



Noncommutative BTZ black hole and Fuzzy space

Abderrahman EL Boukili^{1*}, Mourad Nach¹, Abdellah EL Boukili² and Moulay Brahim Sedra¹

¹Ibn Tofail University, Faculty of Sciences, Physics Departement, (LHESIR), Kénitra, Morocco.

²Mohamed V University, Faculty of Sciences Agdal, (LECA), Rabat, Morocco.

*Corresponding author E-mail: aelboukili@gmail.com

Abstract

We study the effects of noncommutative spaces and fuzzy space on the metric, horizon, entropy and Hawking temperature of a BTZ black hole. The results show deviations from the usual physical properties. The deviations depend on the parameter of noncommutativity.

Keywords: BTZ black hole, entropy, fuzzy space, Hawking temperature, horizon.

1. Introduction

The BTZ black hole in (1+1) dimensions has been considered before in [1] in the context of models of gravitational theories in (1+1) dimensions. We will follow a less ambitious approach and introduce a model of a BTZ black hole by using the formulation of noncommutative geometry.

In fact, the noncommutative space can be realized by the coordinate operators satisfying [2]

$$[x_i, x_j] = i\theta_{ij}; \quad [x_i, p_j] = i\delta_{ij}; \quad [p_i, p_j] = 0 \quad (1)$$

where θ_{ij} is the noncommutativity parameter of dimension (length)². It was noted in [3] that at the replacement

$$x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}p_j; \quad p_j = \hat{p}_j \quad (2)$$

we obtain the standard commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0; \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}; \quad [\hat{p}_i, \hat{p}_j] = 0; \quad (3)$$

The usual product of fields should be replaced by the star-product

$$(f \star g) = \exp\left(\frac{i}{2}\theta_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}\right) f(x)g(y) \Big|_{x=y} \quad (4)$$

However, fuzzy space represents another kind of noncommutative geometry, it is known to correspond to the sphere D2-branes in string theory with a linear background B-field [4]. Moreover, in the presence of constant RR field potential, the D0-branes can expand into a noncommutative fuzzy sphere configuration. The quantum field on the fuzzy space has been studied by many authors [5,6]. Therefore, it is interesting to see how the fuzzy space will change the black hole property.

The paper is organized as follows. In Sec. 2, we review the main features of BTZ Black hole. In Sec. 3, we study the physical properties of BTZ Black hole in noncommutative space. A brief conclusion is reported in Sec. 4.

2. BTZ Black hole

• The Metric

The metric of BTZ black hole is given by [7]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\phi^2 \quad (5)$$

where

$$f(r) = -M + \frac{r^2}{l^2} - \frac{Q^2}{l^2} \ln\left(\frac{r}{l}\right) \quad (6)$$

This known as the lapse function and M and Q are the mass and electric charge of the BTZ black hole, respectively.

• Horizon

For 2-dimensional BTZ-like solutions, the horizons are located at [8]

$$r_{\pm} = l \exp \left\{ -L_w \left(\frac{-l^{-1} \exp\left(-\frac{M}{2^{\frac{1}{2}}Q}\right)}{2^{\frac{1}{2}}Q} \right) - \frac{M}{2^{\frac{1}{2}}Q} \right\} \quad (7)$$

$$r_{\pm} = l \exp \left\{ -L_w \left(-1, \frac{-l^{-1} \exp\left(-\frac{M}{2^{\frac{1}{2}}Q}\right)}{2^{\frac{1}{2}}Q} \right) - \frac{M}{2^{\frac{1}{2}}Q} \right\}$$

where the L_w is LambertW function satisfies $L_w(x) \exp[L_w(x)] = x$ (for more details, see[9]). The event horizon of presented BTZ-like solutions are located at the root(s) of $f(r) = 0$. Thus, the presented $r_{\pm}$ is the radius of the event horizon. The BTZ-like solutions may be interpreted as black hole solutions with inner and outer horizons, an extreme black hole or naked singularity.

• Scalar Ricci Curvature

By definition, the scalar curvature is given by

$$R = -\frac{d^2}{dr^2} f(r) \quad (8)$$

$$= \frac{2}{l^2} + \frac{Q^2}{l^2} \frac{1}{r^2} \approx \frac{Q^2}{l^2} \frac{1}{r^2}$$

• Electric potential

The electric potential of this black hole is [10]

$$\Phi = \frac{\partial M}{\partial Q} = -Q \ln\left(\frac{r}{l}\right) \quad (9)$$

• Hawking temperature

while the Hawking temperature is given as[11]

$$T_H = \frac{1}{4\pi} \left| \frac{df(r)}{dr} \right|_{r=r_+} \quad (10)$$

$$= \frac{r_+}{2\pi l^2} - \frac{Q^2}{8\pi} r_+$$

• **Entropy**

The entropy of BTZ black hole is given by[12]

$$S_{BTZ}^{BH} = \frac{A}{4} = \frac{\pi}{2} r_+ \quad (11)$$

3. Noncommutative BTZ Black hole

• **Metric**

We have the following metric for BTZ black hole in non commutative space

$$dS^2 = -f(\tilde{r}) dt^2 + \frac{1}{f(\tilde{r})} d\tilde{r} d\tilde{r} + \tilde{r} d\varphi^2 \quad (12)$$

where

$$f(\tilde{r}) = -M + \frac{\tilde{r}\tilde{r}}{l^2} - \frac{Q^2}{l^2} \ln \left(\frac{\sqrt{\tilde{r}\tilde{r}}}{l} \right) \quad (13)$$

where $\tilde{r}$ satisfies following Poisson brackets (3). We note that in a new coordinates systems (2) one can treat the noncommutative effects as some perturbations of the commutative counter-part, $f(\tilde{r})$ in terms of the noncommutative coordinates $\tilde{X}_i$ is as

$$f(\tilde{r}) = -M + \frac{\tilde{X}_i \tilde{X}_i}{l^2} - \frac{Q^2}{l^2} \ln \left(\frac{\sqrt{\tilde{X}_i \tilde{X}_i}}{l} \right) \quad (14)$$

Using the new coordinates, we have

$$f(r) = -M + \frac{\left(x_i - \frac{\theta_{ij} p_j}{2}\right) \left(x_i - \frac{\theta_{ik} p_k}{2}\right)}{l^2} - \frac{Q^2}{l^2} \ln \left(\frac{\sqrt{\left(x_i - \frac{\theta_{ij} p_j}{2}\right) \left(x_i - \frac{\theta_{ik} p_k}{2}\right)}}{l} \right) \quad (15)$$

where

$$\theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k \quad (16)$$

therefore

$$f(r) = -M + \frac{r^2}{l^2} - \frac{Q^2}{l^2} \ln \left(\frac{r}{l} \right) - \frac{\vec{L} \cdot \vec{\theta}}{4l^2} + \frac{p^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{16l^2} + \frac{Q^2}{16r^2} \left(\vec{L} \cdot \vec{\theta} - \frac{p^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{4} \right) \quad (17)$$

• Horizon in noncommutative space

The horizon of the noncommutative metric satisfies the following condition

$$f(r) = 0 \quad (18)$$

therefore the horizons of BTZ black hole in non-commutative space are as following

$$r_- = l \exp \left\{ -L_w \left(\frac{-l^{-1} \exp \left(-\frac{M}{2^{\frac{1}{2}} Q} \right)}{2^{\frac{1}{2}} Q} \right) - \frac{M}{2^{\frac{1}{2}} Q} \right\} + \frac{\vec{L} \cdot \vec{\theta}}{8r_-} - \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{32r_-} \quad (19)$$

$$r_+ = l \exp \left\{ -L_w \left(-1, \frac{-l^{-1} \exp \left(-\frac{M}{2^{\frac{1}{2}} Q} \right)}{2^{\frac{1}{2}} Q} \right) - \frac{M}{2^{\frac{1}{2}} Q} \right\} + \frac{\vec{L} \cdot \vec{\theta}}{8r_+} - \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{32r_+}$$

• Scalar Ricci Curvature in noncommutative space

The Scalar Ricci Curvature is given by

$$R \approx \frac{Q^2}{l^2} \frac{1}{\tilde{r}\tilde{r}} \approx \frac{Q^2}{l^2} \frac{1}{\left(x_i - \frac{\theta_{ij} p_j}{2} \right) \left(x_i - \frac{\theta_{ik} p_k}{2} \right)} \quad (20)$$

$$\approx \frac{Q^2}{l^2} \left(r^2 - \frac{\vec{L} \cdot \vec{\theta}}{4} + \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{16} \right)^{-1}$$

• Hawking temperature

The Hawking temperature T_H associated with the outer black hole horizon is

$$T_H = \frac{r_+}{2\pi l^2} - \frac{2Q^2}{r_+}$$

$$= \frac{1}{2\pi l} \exp \left\{ -L_w \left(-1, -\left(2^{\frac{1}{2}} Q l \right)^{-1} \exp \left(-M \left(2^{\frac{1}{2}} Q \right)^{-1} \right) \right) - M \left(2^{\frac{1}{2}} Q \right)^{-1} \right\} + \frac{1}{2\pi l^2} \left(\frac{\vec{L} \cdot \vec{\theta}}{8r_+} - \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{32r_+} \right) \quad (21)$$

$$- 2Q^2 \left[l \exp \left\{ -L_w \left(-1, -\left(2^{\frac{1}{2}} Q l \right)^{-1} \exp \left(-M \left(2^{\frac{1}{2}} Q \right)^{-1} \right) \right) - M \left(2^{\frac{1}{2}} Q \right)^{-1} \right\} + \frac{\vec{L} \cdot \vec{\theta}}{8r_+} - \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{32r_+} \right]^{-1}$$

• Entropy

According to the Bekenstein-Hawking formula, the thermodynamic entropy of a black hole is proportional to the area A of the outer event horizon $S=A/4$. For the charged BTZ black hole we have

$$S = \frac{\pi r_+}{2}$$

$$= \frac{\pi}{2} \left[l \exp \left\{ -L_w \left(-1, -\left(2^{\frac{1}{2}} Q l \right)^{-1} \exp \left(-M \left(2^{\frac{1}{2}} Q \right)^{-1} \right) \right) - M \left(2^{\frac{1}{2}} Q \right)^{-1} \right\} + \frac{\vec{L} \cdot \vec{\theta}}{8r_+} - \frac{\vec{p}^2 \theta^2 - \vec{p} \cdot \vec{\theta}}{32r_+} \right] \quad (22)$$

4. Fuzzy BTZ Black hole

The static metric ansatz of non-rotating BTZ black hole is described by [13]

$$ds^2 = -g_{00} dt^2 + g_{00}^{-1} dr^2 + r^2 d\varphi^2 \quad (23)$$

where

$$g_{00} = \frac{r^2}{\ell^2} - 16\pi G \int_0^r dr r \rho_h(r) \quad (24)$$

The matter density is given by the following smeared distribution function [14-16]

$$\rho_h(r) = \frac{M}{2\pi h^2} \exp\left(-\frac{r}{h}\right) \quad (25)$$

The Hawking temperature T_H and the black hole entropy S are calculated by [17,18]

$$T_H \equiv \left(\frac{1}{4\pi} \frac{dg_{00}}{dr} \right)_{r=r_H} \quad (26)$$

$$S = \int \frac{dM}{T_H} = \int_{r_H}^{\infty} \frac{1}{T_H} \frac{dM}{dr_H} dr_H$$

we have to integrate the radius r_H from zero, as the minimum radius of fuzzy BTZ black hole is zero.

5. Conclusion

In this paper, we have study the effects of noncommutative spaces on the metric, horizon, entropy and Hawking temperature of a BTZ black hole. Through our paper, the results show deviations from the usual physical properties. The deviations depend on the parameter of noncommutativity and we find the classical results when θ tends to 0. However, we have started the study of BTZ black hole in fuzzy space by giving the metric, Hawking temperature and entropy. we will study in detail this part in our next paper.

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