestion lemma): Let G be an r-regular graph and let f: G → H be an embedding. Let the graph H \ E has the components H_i, i =1, 2 and $G_i = f^{-1}(H_i)$ then the edge cut E of H has the following properties:*

1) The path $P_f(f(x), f(y))$ has no edges in E for every edge (x, y) $\in G_i$, i = 1, 2.
2) The path $P_f(f(x), f(y))$ has exactly one edge in E for every edge (x, y) in G with x ∈ $G_1$and y ∈ G_2.
3) G_1 is a maximum subgraph of G.

Then $EC_f(E)$, is minimum and $EC_f(E) = r|V(G_1)|-2|E(G_1)|$ as proved in [6].

Lemma 2.8: (*Partition Lemma): Let f: G→H be an embedding. Let {E_1,E_2,…,E_p} be a partition of E (H) such that each E_i is an edge cut of H. Then* $WL_f(G,H) = \sum_{i=i}^{p} EC_f(E_i)$ *as proved in [6].*

Lemma 2.9: (*k-partition Lemma): Let f be an embedding of G into H. Let {E_1,E_2,…,E_p} be a partition of k[E(H)] such that each E_i is an edge cut of H. Then* $WL_f(G,H) = \frac{1}{k}\sum_{i=i}^{p} EC_f(E_i)$ *as proved in [13].*

Lemma 2.10: (*Generalized partition Lemma): Let f be an embedding of G into H. For 1 ≤ i ≤ k, suppose $S_i = \{S^i_1, S^i_2, .., S^i_{pi}\}$ partitionsE(H)\$F_i$ for mutually disjoint F_i's such that S^i_j, 1 ≤ j ≤ p_i, 1 ≤ i ≤ k and* $S = \bigcup_{i=1}^{k} F_i$ *are all edge cuts of H. Then* $WL_f(G,H) = \frac{1}{k}\left[\sum_{i=1}^{k}\sum_{j=1}^{p_i} EC_f(S^i_j) + EC_f(S)\right]$ *as proved in [5].*

Lemma 2.11: *(Monophonic congestion Lemma): Let G be an r-regular graph with n vertices. Let H be a finite graph with n vertices. Let f_m: G → H be a monophonic embedding of G into H. Let the graph H \ E_j, j = 1,2,…,p; 0 <p <|E(G)|, have the components H_i, i =1, 2 and $G_i = f_m^{-1}(H_i)$, where E_j's are the edge cuts of H, form a partition in H and have the following properties:*

1) For m ≥ 0, there are m edges (x, y) $\in G_i$, i = 1,2, such that the monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly two edges in E_j.
2) The monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly one edge in E_j for every (x, y) ∈ G with x ∈ G_1& y ∈ G_2.

Where

G_1 is a maximum subgraph of G. Then $MEC_{f_m}(E_j)$ is monophonic and the monophonic wirelength of f_m of G into H is given by$MWL_{f_m}(G,H) = \sum_{j=1}^{p} MEC_{f_m}(E_j)$, where the monophonic edge congestion, $MEC\{f_m\}(E_j) = r|V(G_1)|-2|E(G_1)|+2m$, $m \geq 0$.

Theorem 2.12:*Let f : G→ H be an embedding, where G be the circulant graph G[2n, ±S], S ⊆ {1,2,3,….,n} and H be the grid M[n × 2]. If d is the diameter of the embedding of G into H, then,*

1) WL(G[2n,{1,2,…,n - 1}], M[n × 2]) = 2n|S| + 1/3 [d(d-1)(2d-1)].
2) WL(G[2n,{1,2,…,n}], M[n × 2]) = (2n|S|1) - n + 1/3 [d(d-1)(2d-1)].
3) WL(G[2n,{1,2,..,n-2}], M[n × 2]) = 2(|S|-1)(n-1) + 1/3[d(d-1)(2d-1)].

3. Average wirelength

The average wirelength of an embedding f: G → H denoted by $\mu W_f(G, H)$,is the expected edge congestion of G into H between a randomly chosen pair of distinct vertices f(x), f(y) ∈ H such that (x, y) ∈ E(G). Therefore$\mu W_f(G,H) = \frac{1}{|E(G)|}\sum_{(x,y)\in E(G)} d_H(f(x), f(y)) = \frac{1}{|E(G)|}\sum_{e\in E(H)} EC_f(G, H(e))$

In the same manner, the average monophonic wirelength of f_m: G → H is given by,

$\mu MW_{f_m}(G,H) = \frac{1}{|E(G)|}\sum_{e\in E(H)} MEC_{f_m}(G, H(e))$.

3.1. Bounds on average wirelength and average monophonic wire length

For n ≥ 2,

2 ≤ μW_f*(G[2n, {1,2,3,…,n-1}], M[n × 2]) ≤ [n/2]and*

$$2 \leq \mu MW_{f_m}(G[2n,\{1,2,3,\ldots.,n-1\}], M[n \times 2]\} \leq \left\lceil \frac{n+1}{2} \right\rceil$$

For n ≥ 3,

2 ≤ μW_f*(G[2n, {1,2,3,…,n-2}], M[n × 2]) ≤ [n/4] and*

$$2 \leq \mu MW_{f_m}(G[2n,\{1,2,3,\ldots.,n-2\}], M[n \times 2]\} \leq \left\lceil \frac{n}{3} \right\rceil.$$

For n ≥ 4,

2 ≤ μW_f*(G[2n, {1,2,3,…,n-3}], M[n × 2]) ≤* $\left\lceil \frac{n-1}{4} \right\rceil$ *and*

$$2 \leq \mu MW_{f_m}(G[2n,\{1,2,3,\ldots.,n\text{-}3\}], M[n \times 2]\} \leq \left\lceil \frac{n\text{-}1}{3} \right\rceil.$$

4. Monophonic wirelength of circulant networks into cycles

Definition 4.1:*A connected undirected graph represented by G [n, ±S] where S ⊆ {1,2,3,…,[n/2]}, n ≥ 3 is said to be a circulant graph if it consists of the vertex set.*

V = {0,1,2…,n-1} and the edge set

E= {(x, y): |x-y| ≅s (mod n), s ∈S} as defined in [5].

Definition 4.2:*A graph with a closed walk consisting of n points is called a cycle, denoted by C_n as defined in [14].*

Theorem 4.3:*A set of k consecutive vertices of G[n, ±1];*

1 ≤ k ≤ n induces a maximum subgraph of G[n,±S]

Where S ⊆{1,2,3,…,j}, 1 ≤j ≤ [n/2], n ≥ 3 as defined in [5].

In this section, G denotes the circulant graph G[n, ±S] where S ⊆{1,2,3,…,[n/2]} of n vertices and H denotes the cycle graph C_n with n vertices and f_m, the monophonic embedding from G to H.

Theorem 4.4: If *f_m: G → H is a monophonic embedding, then the wirelength induced by f_m from the Circulant graph to Cycle graph is monophonic.*

Proof:

We prove this for two cases.

Case (i) (n is odd)

Let $P_1 = \{P_1^1, P_2^1, \ldots., P_{\frac{n-1}{2}}^1\}$and $P_2 = \left\{P_1^2, P_2^2, \ldots., P_{\frac{n-1}{2}}^2\right\}$where $P_s^1 = \left\{(s-1, s), \left(\frac{(n-3)}{2} + s, \frac{(n-1)}{2} + s\right)\right\}$and $P_s^2 = \left\{(s-1, s), \left(\frac{(n-1)}{2} + s, \frac{(n+1)}{2} + s\right)\right\}$, $1 \leq s \leq \frac{(n-1)}{2}$, taken modulo n.

Consider the edges, $R_1 = \{(n\text{-}1, 0)\}$ and$R_2 = \left\{\left(\frac{(n-1)}{2}, \frac{n+1}{2}\right)\right\}$, clearly P_t partitions E(H)\$R_t$, t=1,2 and the sets R_1 and R_2 are disjoint and their union is an edge cut of H. For each s, E(H)\$P_s^t$has two components $H_{s_1}^t$and $H_{s_2}^t$induced by consecutive vertices on H with $|H_{s_1}^t| = \left\lceil \frac{n}{2} \right\rceil = |H_{s_2}^t|$.Let $G_{s_1}^t = f_m^{-1}(H_{s_1}^t)$ and $G_{s_2}^t = f_m^{-1}(H_{s_2}^t)$.Then $G_{s_1}^t$is on $\left\lceil \frac{n}{2} \right\rceil$consecutive vertices of G[n, ±1] and these vertices induce a maximal subgraph of G[n, ±S] by theorem 4.3. Hence P_s^t satisfies the monophonic congestion lemma and therefore $MEC_{f_m}(P_s^t)$is monophonic and thus the wire length induced by f_m is monophonic.

Case (ii) (n is even)

Let P_t, t = 1,2,…,n/2 be the edge cuts of H, form a partition in H. For each t, let $P_t = \{(t-1,t),(\frac{n}{2}+t-1),(\frac{n}{2}+t)\}$ $1 \le t \le n/2$ where the vertices are taken modulo n. That is P_t has two direct opposite edges of the cycle graph H. Let H_{t_1} and H_{t_2} are the components of H\$P_t$ for each t. Let $G_{t_1} = f_m^{-1}(H_{t_1})$ and $G_{t_2} = f_m^{-1}(H_{t_2})$. Then the graphs G_{t_1} and G_{t_2} are on $\lceil \frac{n}{2} \rceil$ consecutive points of G[n, ±1] and these points induce a maximal subgraph of G[n, ±S] by theorem 4.3. Hence, P_t satisfies the monophonic congestion lemma and therefore $MEC_{f_m}(P_t)$ is monophonic and thus the wirelength induced by f_m is monophonic.

5. Monophonic Embedding Algorithm

Aim: To find a monophonic embedding $f_m : G \to H$ that produces the monophonic wirelength $MEC_{f_m}(G,H)$ where G is the family of circulant graph with n vertices of r-regular and H is the cycle graph C_n, $n \ge 4$.

5.1. Monophonic Algorithm

1) Name the vertices of G[n, ±S], S ⊆ {1,2,3,…,n/2} as the vertices of Cycle from 0,1,2,…,n-1.
2) Name the vertices of C_n as 0,1,2,…,n-1 in clockwise.

Input: A family of circulant graphs G[n, ±S], S ⊆ {1,2,3,…,n/2} and the cycle graph C_n.

Output: A monophonic embedding f_m from G[n, ±S], into C_n given by $f_m(x) = x$ with monophonic wirelength MWL(G[n, ±S], C_n) = $\lceil \frac{n}{2} \rceil [MEC_{f_m}(P)]$.

Proof: In theorem 4.4, it is proved that the wirelength induced by f_m from G[n, ±S] to C_n is monophonic. We prove two cases.

Case (i) (n is odd)

Using the notations used in case (i) of theorem 4.4, we have by generalized partition lemma,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{\sum_{t=1}^{2}\sum_{s=1}^{\frac{(n-1)}{2}} MEC_{f_m}(P_s^t) + MEC_{f_m}(P)\}$$

As H\P is isomorphic to H\P_s^t,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{(n-1)MEC_{f_m}(P) + MEC_{f_m}(P)\} = \frac{n}{2}MEC_{f_m}(P)$$

Case (ii) (n is even)

Using the notations used in case (ii) of theorem 4.4, we have by generalized partition lemma,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{\sum_{t=1}^{\frac{n}{2}} MEC_{f_m}(P_t)\} = \frac{1}{2}\{nMEC_{f_m}(P)\} = \frac{n}{2}MEC_{f_m}(P)$$

5.1.1. Monophonic algorithm (i)

Input: A family of circulant graph $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-1 \rceil\}]$ and the cycle graph C_n, $n \ge 4$.

Output: A monophonic embedding f_m from $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-1 \rceil\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G[n,\{1,2,\ldots,\lceil \tfrac{n}{2}-1 \rceil\}],C_n) = \begin{cases} \frac{n}{2}(3|s|^2-|s|)\ if n is even \\ \frac{n}{2}(3|s|^2-3|s|+2)\ if n is odd \end{cases}$$

5.1.2. Monophonic algorithm (ii)

Input: A family of circulant graph $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-2 \rceil\}]$, and the cycle graph C_n, $n \ge 5$.

Output: A monophonic embedding f_m from $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-2 \rceil\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G[n,\{1,2,\ldots,\lceil \tfrac{n}{2}-2 \rceil\}],C_n) = \begin{cases} \frac{n}{2}(3|s|^2-3|s|-4\ if n is even \\ \frac{n}{2}(3|s|^2-|s|-2 if n is odd \end{cases}$$

5.1.3. Monophonic algorithm (iii)

Input: A family of circulant graph $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-3 \rceil\}]$, and the cycle graph C_n, $n > 6$.

Output: A monophonic embedding f_m from $G[n,\{1,2,\ldots,\lceil \frac{n}{2}-3 \rceil\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G[n,\{1,2,\ldots,\lceil \tfrac{n}{2}-3 \rceil\}],C_n) = \begin{cases} \frac{n}{2}(3|s|^2+7|s|-8)\ if n is even \\ \frac{n}{2}(3|s|^2+5|s|-6)\ if n is odd \end{cases}$$

6. Conclusion

In this paper we have applied the monophonic idea on graph embedding of two graphs and also we have given a monophonic algorithm for finding the monophonic wirelength of circulant networks into cycles.

Acknowledgement

The authors thank the anonymous referee for the valuable suggestions in revising the manuscript.

References

[1] F. Buckley and F. Harary, Distance in Graph, *Addison- Wesley*, California, 1990.
[2] A.P. Santhakumaran and P. Titus, "Monophonic Distance in Graphs", *Discrete Mathematics Algorithms Application*, Vol.3, No.2, (2011), pp.159-169, https://doi.org/10.1142/S1793830911001176.
[3] P. Maya and T. Nicholas, "Some Results on Integer Edge Cordial Graph", *DJ Journal of Engineering and Applied Mathematics*, Vol.3, No.1, (2017), pp. 1-12, https://doi.org/10.18831/djmaths.org/2017011001.
[4] P. Manuel, "Minimum Average Congestion of Enhanced and Augmented Byper-Cube into Complete Binary Tree", *Discrete Applied Mathematics*, Vol.159, No.5, (2010), pp.360-366, https://doi.org/10.1016/j.dam.2010.12.001.
[5] IndraRajasingh, Paul Manuel, M. Arockiaraj and BharatiRajan, "Embeddings of Circulant Networks", *Journal of Combinatorial Optimization*, Vol.26, No.1, (2013), pp.135-151, https://doi.org/10.1007/s10878-011-9443-x.
[6] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, "Exact Wirelength of Hypercube on a Grid", *Discrete Applied Mathematics*, Vol.157,

No.7, (2009), pp.1486-1495.https://doi.org/10.1016/j.dam.2008.09.013.
[7] Jess Mathew, "Subdivided Stars Super (b, d) - Edge-Antimagic Total Graph Labelling", *DJ Journal of Engineering and Applied Mathematics*, Vol.1, No.1, (2015), pp.30-34, https://doi.org/10.18831/djmaths.org/2015011005.
[8] S.L. Bezrukov, J.D. Chavez, L. H. Harper, M. Rottger and U. P.Schroeder, "Embedding of Hypercubes into Grids", *International Symposium on Mathematical Foundations of Computer Science*, Vol.1450, (1998), pp.693-701, https://doi.org/10.1007/BFb0055820.
[9] L. Bezrukov, "Embedding Complete Trees into the Hypercube", Discrete Applied Mathematics, Vol.110, No.2-3, (2001), pp.101-119,https://doi.org/10.1016/S0166-218X(00)00256-0.
[10] JasinthaQuadras, S. Sarah Surya, "Linear Wirelength of Circulant Networks", *International Journal of Pure and Applied Mathematics*, Vol.86, No.6, (2013), pp.883-891,
[11] P. Manuel, M. Arockiaraj, I. Rajasingh and B. Rajan, "Embedding Hypercubes into Cylinders, Snakes and Caterpillars for Minimizing Wirelength", *Discrete Applied Mathmetics*, Vol.159, No.17, (2011), pp.2109-2116, https://doi.org/10.1016/j.dam.2011.07.003.
[12] I. Rajasingh, J. Quadras, P.Manuel and A.William, "Embedding of Cycles and Wheels into Arbitrary Trees", *Networks*, Vol.44, (2004), pp.173-178, https://doi.org/10.1002/net.20027.
[13] Arokiaraj, Paul Manuel, IndraRajasingh and BharatiRajan, "Wirelength of 1-Fault Hamiltonian Graphs into Wheels and Fans", *Information Processing Letters*, Vol.111, No.18, (2011), pp.921-925,https://doi.org/10.1016/j.ipl.2011.06.011.
[14] F. Harary, Graph Theory, *Addison- Wesley*, 1969.https://doi.org/10.21236/AD0705364.