International Journal of Engineering & Technology, 7 (2.14) (2018) 9-12

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research Paper

Enthalpy method for one dimensional heat conduction

Farah Suraya Md Nasrudin[1]*, Shafaruniza Mahadi[2]

[1,2]*Universiti Teknologi MARA, 85000 Johore, Malaysia*
**Email: farah53@johor.uitm.edu.my*

Abstract

In this paper, the Enthalpy Method is employed to compute an approximate solution of the system of nonlinear differential equations focusing on the simulation of moving boundary for one dimensional heat conduction. This paper is only considered in the problem of a technical grade paraffin's melting process. In order to seek the solution in term of temperature distribution, Finite Difference Method will be used. The results obtained are compared between solving with enthalpy and without enthalpy. The enthalpy method is more versatile, convenient, adaptable and easily programmable.

Keywords: *Enthalpy Method, Finite Difference Method, heat conduction, Stefan problem, temperature distribution*.

1. Introduction

In mathematics and its applications, a Stefan problem is a particular kind of boundary value problem for a partial differential equation (PDE), adapted to the case in which a phase boundary can move with time. Hence, Stefan problems are examples of moving boundary problems (Alexiades & Solomon, 1993). The melting of ice is an example of classical Stefan problem. Heat diffusion with phase change of diffusing medium appears in variety of the natural phenomena and technological processes and is frequently coupled with other heat transfer phenomena. In phase change process, the boundaries are explicit in the different phases of the process (Frank & David, 1990).

Only a small number of realistic problems can be solved analytically (Zhao et.al., 2017), (Myers & Font, 2015), (Mitchell, 2015), (Li & Sun, 2015). Therefore, the problems are forced to seek approximation or numerical solutions. They come in two very distinct varieties; analytical approximation and numerical approximation. The widely used analytical method is heat balance method. This paper will focus on a numerical method to solve two phase Stefan problem with a certain boundary condition.

Analytical methods for Stefan problems are limited to simple problems only. The one that can be applied to two-phase Stefan problems is that of heat balance integral method. However, depending on the problem and boundary conditions at hand, the solution process may get very complicated or cannot be used at all (Ockendon & Hodgkins, 1975).

With regard to numerical approach, several schemes are available for the solution of Stefan problems. However, some of them are not really easy to implement especially when one has to track down the location of the moving front (Sarit, 2005). To avoid this procedure, it is preferable to use enthalpy formulation when the position of the interface is not important. The aim of this research is to study and implement solutions of one dimensional Stefan problem using numerical enthalpy method.

This paper focused on the simulation of moving boundary for one dimensional heat conduction in the problem of melting. It considered numerical enthalpy method to solve this problem. That is, to seek the solution of two phase Stefan problem with regard to temperature distribution as we move forward in time. Finite Difference Method will be employed.

Matlab has been chosen as the tool for the simulation in this paper. Other than that, Microsoft Excel also has been used to analyzed data in the calculation of errors between subjects of concerned.

2. Mathematical model

The computational stage of all numerical methods for solving problems of any complexity generally involves a great deal of arithmetic. It is usual therefore to arrange, whenever possible, for one solution for a variety of different problems. This can be done by expressing all equations in terms of dimensionless (non-dimensional) variables (Smith, 1987), (Zerroukat & Chatuin, 1994). Then all problems with the same dimensionless mathematical formulation can be dealt with by means of one solution.

2.1. Solution without enthalpy

In solid, heat equation is given by

$$\frac{\partial T_S}{\partial \tau} = \kappa_S \frac{\partial^2 T_S}{\partial \overline{X}^2} \tag{1}$$

for $0 < x < X(\tau), \tau > 0$, where $\kappa_S = \dfrac{\lambda_S}{\rho_S c_S}$

and denoted the dimensionless variables as follow:

$$T_S^* = \frac{T_S - T_\infty}{T_m - T_\infty}, \quad t = \frac{\kappa_S \tau}{l^2}, \quad x = \frac{\overline{X}}{l}, \quad U = \frac{T_S^*}{T_o}$$

and we get

$$\left(T_m - T_\infty\right)\frac{\kappa_S}{l^2}\frac{\partial T_S^*}{\partial t} = \kappa_S \frac{T_m - T_\infty}{l^2}\frac{\partial^2 T_S^*}{\partial x^2} \tag{2}$$

and reduced to $\frac{\partial T_S^*}{\partial t} = \frac{\partial^2 T_S^*}{\partial x^2}$. (3)

But $T_S^* = UT_o$, so that (3) is reduced to dimensionless form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (4)$$

In liquid, heat equation is given by

$$\frac{\partial T_L}{\partial \tau} = \kappa_L \frac{\partial^2 T_L}{\partial \overline{X}^2} \quad (5)$$

for $\overline{X}(t) < x < l$, $\tau > 0$, where $\kappa_L = \frac{\lambda_L}{\rho_L c_L}$

and denoted the dimensionless variables as follow:

$$T_L^* = 1 + \frac{T_L - T_m}{T_{\max} - T_m}, x = \frac{\overline{X}}{l}, t = \frac{\kappa_S \tau}{l^2}, U = \frac{T_L^*}{T_o}.$$

$$(T_{\max} - T_m)\frac{\kappa_S}{l^2}\frac{\partial T_L^*}{\partial t} = \kappa_L \frac{T_{\max} - T_m}{l^2}\frac{\partial^2 T_L^*}{\partial x^2}$$

and reduced to

$$\frac{\partial T_L^*}{\partial t} = \frac{\kappa_L}{\kappa_S}\frac{\partial^2 T_L^*}{\partial x^2}. \quad (6)$$

But $T_L^* = UT_o$, so that the equation is reduced to dimensionless form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (7)$$

Using explicit forward difference formula in time and centered difference formula for spatial coordinate,

$$\frac{U_i^{m+1} - U_i^m}{\Delta t} = \frac{U_{i+1}^m - 2U_i^m + U_{i-1}^m}{(\Delta x)^2} \quad (8)$$

Let $r = \frac{\Delta t}{(\Delta x)^2}$

$$U_i^{m+1} = rU_{i-1}^m + (1-2r)U_i^m + rU_{i+1}^m \quad (9)$$

2.2. Solution with enthalpy

If $E < 1$, $U = E$, if E in solid phase.

If $E > 1 + Ste$, $U = \frac{\kappa_L}{\kappa_S}[E - 1 - Ste] + 1$, if E in liquid phase.

Else, $U = 1$, at melting temperature.

Partial differential equation of heat equation in solid is given by

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad \text{for } X(t) < x < l,\ t > 0 \quad (10)$$

we get the dimensionless form

$$\frac{\partial E}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (11)$$

Partial differential equation of heat equation in liquid is given by

$$\frac{\partial U}{\partial t} = \frac{\kappa_L}{\kappa_S}\frac{\partial^2 U}{\partial x^2} \text{ for } 0 < x < X(t),\ t > 0 \quad (12)$$

also we get $\frac{\partial E}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (13)

Using explicit forward difference formula in time and centered difference formula for spatial coordinate.

$$\left[\frac{E_i^{m+1} - E_i^m}{\Delta t}\right] = \left[\frac{U_{i+1}^m - 2U_i^m + U_{i-1}^m}{(\Delta x)^2}\right] \quad (14)$$

Let $r = \frac{\Delta t}{\Delta x^2}$, so we get

$$E_i^{m+1} = E_i^m + \left[rU_{i+1}^m - 2rU_i^m + rU_{i-1}^m\right] \quad (15)$$

3. Experimental result

In this paper, Matlab is used to calculate the numerical solutions. In this section, the temperature distribution is analyzed in certain time when heat source is added at a point. Several initial conditions have been applied. In this model, thermophysical properties of the technical grade paraffin is used (Anica, 2005), (Zhang, et. al., 2016).

Table 1: Specifications for Technical Grade Paraffin

Melting/solidification temperature (K)	300.7
Latent heat capacity (kJ kg^{-1})	206
Thermal conductivity (W m^{-1} K^{-1}) solid/liquid	0.18/0.19
Specific heat (kJ kg^{-1} K^{-1}) solid/liquid	1.8/2.4
Density (kg m^{-3}) solid/liquid	789/750

We chose boundary conditions (Dirichlet type)

$u(0,t) = 0, \quad u(1,t) = 0$

with the initial condition

$u(x,0) = f(x)$ where $f(x) = x$

3.1. Without enthalpy

From the previous section, explicit finite difference method is applied onto the dimensionless heat equation and the solution is given by (9) and this was stimulated by using Matlab for various boundary and initial condition, but for this paper, we only show the data for the boundary and initial condition mentioned before.
We chose to show the data in graph and table for heat source at point x(1.2).

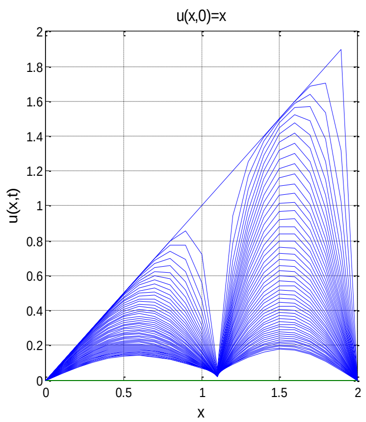

Fig. 1: Graph $f(x) = x$ without enthalpy and with heat source at $x_{1.2}$

The graph above showed the temperature distribution at every points x from 0 to 2. We can see how the temperature arise when we put heat at point x(1.2).
To make it clearer to analyze, next, we show the data in the table below. For this, we chose to discretized the data at point x(1.0) which is nearer to heat source and x(1.9) which is more far from heat source to see the differences.

Table 2: Temperature Distribution at x_{10} and x_{19} Without Enthalpy at Certain Time Levels.

time, t	x (1.0)	x (1.9)
0.0080	0.7733	1.5369
0.0160	0.6221	1.2543
0.0240	0.5167	1.0594
0.0320	0.4419	0.9185
0.0400	0.3859	0.8092
0.0480	0.3424	0.7199
0.0560	0.3072	0.6443
0.0640	0.2781	0.5789
0.0720	0.2535	0.5215
0.0800	0.2325	0.4706
0.0880	0.2142	0.4253
0.0960	0.1982	0.3847
0.1040	0.1841	0.3482
0.1120	0.1716	0.3154
0.1200	0.1604	0.2858
0.1280	0.1503	0.2591
0.1360	0.1414	0.2349
0.1440	0.1333	0.2132
0.1520	0.1260	0.1935
0.1600	0.1195	0.1757
0.1680	0.1137	0.1596
0.1760	0.1085	0.14516
0.1840	0.1039	0.1319
0.1920	0.0998	0.1201
0.2000	0.0962	0.10947

The result in the table above showed the temperature distribution as time increased. We chose time interval of 0.008.

3.2. With enthalpy

From the previous section, explicit finite difference method is applied onto the dimensionless enthalpy-temperature equation and the solution is given by (15) and this was stimulated by using Matlab using the same boundary and initial condition.
We chose to show the data in graph and table for heat source at point x(1.2).

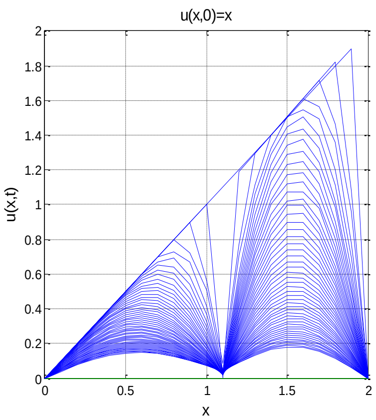

Fig. 2: Graph $f(x) = x$ with enthalpy and heat source at $x_{1.2}$

The graph above showed the temperature distribution at every points x from 0 to 2. We can see how the temperature arise when we put heat at point x(1.2).
To make it clearer to analyze, next, we show the data in the table below. For this, we chose to discretized the data at point x(1.0) which is nearer to heat source and x(1.9) which is more far from heat source to see the differences.

Table 3: Temperature Distribution at x_{10} and x_{19} With Enthalpy at Certain Time Levels.

time, t	x (1.0)	x (1.9)
0.0080	0.7240	1.4644
0.0160	0.6676	1.2006
0.0240	0.5418	1.0231
0.0320	0.4596	0.9029
0.0400	0.4002	0.8025
0.0480	0.3545	0.7168
0.0560	0.3179	0.6477
0.0640	0.2878	0.5881
0.0720	0.2625	0.5325
0.0800	0.2409	0.4822
0.0880	0.2222	0.4367
0.0960	0.2058	0.3957
0.1040	0.1913	0.3586
0.11200	0.17844	0.32527
0.1200	0.1669	0.2949
0.1280	0.15676	0.2675
0.1360	0.14743	0.2427
0.1440	0.13912	0.2203
0.1520	0.1317	0.1999
0.1600	0.1249	0.1816
0.1680	0.1189	0.1650
0.1760	0.1135	0.1500
0.1840	0.1086	0.1364
0.1920	0.1043	0.12426
0.2000	0.1006	0.1131

To analyze both temperature distributions with enthalpy and without enthalpy, temperatures at each time were compared and then the error for both points $x_{1.0}$ and $x_{1.9}$ were determined.

Table 4: Difference in Temperature Between Solutions With Enthalpy and Solution Without Enthalpy at points $x_{1.0}$ and $x_{1.9}$.

time, t	error / difference $\left\lvert u_{\text{enthalpy}}(i) - u_{\text{without enthalpy}}(i) \right\rvert$	
	$x_{1.0}$	$x_{1.9}$
0.0080	0.0493	0.0725
0.0160	0.0456	0.0536
0.0240	0.0251	0.0364
0.0320	0.0177	0.0155
0.0400	0.0142	0.0067
0.0480	0.0121	0.0030
0.0560	0.0107	0.0034
0.0640	0.0098	0.0092
0.0720	0.0090	0.0110
0.0800	0.0084	0.0115
0.0880	0.0079	0.0114
0.0960	0.0075	0.0109
0.1040	0.0072	0.0104
0.1120	0.0069	0.0097
0.1200	0.0066	0.0091
0.1280	0.0063	0.0084
0.1360	0.0061	0.0077
0.1440	0.0058	0.0071
0.1520	0.0056	0.0065
0.1600	0.0054	0.0059
0.1680	0.0052	0.0054
0.1760	0.0049	0.0049

From the analysis in the table above, it is found that the absolute errors for both problems are less than 8%. Other than that, the absolute error is decreasing with time. The location of heat source does not really matter in the calculation of temperature.

4. Conclusion

In this paper, the numerical technique is used to get the values of temperature distributions for two phase Stefan Problem. The result of this research provides information in determining the relation between various physical constants and external parameters. The enthalpy method is discretized by finite difference method and is more versatile, convenient, adaptable and easily programmable numerical method available for phase change problem.

The results of this study give benefits to the field of mathematics, engineering and physics. This paper will lead to further investigations on the heat transfer especially in melting process that involves phase change. This paper also provided a constructive solution to problems in searching the temperature distribution along the medium. Therefore, this paper can be improved by extend this study by using thermophysical properties of other substances. Also, this paper can be further this by solving it using other numerical methods such as the implicit methods. Then, the convergence and stability of the explicit method, etc. can be implement.

Acknowledgement

This is partially supported by Universiti Teknologi MARA Johore Branch.

References

[1] Alexiades, V., Solomon, A.D., *Mathematical Modelling of Melting and Freezing Processes*, Hemisphere Pub. Corp., Washington, (1993)

[2] Anica, T., "An Experimental and Numerical Investigation of Heat Transfer during Technical Grade Paraffin Melting and Solidification ini a Shell-and-tube Latent Thermal Energy Storage Unit", Vol. 79, Issue 6, (2005), pp 648-660.

[3] Frank P. I., David P. D., *Fundamental of Heat and Mass Transfer*, 3rd Ed., John Wiley & Sons Inc., New York, (1990)

[4] Li, X., Sun, X., "Similarity Solution for Phase Change Problems with Fractional Governing Equations", *Applied Mathematics Letters*, Vol. 45, (2015), pp 7-11.

[5] Mitchell, S.L., "Applying the Combined Integral Method to Two-phase Stefan Problems with Delayed Onset of Phase Change", *Journal of Computational and Applied Mathematics*, Vol. 281, (2015), pp 58-73

[6] Myers, T.G., Font, F., "On the One-phase reduction of the Stefan Problem with a Variable Phase Change Temperature", *International Communications in Heat and Mass Transfer*, Vol. 61, (2015), pp 37-41

[7] Ockendon, J. R., Hodgkins, W. R., *Moving Boundary Problem in Heat Flow and Diffusion*, Clarendon Press, Oxford, London, (1975)

[8] Sarit K. D., *Process Heat Transfer*, Alpha Science International Ltd.,Harrow,U.K., (2005)

[9] Smith, G.D., *Numerical Solution of Partial Differential Equations*, Clarendon Press, Oxford, New York., (1984)

[10] Zerroukat, M., Chatuin, C.R., *Computational Moving Boundary Problems*, John Wiley & Sons Inc., New York., (1994)

[11] Zhang, Y., et. al., "Form-stable Phase Change Materials with High Phase Change Enthalpy from the Composite Paraffin and Cross-linking Phase Change Structure", *Applied Energy*, Vol. 184, (2016), pp 241-246.

[12] Zhao, Y., Zhao, C.Y., Xu, Z. G., "Numerical study of Solid-liquid Phase Change by Phase Field Method", *Computer and Fluids*, (2017), available online: https://doi.org/10.1016/j.compfluid.2017.05.032