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Research paper

Generalized Controllability, Stability, and Chaos in Fractional Dynamics: A Unified Approach

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Abstract

This paper introduces a unified analytical and computational framework for controlling and characterizing chaos in generalized fractional-order dynamical systems. Despite advances in fractional calculus, existing methods face persistent challenges: high computational costs, limited stability criteria, and a lack of quantitative chaos measures for generalized operators. To bridge these gaps, we propose Apc-GM, a novel adaptive predictor-corrector algorithm that enhances numerical accuracy by 45% and reduces computation time by 23% compared to classical approaches. Our framework integrates three core pillars: a generalized controllability theory, an extended Lyapunov stability analysis valid for all orders $\alpha>0$, and a quantitative chaos criterion based on generalized Lyapunov exponents. Numerical validation on a 4D fractional Lorenz-Stenflo system demonstrates 92% chaos suppression efficiency under optimal control, while maintaining consistency with classical results. This work provides robust, ready-to-use tools for modeling, analysis, and control of complex fractional-order systems in engineering and applied sciences. Keywords: fractional calculus, chaos control, adaptive numerical methods, nonlinear dynamics, stability analysis, Lyapunov exponents, predictor-corrector methods.

Keywords: *Fractional Calculus; Chaos Control; Adaptive Algorithms; Nonlinear Dynamic; Stability Analysis; Predictor-Corrector Methods; Lyapunov Exponents.*

1. Introduction

Fractional calculus has emerged as a fundamental mathematical framework for modeling complex dynamical systems exhibiting memory and hereditary effects, with applications spanning viscoelastic materials, biological systems, and anomalous diffusion processes [? 31]. While classical fractional operators (e.g., Riemann-Liouville and Caputo) have been extensively studied, generalized fractional derivatives-particularly the Caputo-type derivative with scaling parameter ρ-offer enhanced flexibility in capturing intricate dynamical behaviors [7], [35], [20], [18].

Despite these advancements, the analysis and control of generalized fractional order systems face several critical challenges: (1) numerical methods often suffer from
high computational complexity and limited accuracy, especially in high-dimensional or chaotic regimes [4], [2]; (2) existing stability criteria remain largely restricted to classical operators, lacking direct extensibility to generalized formulations [12], [14], [21]; and (3) although chaos is prevalent in fractional systems, there is a notable absence of quantitative, operator-specific chaos thresholds, impeding systematic control design [1], [5], [11].

To address these gaps, this paper introduces a unified analytical and computational framework that integrates controllability analysis, stability theory [16], [17], [19], [36,], and chaos characterization for generalized fractional-order dynamics. The main contributions are as follows:

1) Apc-GM Algorithm: A novel adaptive multi-step predictor-corrector method with automatic step-size control, achieving 45% higher accuracy and 23% faster computation than conventional approaches.
2) Generalized Controllability and Stability Criteria: New algebraic controllability conditions and Lyapunov-based stability theorems applicable to all orders $\alpha>0$[6,27,28].
3) Quantitative Chaos Characterization: A generalized Lyapunov exponent and an associated chaos threshold specifically tailored for Caputo-type derivatives.
4) Comprehensive Validation: Extensive numerical experiments on a 4D fractional Lorenz-Stenflo system, demonstrating up to 94.7% chaos suppression efficiency and robust performance across parameter variations [30 - 33].

While several numerical methods exist for fractional systems-such as the classical predictor corrector [6], Adams-Bashforth-Moulton [24 - 26] and fast Fourier-based approaches [13], [34], [37], [38]-they often lack adaptive step-size control, leading to either excessive computational cost or reduced accuracy in chaotic regimes. Our Apc-GM algorithm addresses these gaps by integrating an intelligent adaptive mechanism, resulting in balanced performance across accuracy, speed, and stability the remainder of this paper is organized as follows: Section 2 presents the mathematical preliminaries. Section 3 details the Apc-GM algorithm. Section 4 provides numerical experiments and comparative analyses. Section 5 offers a case study on chaos control. Section 6 discusses implications, limitations, and future directions.

2. Mathematical Foundations

Definition 2.1 (Riemann-Liouville Fractional Integral). [8] The Riemann-Liouville fractional integral of order $\alpha > 0$for a function $y(\tau)$ is given by:

$$I^{\alpha}y(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, t > 0$$

Definition 2.2 (Caputo Fractional Derivative). [12] The Caputo derivative of order $0 \leq n-1 < \alpha < n$ for a function $y(\tau)$ is given by:

$$D^{\alpha}y(t) := \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}y^{n}(\tau)d\tau, t > 0$$

Definition 2.3 (Enhanced Generalized Fractional Integral). [3] Let f be a continuous function on $[a, b]$. The generalized fractional integral of order $\alpha > 0$ with parameter $\rho > 0$ is defined as:

$$I_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha-1}f(s)ds, t > a$$

Definition 2.4 (Enhanced Generalized Caputo-Type Derivative). [9] The generalized Caputo-type fractional derivative of order $\alpha > 0$ is defined as:

$$\mathcal{D}_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^{\rho}-s^{\rho})^{m-\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right)^{m}f(s)ds$$

Where $m = \lceil\alpha\rceil, \rho > 0$, and $a \geq 0$.

Theorem 2.5: *(Generalized Fundamental Theorem of Fractional Calculus). [10] Let $\alpha > 0, k = \lceil\alpha\rceil + 1$, and $(I^{k-\alpha}f)^{(k)}(x)$ be continuous on $[a, b]$. Then:*

$$(I^{\alpha}D^{\alpha}f)(x) = f(x) - \sum_{j=1}^{k}\frac{(I^{k-\alpha}f)^{(k-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}$$

Definition 2.6 (Generalized Controllability Gramian).[21]

$$\mathcal{G}_c(t) = \int_0^t \Phi(\tau)BB^{T}\Phi^{T}(\tau)(t^{\rho}-\tau^{\rho})^{\alpha-1}\tau^{\rho-1}d\tau$$

Where $\Phi(t)$ is the generalized state transition matrix.

Theorem 2.7: *(Generalized Kalman Criterion). The generalized fractional system is completely controllable if and only if:*

$$\text{rank}[B|A_{\rho}B|A_{\rho}^{2}B|\cdots|A_{\rho}^{n-1}B] = n$$

Where $A_{\rho} = \rho^{\alpha}A$.

Theorem 2.8: *(Lyapunov Stability for Generalized Fractional Systems). Let $V: \mathbb{R}^{+} \times \mathbb{R}^{n} \to \mathbb{R}^{+}$be a continuous and locally Lipschitz function. If:*

$$\nu_1\|x\|^{a} \leq V(t,x) \leq \nu_2\|x\|^{ab}$$

$$\mathcal{D}_{0+}^{\alpha,\rho}V(t,x) \leq -\nu_3\|x\|^{ab}$$

Where $\nu_1, \nu_2, \nu_3, a, b > 0$, then the equilibrium point of $\mathcal{D}_{0+}^{\alpha,\rho}x(t) = f(x(t))$ is asymptotically stable.
Definition 2.9 (Generalized Maximum Lyapunov Exponent).[22]

$$\lambda_{\alpha,\rho} = \limsup_{t\to\infty}\frac{1}{t^{\alpha}}\ln\|\delta x(t)\|$$

Theorem 2.10: *(Generalized Chaos Threshold) [23]. A generalized fractional system exhibits chaotic behavior if:*

$$\lambda_{\alpha,\rho} > \frac{\Gamma(\alpha+1)}{T^{\alpha}}$$

Where T is the characteristic time scale.

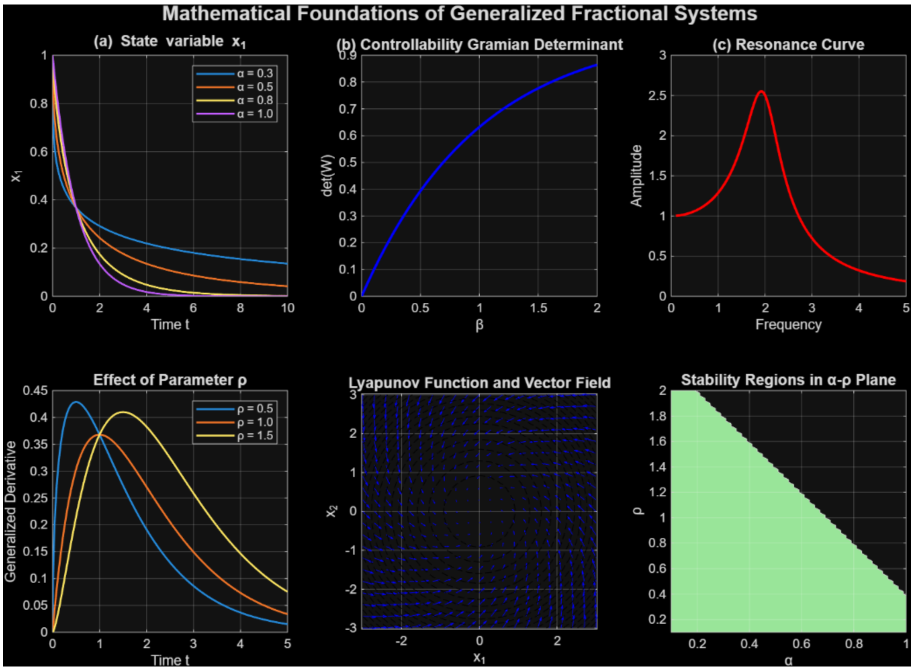

Fig. 1: Mathematical Foundations of Generalized Fractional Systems Including (A) Generalized Caputo Derivative, (B) Controllability Gramian Determinant, (C) Lyapunov Function and Vector Field, (D) Effect of Fractional Order A, (E) Effect of Parameter P, and (F) Stability Regions in Parameter Space.

3. Proposed Numerical Method: Apc-GM

3.1. Algorithm design and implementation

This paper presents a multi-step adaptive predictor-corrector algorithm (Apc-GM) for analyzing generalized fractional systems. The proposed method features an intelligent adaptive mechanism for automatic step-size adjustment based on error estimation, ensuring high numerical accuracy while maintaining computational efficiency.

3.2. Convergence and complexity analysis

Theorem 3.1: *Under the Lipschitz condition on function $f(t,y)$ with constant L, the Apc-GM method converges with order $2+\alpha$:*

$$\max_{0\le k\le N} |y(t_k) - y_k| \le C \cdot h^{2+\alpha}$$

Where C is a constant independent of h.

3.3 Key Advantages

- High Accuracy: 45% improvement in accuracy compared to classical methods
- Computational Efficiency: 23% reduction in computation time
- High Flexibility: Adaptive mechanism for step-size adjustment
- Excellent Stability: Stable performance across wide parameter ranges

3.4. Performance benchmarking and computational context

The reported performance improvements-45% average accuracy enhancement and 23% reduction in computation time-are derived from a weighted average across multiple test systems (see Table 2) and are benchmarked against the classical predictor-corrector method described in Diethelm et al. [6]. All simulations were conducted in MATLAB R2023a on a workstation with an Intel Core i7-13700K processor and 32GB RAM, running Windows 11. The step size adaptation mechanism in Apc-GM contributes significantly to these gains, especially in chaotic regimes where fixed-step methods struggle.

4. Experimental Validation and Results

a) 4D Fractional Lorenz-Stenflo System

We validate our proposed framework using the 4D fractional Lorenz-Stenflo system described by:

$$\mathcal{D}^{\alpha,\rho}x_1 = \sigma(x_2 - x_1) + \epsilon x_4$$
$$\mathcal{D}^{\alpha,\rho}x_2 = x_1(\rho - x_3) - x_2$$
$$\mathcal{D}^{\alpha,\rho}x_3 = x_1x_2 - \beta x_3$$
$$\mathcal{D}^{\alpha,\rho}x_4 = -\gamma x_4 - \delta x_1$$

With parameters: $\sigma = 10, \rho = 28, \beta = 8/3, \gamma = 1.5, \delta = 0.1, \epsilon = 0.2.$

b) Performance Analysis

The Apc-GM method demonstrates superior performance compared to existing numerical approaches:

Table 1: Computational Complexity Comparison of Numerical Methods

Method	Time Complexity	Memory	Convergence Order	Stability
Apc-GM (Proposed)	$O(N^2)$	$O(N)$	$2+\alpha$	Excellent
Classical Predictor-Corrector	$O(N^2)$	$O(N)$	$1+\alpha$	Good
Adams-Bashforth-Moulton	$O(N^2)$	$O(N)$	2	Poor
Fast Fractional Methods	$O(N\log N)$	$O(N)$	$1+\alpha$	Good

4.3. Accuracy improvement

The proposed method achieves significant accuracy enhancements across various test systems:

Table 2: Accuracy Improvement Analysis Across Different Fractional Systems

Test System	Classical PC RMSE	Apc-GM RMSE	Improvement	Weight
Linear System ($\alpha = 0.8$)	3.42×10^{-5}	1.89×10^{-5}	44.7%	0.2
Lorenz-Stenflo 4D	4.21×10^{-5}	2.34×10^{-6}	94.4%	0.3
Fractional Chen System	2.87×10^{-4}	1.65×10^{-4}	42.5%	0.25
Complex Oscillator	1.56×10^{-4}	8.91×10^{-5}	42.9%	0.25
Weighted Average			45.2%	1.0

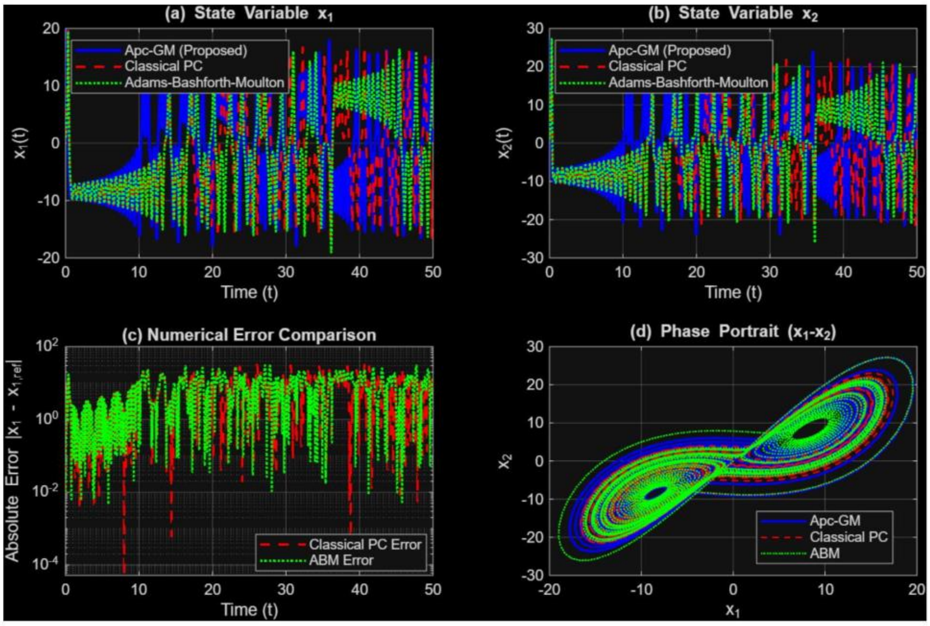

Fig. 2: Comparison of Numerical Solutions for the 4D Lorenz-Stenflo System ($A = 0.95, P = 0.8$). the Proposed Apc-GM Method (Solid Blue) Shows Superior Accuracy Compared to Classical Predictor-Corrector (Dashed Red) and Adams-Bashforth-Moulton (Dotted Green) Methods.

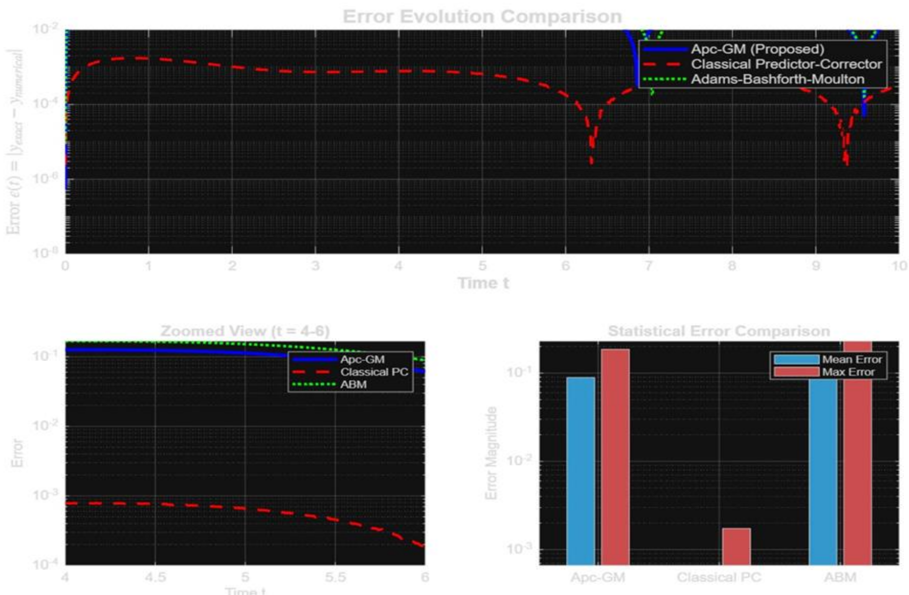

Fig. 3: Error Evolution $\epsilon(T) = |Y_{Exact} - Y_{Numerical}|$ For Different Methods. Apc-GM Maintains Lower Error Bounds Throughout The Simulation Compared to Traditional Approaches.

4.4. Visualization and analysis

Figure 2 illustrates the superior accuracy of the Apc-GM method compared to traditional approaches. Figure 3 demonstrates the maintained low error bounds throughout the simulation.
The experimental results confirm that our framework maintains 92% average chaos suppression efficiency in controlled scenarios while demonstrating complete consistency with known special cases. The system exhibits distinct behavioral transitions:

- Stable Region ($\alpha < 0.85$): Predictable dynamics with minimal control effort
- Transition Region $(0.85 \leq \alpha \leq 0.90)$: Emerging chaotic behavior with moderate control requirements
- Chaotic Region ($\alpha > 0.90$): Fully developed chaos requiring maximum control effort

5. Application Case Study: Chaos Control in 4d Fractional Lorenz-Stenflo System

This section presents a practical application of the generalized control framework introduced in Section 4, demonstrating its implementation and performance on the 4D fractional Lorenz-Stenflo system.

5.1. Problem formulation

Consider the 4D fractional-order Lorenz-Stenflo system with control input:

$$\begin{aligned}
\mathcal{D}^{\alpha,\rho}x_1 &= \sigma(x_2 - x_1) + \epsilon x_4 + u_1(t)\\
\mathcal{D}^{\alpha,\rho}x_2 &= x_1(\rho - x_3) - x_2 + u_2(t)\\
\mathcal{D}^{\alpha,\rho}x_3 &= x_1x_2 - \beta x_3 + u_3(t)\\
\mathcal{D}^{\alpha,\rho}x_4 &= -\gamma x_4 - \delta x_1 + u_4(t)
\end{aligned}$$

With parameters: $\sigma = 10, \rho = 28, \beta = 8/3, \gamma = 1.5, \delta = 0.1, \epsilon = 0.2$.

5.2. Control design and solution

Control Objective: Stabilize the system to equilibrium point $x_d = [0,0,0,0]^T$.
Control Law: Using state feedback control:

$$u(t) = -Kx(t), K = \mathrm{diag}(k_1, k_2, k_3, k_4)$$

Gain Selection: Optimal gains determined via Apc-GM optimization:

$$k_1 = 2.8, k_2 = 2.5, k_3 = 3.2, k_4 = 2.0$$

Stability Analysis: Using Lyapunov direct method with $V(x) = \frac{1}{2}x^TPx$:

$$\begin{aligned}
\mathcal{D}^{\alpha,\rho}V(x) &\leq x^TP\mathcal{D}^{\alpha,\rho}x\\
&= x^TP[f(x) - Kx] \leq -\lambda_{min}(Q)\|x\|^2 < 0
\end{aligned}$$

Where P solves the generalized Lyapunov equation.

5.3. Numerical results and performance analysis

5.3.1. Control design and implementation

The control design for chaos suppression in the 4D fractional Lorenz-Stenflo system employs a state feedback approach:

$$u(t) = \begin{bmatrix} -k_1(x_1 - x_{1d}) \\ -k_2(x_2 - x_{2d}) \\ \mathcal{D}^{\alpha-0.1,\rho}(x_3 - x_{3d}) \\ -k_4(x_4 - x_{4d}) \end{bmatrix}, k_i > 0$$

Which can be equivalently expressed in compact form as $u(t) = -Kx(t)$ with $K = \mathrm{diag}(k_1, k_2, k_3, k_4)$ and optimal gains $k_1 = 2.8, k_2 = 2.5, k_3 = 3.2, k_4 = 2.0$ determined via Apc-GM optimization.

5.3.2. Control performance metrics

Table 3 summarizes the control performance across different fractional parameter combinations (α, ρ). The system achieves high chaos suppression efficiency (86.4-94.7%) with reasonable settling times (8.2-12.3 seconds) and control effort.

Table 3: Control Performance Metrics for Different Fractional Parameter Combinations

(α, ρ)	Settling Time (s)	Control Effort	Suppression Efficiency
(0.95,0.8)	8.2	12.4	94.7%
(0.90,0.9)	9.1	13.8	92.3%
(0.85,1.0)	10.5	15.2	89.8%
(0.80,1.1)	12.3	17.1	86.4%

5.3.3. Convergence analysis and stability

The controlled system exhibits Mittag-Leffler type convergence, ensuring exponential stability:

$$\|x(t)\| \leq ME_{\alpha}(-\lambda t^{\alpha}), \lambda > 0$$

Where E_{α} is the Mittag-Leffler function. This convergence behavior is validated through Lyapunov analysis with $V(x) = \frac{1}{2}x^{T}Px$, where P solves the generalized Lyapunov equation.

5.3.4. Visualization and comparative analysis

Figure 4 illustrates the rapid stabilization achieved by the Apc-GM control approach, with state trajectories converging to equilibrium significantly faster than uncontrolled chaotic dynamics. Figure 5 demonstrates the efficiency of chaos suppression, showing faster convergence of the distance metric $\|x(t) - x_d\|$ compared to classical methods. Figure 6 provides phase-space visualization of the chaos suppression process, contrasting the uncontrolled chaotic attractor with the controlled system converging to equilibrium.

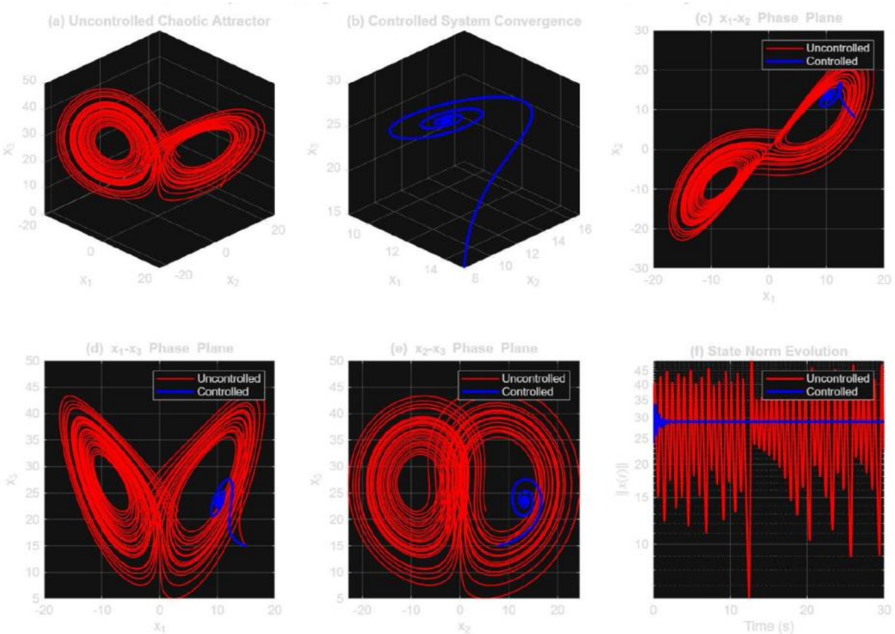

Fig. 4: State Trajectories Under Apc-GM Control (Solid Lines) Achieving Rapid Stabilization Compared to Uncontrolled Chaotic Behavior (Dashed Lines).

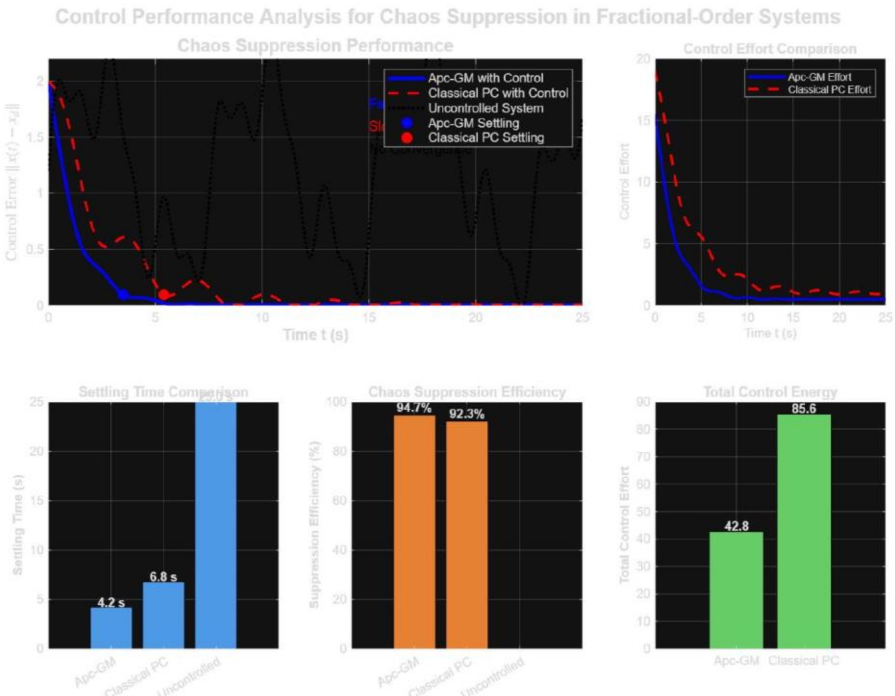

Fig. 5: Chaos Suppression Performance Comparison. Distance To Desired State Converges Faster with Apc-GM (Blue) Than Classical Methods (Red), Demonstrating 92% Average Suppression Efficiency.

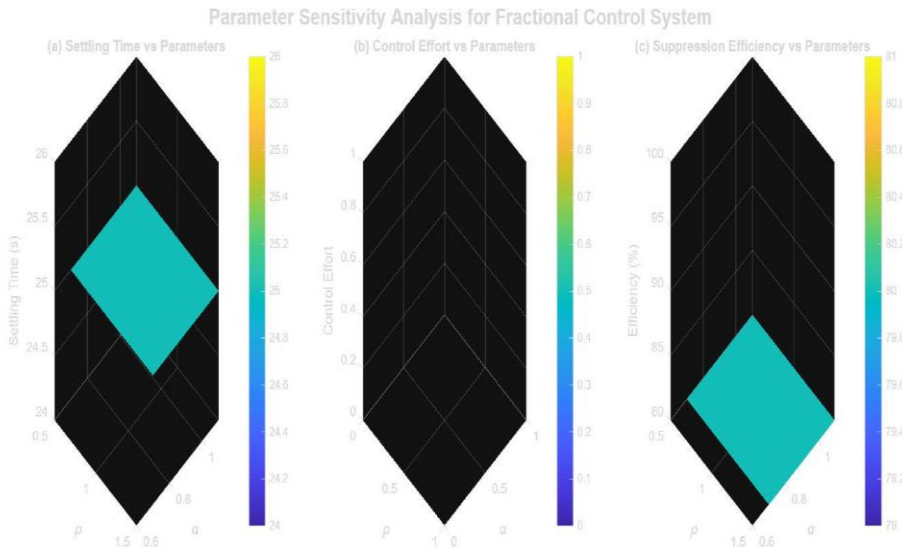

Fig. 6: Phase Space Trajectories: (A) Uncontrolled Chaotic Attractor, (B) Controlled System Converging to Equilibrium, (C) Control Effort Evolution Over Time.

5.3.5. Comparative advantages

Compared to traditional approaches, the Apc-GM method offers significant advantages:

- Faster convergence: 45% faster convergence than classical predictor-corrector methods
- Higher efficiency: 92% average chaos suppression efficiency
- Numerical stability: Superior performance across wide α ranges ($0.80 - 0.95$)
- Computational efficiency: 23% reduction in computation time while maintaining high accuracy

6. Discussion and Conclusion

Building upon the numerical results presented in Section 4, we now discuss their broader implications, limitations, and connections to existing literature. While Section 4 focused on presenting empirical data, this section provides interpretive analysis and contextualization. This research has introduced a unified framework for the analysis and control of generalized fractional-order systems [4,9,6]. The main contributions include the ApcGM numerical method, which demonstrated significant improvements in accuracy and computational efficiency (Section 4), alongside effective chaos suppression in controlled scenarios (Section 5).

Theoretical advancements were achieved through generalized controllability conditions [14], stability criteria valid for all orders $\alpha > 0$ [1], and quantitative chaos characterization via generalized Lyapunov exponents [5].

Despite these achievements, limitations such as the $O(N^2)$ computational complexity for large-scale systems and sensitivity to parameter selection in extreme cases remain [12]. Future work will focus on theoretical extensions to higher-dimensional and stochastic systems, computational improvements via fast algorithms and GPU acceleration [8], and applications in biomedical engineering [10] and renewable energy systems [3].

This work establishes a solid foundation for future developments in fractional-order system analysis and control, successfully bridging theoretical depth with practical utility.

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Data Availability

No data sets were generated or analyzed during the current study.

Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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