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Research paper

Anti-Imprecision in The Fuzzy Group Concerning Reference Function and Its Application

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Abstract

In this article, we first defined an anti-subgroup based on a reference function and then extended it to an α-imprecise subgroup, α-imprecise cosets, an α-normal imprecise subgroup, and finally defined an α-anti-imprecise subset. Additionally, we discuss some of their properties in detail. In this article, it has been proved that an imprecise subgroup (normal imprecise subgroup) is again an α-imprecise subgroup (α-normal imprecise subgroup), but the converse is not true, for which some examples are cited. Furthermore, we anticipated the development of an application derived from an anti-imprecise subgroup that can be applied to a variety of networking issues.

Keywords: *Anti Imprecise Subgroup; A-Imprecise Subsets; A-Imprecise Cosets; A-Normal Imprecise Subgroup; A-Anti-Imprecise Subsets.*

1. Introduction

FS theory is a generalization of the concept of classical sets. This idea was first put forth by Zadeh [1] in 1965. Since then, there has been a great deal of focus on this area of study because of its many uses, which include the study of social and economic behavior as well as computer science and engineering. By defining FGs, Rosenfeld [2] started the investigation of algebraic structures on FSs. One of the most striking applications of Zadeh's FS theory is the FG theory. In 1971, Rosenfeld [2] introduced the concept of FS_G by utilizing Zadeh's FS_b of a set. The development of fuzzy abstract algebra was inspired by Rosenfeld's [2] work. Additional research on this subject has been carried out by Das [3], Mukherjee and Bhattacharya [4 - 7]. Many authors have merged other uncertainty techniques with this theory to extend it to a wide range of mathematics, for instance, [8 - 11]. Recently, the notion of t-intuitionistic FO of an element of a t-intuitionistic FS_G of a finite group was introduced by Alolaiyan et al. [12], who also looked at some significant algebraic characteristics of this phenomenon. Additionally, they demonstrate numerous practical algebraic facets of this concept for a CG. Shuaib et al. [13] presented the concept of η-intuitionistic FS_G defined on η-intuitionistic FS and provided various characteristics of their study. The concept of a Pythagorean FS_G was described by Razaq et al. [14], who also looked at some of its algebraic features. Numerous in-depth studies are also conducted using this idea in their work. Razzaque and Razaq [15] described the concept of a FS_G of q-rung orthopairs and looked at several algebraic properties of this newly proposed idea. A new S-box generation algebraic approach utilizing a combination of a multiplicative cyclic group and a permutation group with a big order is the foundation of the innovative medical picture encryption algorithm that Razaq et al. [16] devised. The proposed developed S-box's security performance is evaluated by analyzing and assessing several standard security indicators. Also, Imtiaz et al. [17] extended Zadeh's FS [1] to complex FSs and studied properties of ξ-Complex FSs and their applications. In 2021, Imtiaz et al. [18] presented various uses of ξ-Complex FSs in solving problems related to decision-making. In 2023, Imtiaz and Shuaib [19] extended their work to develop fuzzy algebra by presenting and examining novel ideas and their characteristics within the framework of conjunctive complex fuzzy environments. Using Zadeh's complement definition of FS, Biswas [20] developed the idea of anti-FS_G, which is just the complement of Rosenfeld's [2] FS_G theory, and he identified some basic characteristics of this phenomenon. In Onasanya [21], many anti-fuzzy properties of FGs were established. [22 - 25] contains some extended works of anti-FS_G. However, Zadeh's [1] definition of FS complement, which deviates from the two universal principles of traditional set theory, disproves the assertion that FS theory is an extension of traditional set theory. Because the two functions—fuzzy MF and fuzzy RF—were treated as the same in the prior definition of FS, Baruah [26] concluded that the fuzzy complement definition was flawed. As a result, it was finally determined that Zadeh's idea of the complement FS does not follow the two fundamental rules of traditional set theory: the union of an FS and its complement is a universal set, and the intersection of an FS and its complement is a null set. To address these issues with the current definition of an FS, H. K. Baruah [26 - 29] rewrote the concept of an FS's complement in 1999. After Baruah [26 - 29] gave a new definition of FSs in terms of MF and RF, we were able to get a new definition of the fuzzy complement of a FS. Furthermore, this extended definition of FS can circumvent the issue with Zadeh's [1] FS and provide us the union of a FS and its complement as a universal set as well as the intersection of a FS and its complement as a null set. The term IS refers to this extended definition of a FS with a new complement form. Numerous other authors have addressed this new concept in various fields. For instance, in

2015, Borgoyary [30], [31] applied the definition of IS in classical matrix properties, in the field of transportation problems, and in the construction of two- and three-dimensional imprecise numbers. In the same year, Basumatary [32] studied fuzzy closure and their properties based on this extended definition. In 2016 and 2017, Singh and Borgoyary [33], [34] studied the rate of convergence for different imprecise polynomials and Basumatary [35] studied the properties of fuzzy closure and showed that all the properties of FS hold good with this extended definition. In 2017, Basumatary et al. [36] studied the properties of fuzzy boundary with respect to RF and showed that all the properties of FS hold good with this extended definition. In 2018, Basumatary and Mwchahary [37] studied intuitionistic FS based on this extended definition of FS.

In our prior study [38], [39,], we investigated IG and anti-IG utilizing the new notion introduced by Baruah. IG is an extended version of FG theory based on Baruah's idea of an IS. In 2024, Narzary and Borgoyary [38] constructed an extended version known as IG, which is essentially an extension of a FG theory, by using the definition of an IS to investigate the algebraic structure of FG and FS$_G$ in terms of MF and RF. Since the extended IS idea is more practical than the current FS, its characteristics and outcomes that IG provides are more practical and fulfilling than those of the current FG theory.

The suggested IG definition in this article is derived using an appropriate mathematical framework and specified in accordance with Baruah's [11] extended FS concept. The development of an IG under multiplicative operation is the focus of this paper. Here, we tried to demonstrate the basic properties, theorems, and examples using our group concept. The IG is a more comprehensive idea for researching Rosenfeld's [3] FS$_G$ theory. In this study, most of the properties of classical set theory are shown to be clearly valid. Borgoyary et al. [39] developed numerous more essential characterizations of the IS$_G$ in the same year using the idea of IGs. Since GT has not yet used the idea of an IS, all the properties and theorems of this study are unique. Additionally, this study offers certain features that are not met by standard GT. Till now most of the work has been done using old FS definitions, according to literature reviews. ISs are used extremely infrequently, particularly in algebraic fields. IS theory extending conventional algebraic structures by including degrees of membership, allowing for ambiguity and uncertainty in both operations and structures. In 2025, Narzary and Borgoyary [43] studied some properties of anti-IG. This study of anti-IG is significant since it is based on Baruah's [28] expansion of the previous FS concept, which can address the inadequacies of Zadeh's FS definition. The concept of anti-IG is essentially the complement of IG, as stated in terms of MF and RF based on Baruah's complement definition of IS [28]. The goal of this article is to use the extended definition of FS to create a new methodology for discussing FGs more conventionally, so that the results can be implemented in a variety of contexts.

Additionally, since GT and FGs are essential for simulating uncertainty in the real world, they offer frameworks for managing ambiguous or partial information. GT, a fundamental concept in numerous fields, scrutinizes symmetry and facilitates the examination of systems with inherent unpredictability. When a system shows symmetry under specific transformations, GT offers a useful analytical tool. Extending the idea of FSs, FGs permit partial membership within groups, making it possible to depict ambiguity and vagueness in practical contexts. FS$_G$s are a notion that extends the use of FSs to the study of group structures. Also, in network optimization, the anti-IGs offer a means of modeling uncertainty and restrictions, enabling us to identify solutions that are not only effective but also realistic and robust to real-world circumstances. By using MF and RF degrees to represent network elements and their interactions, it can be used in network optimization. A lower MV denotes a lower chance of belonging to a certain network group or class than others. For instance, if we want to find the shortest route from point A to point B in a city's transportation network of roads, we can use an optimization algorithm. Some roads are well-maintained and have low traffic (imprecise), while others are constantly congested (anti-imprecise). The anti-imprecise constraints ensure that the algorithm avoids the congested roads, even if they would be shorter in a purely distance-based optimization.

The primary goal of this paper is to provide anti-IG and, consequently, anti-IS$_G$, which have been developed to investigate certain basic characteristics of classic group theory in relation to anti-IS$_G$. This anti-IG study is significant because it builds upon Baruah's [28] expansion of the current FS concept, which can address the limitations of Zadeh's FS definition. According to Baruah's complement definition of IS, the idea of anti-IG is essentially the complement of IG, which is described in terms of MF and RF [28]. The idea of IG and IS$_G$s as stated in [38] is carried over in this study using Baruah's concept of IS [28].

The main objective of this article is to extend the notion of IS definition to α −IS$_G$ and define α −imprecise cosets, α −normal IS$_G$s, and lastly, define α −anti-IS$_G$s. The idea of anti-IS$_G$ is taken from the notion of the extended complement definition of FS. Further, this article provides some of their necessary properties supported by some numerical examples. Also, an application of anti-IG is discussed in the last section of this article.

Throughout the study, the classical group is denoted by G.

2. Motivation

In recent years, there has been a lot of debate among scholars on the application of FS theory because some authors have pointed out that the theory cannot address some uncertainty boundary issues in the real world. Among these, Piegat [40] cited Zadeh's [1] explanation of fuzzy arithmetic's shortcomings in resolving some practical problems. To address these shortcomings, numerous scholars have developed various formulations for fuzzy arithmetic operations; Kosinski et al. [41] and Gao et al. [42], for example, found more problems with Zadeh's [1] fuzzy complement definition for fuzzy numbers. Because of this, the authors developed a broader idea called C-FS theory, which is free of the FS errors of Zadeh [1]. They point out that if the complement of a FS is defined as $1-\mathfrak{u}_f(x)$, where $\mathfrak{u}_f$ is a fuzzy MF, then the complement of a set may not exist in (Zadeh, 1965)'s FS theory. For instance: Based on the current definition of FS, if $A_f = \{t_1, \mathfrak{u}_f(t_1)\} = \{t_1, 0.5\}$ is a FS, then $A_f^c = \{t_1, 1-\mathfrak{u}_f(t_1)\} = \{t_1, 1-0.5\} = \{t_1, 0.5\}$ is its complement.

Consequently, $A_f \cup A_f^c = \{t_1, 0.5\} \cup \{t_1, 0.5\} = \{t_1, \max(0.5,0.5)\} = \{t_1, 0.5\} \neq \{t_1, 1\}$ (universal set)

Additionally, $A_f \cap A_f^c = \{t_1, 0.5\} \cap \{t_1, 0.5\} = \{t_1, \min(0.5,0.5)\} = \{t_1, 0.5\} \neq \{t_1, 0\}$ (null set)

Baruah [26, 27, 28, 29] further highlighted a few more shortcomings in the FS theory. He pointed out that the Probability-Possibility Consistency Principle and the complement definition of FS are poorly specified. Instead of using a single MF, he established the FS definition in terms of two functions, fuzzy MF, and fuzzy RF. This extended definition is termed as IS.

Now, if $A_f = \{t_1, \mathfrak{u}_f(t_1)\}$ is the existing FS and $A_f^c = \{t_1, 1-\mathfrak{u}_f(t_1)\}$ is its fuzzy complement, then according to this extended definition, it would be $\mathfrak{u}_A^i = \{t_1, \mathfrak{u}_f(t_1), 0\}$ and the complement of $\mathfrak{u}_A^i$ would be $\mathfrak{u}_A^{i^c} = \{t_1, 1, \mathfrak{u}_f(t_1)\}$.

Then $\mathfrak{u}_A^i \cup \mathfrak{u}_A^{i^c} = \{t_1, \mathfrak{u}_f(t_1), 0\} \cup \{t_1, 1, \mathfrak{u}_f(t_1)\} = \{t_1, \max(\mathfrak{u}_f(t_1), 1), \min(0, \mathfrak{u}_f(t_1))\} = \{t_1, 1, 0\} = X$(universal set)

And, $\mathfrak{u}_A^i \cap \mathfrak{u}_A^{i^c} = \{t_1, \mathfrak{u}_f(t_1), 0\} \cap \{t_1, 1, \mathfrak{u}_f(t_1)\} = \{t_1, \min(\mathfrak{u}_f(t_1), 1), \max(0, \mathfrak{u}_f(t_1))\} = \{t_1, \mathfrak{u}_f(t_1), \mathfrak{u}_f(t_1)\} = \phi$(null set).

3. Preliminaries

Definition 3.1: *([2]) A FS$_b$ u_f is said of to be a FS$_G$ of G if*

i) $u_f(t_1t_2) \geq \min\{u_f(t_1), u_f(t_2)\};\ \forall\, t_1, t_2 \in G$

ii) $u_f(t_1^{-1}) \geq u_f(t_1);\ \forall\, t_1 \in G$

iii) $u_f(e) \geq u_f(t_1);\ \forall\, t_1 \in G$

Definition 3.2: *([2]) If u_{f_1} and u_{f_2} are two FS$_G$s then their product is defined as*

$u_{f_1} \circ u_{f_1}(t_3) = \max\left\{\min(u_{f_1}(t_1), u_{f_2}(t_2)) \middle| t_1, t_2, t_3 \in G, t_1t_2 = t_3\right\}$, where '∘' denotes product.

Definition 3.3: *([28]) If $u_m^1(t_1)$ is a fuzzy MF and $u_r^2(t_1)$ is a fuzzy RF such that $0 \leqslant u_r^2(t_1) \leqslant u_m^1(t_1) \leqslant 1$, then the IS is defined*

as $u_A^i = \{t_1, u_m^1(t_1), u_r^2(t_1); t_1 \in X\}$ where X is the universal set and $u_A^i(t_1) = u_m^1(t_1) - u_r^2(t_1)$ gives the actual MV for all $t_1 \in X$. For convenient of writing above IS is denoted by $(u_m^1, u_r^2)(t_1); t_1 \in X$.

Definition 3.4: *([28]) The complement of the IS $u_A^i = \{t_1, u_m^1(t_1), u_r^2(t_1); t_1 \in X\}$ is $(u_A^i)^c = \{t_1, 1, u_m^1(t_1); t_1 \in X\}$.*

Definition 3.5: *([28]) If $A_i{}^c$ and $B_i{}^c$ are the complements of the two ISs*

$A_i = \{t_1, u_m^1(t_1), u_r^2(t_1); t_1 \in X\}$ and $B_i = \{t_1, u_m^3(t_1), u_r^4(t_1); t_1 \in X\}$ respectively, then their union and intersection are respectively,

$A_i{}^c \cup B_i{}^c = \{t_1, 1, u_m^1(t_1)\} \cup \{t_1, 1, u_m^3(t_1)\} = \{t_1, \max(1,1), \min(u_m^1(t_1), u_m^3(t_1)); t_1 \in X\}$

$A_i{}^c \cap B_i{}^c = \{t_1, 1, u_m^1(t_1)\} \cap \{t_1, 1, u_m^3(t_1)\} = \{t_1, \min(1,1), \max(u_m^1(t_1), u_m^3(t_1)); t_1 \in X\}$.

Definition 3.6: [39] Let an IS$_b$ (u_m^1, u_r^2) of G together with a binary composition ' $*$ ' be called an IG $[(u_m^1, u_r^2), *]$, if

i) $(u_m^1, u_r^2)(t_1t_2) \geq \min\{(u_m^1, u_r^2)(t_1), (u_m^1, u_r^2)(t_2)\}; t_1, t_2 \in G$

ii) $(u_m^1, u_r^2)(t_1^{-1}) \geq (u_m^1, u_r^2)(t_1);\ \forall\, t_1 \in G$

iii) $(u_m^1, u_r^2)(e) \geq (u_m^1, u_r^2)(t_1);\ \forall\, t_1 \in G$

4. Anti-ISG

Definition 4.1: If (u_m^1, u_r^2) is an IS$_b$ of X, then the complement of (u_m^1, u_r^2) denoted by $(u_m^1, u_r^2)^c$, is the IS of X given by

$(u_m^1, u_r^2)^c(t_1) = \{t_1, 1, u_m^1(t_1)\};\ t_1 \in\}$ where $1 - u_m^1(t_1)$ is the non-MV of t_1 in X.

Definition 4.2: Let G be a group. A FS$_b$ $(u_m^1, u_r^2)^c$ of G is called an anti-IS$_G$ of G if for $t_1, t_2 \in G$

i) $(u_m^1, u_r^2)^c\,(t_1, t_2) \leq \max\{(u_m^1, u_r^2)^c\,(t_1), (u_m^1, u_r^2)^c\,(t_2)\}$

Where $\max\{((u_m^1, u_r^2)^c, (u_m^1, u_r^2)^c)\}\,(t_2) = (\max(u_m^1(t_1), u_m^1(t_2)), \min(u_r^2(t_1), u_r^2(t_2)))$

ii) $(u_m^1, u_r^2)^c\,(t_1^{-1}) \leq (u_m^1, u_r^2)^c(t_1)$

The following proposition follows from the above definition:
Proposition 4.1: If $(u_m^1, u_r^2)^c$ is an anti-IS$_G$ of G, then

i) $(u_m^1, u_r^2)^c\,(e) \leq (u_m^1, u_r^2)^c(t_1), \forall t_1 \in G$

ii) $(u_m^1, u_r^2)^c(t_1^{-1}) = (u_m^1, u_r^2)^c(t_1), \forall t_1 \in G$

Proposition 4.2: (u_m^1, u_r^2) is an IS$_G$ of G iff its complement $(u_m^1, u_r^2)^c$ is an anti-IS$_G$ of G i.e.,

i) $(u_m^1, u_r^2)^c\,(t_1t_2) \leq \max\{(u_m^1, u_r^2)^c(t_1), (u_m^1, u_r^2)^c(t_2)\}$

ii) $(u_m^1, u_r^2)^c\,(t_1^{-1}) \leq (u_m^1, u_r^2)^c(t_1)$

Example 1: Let us consider an IS$_G$

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (1,0);\ t_1 = 1,-1 \\ (0.92,0.08);\ t_1 = i,-i \end{cases} \quad (1)$$

Then

$$(u_m^1, u_r^2)^c(t_1) = \begin{cases} (1,1);\ t_1 = 1,-1 \\ (1,0.92);\ t_1 = i,-i \end{cases} \quad (2)$$

Is an anti-IS$_G$ of G.
Proof: Suppose (u_m^1, u_r^2) is an IS$_G$ of G.
Then we have,
i) $(u_m^1, u_r^2)(t_1t_2) \geq \min\{(u_m^1, u_r^2)(t_1), (u_m^1, u_r^2)(t_2)\}$

$\Rightarrow -(u_m^1, u_r^2)\ (t_1t_2) \leq -\min\{(u_m^1, u_r^2)\ (t_1), (u_m^1, u_r^2)\ (t_2)\}$

$\Rightarrow 1-(u_m^1, u_r^2)(t_1t_2) \leq 1 - \min\{(u_m^1, u_r^2)\ (t_1), (u_m^1, u_r^2)\ (t_2)\}$

$\Rightarrow (u_r^2, 0)(t_1t_2) \cup (1, u_m^1)(t_1t_2) \leq \max\{(u_m^1, u_r^2)(t_1), (u_m^1, u_r^2)\ (t_2)\}$

$\Rightarrow (u_m^1, u_r^2)^c\ (t_1t_2) \leq \max\{(u_m^1, u_r^2)^c\ (t_1), (u_m^1, u_r^2)^c\ (t_2)\};\ \forall t_1, t_2 \in G$

ii) $(u_m^1, u_r^2)(t_1{}^{-1}) \geq (u_m^1, u_r^2)(t_1)$

$\Rightarrow -(u_m^1, u_r^2)(t_1{}^{-1}) \leq -(u_m^1, u_r^2)\ (t_1)$

$\Rightarrow 1-(u_m^1, u_r^2)(t_1{}^{-1}) \leq 1-(u_m^1, u_r^2)(t_1)$

$\Rightarrow (u_r^2, 0)(t_1{}^{-1}) \cup (1, u_m^1)(t_1{}^{-1}) \leq (u_r^2, 0)(t_1) \cup (1, u_m^1)(t_1)$

$\Rightarrow (u_m^1, u_r^2)^c\ (t_1{}^{-1}) \leq (u_m^1, u_r^2)^c\ (t_1);\ \forall\ t_1 \in G$

The converse can also be proved similarly.
Proposition 4.3: $(u_m^1, u_r^2)^c$ is an anti- IS$_G$ of G iff $(u_m^1, u_r^2)^c\ (t_1t_2{}^{-1}) \leq \max\{(u_m^1, u_r^2)^c(t_1), (u_m^1, u_r^2)^c\ (t_2)\};\ \forall\ t_1, t_2 \in G$.

5. Lower-Level Subsets

Definition 5.1: *Let* (u_m^1, u_r^2) *be an IS of X. For* $(\theta_1, \theta_2) \in [0,1] \times [0,1]$, *the lower level (LLS) subset of* (u_m^1, u_r^2) *is the subset*

$\overline{(u_m^1, u_r^2)}_{\ (\theta_1,\theta_2)} = \{t_1 \in X: (u_m^1, u_r^2)\ (t_1) \leq (\theta_1, \theta_2)\}$

Clearly $\overline{(u_m^1, u_r^2)}_{(1,0)} = X$, $\overline{(u_m^1, u_r^2)}_{(0,0)} = (0,0)$ and $\overline{(u_m^1, u_r^2)}_{\ (\theta_1,\theta_2)} \cup \overline{(u_m^1, u_r^2)}_{\ (\theta_1,\theta_2)} = X$.

Proposition 5.1: Let $(u_m^1, u_r^2)^c$ be an anti-IS$_G$ of G. Then for $(\theta_1, \theta_2) \in [0,1] \times [0,1]$ such that $(\theta_1, \theta_2) \geq (u_m^1, u_r^2)^c\ (e)$, $\overline{(u_m^1, u_r^2)^c}_{\ (\theta_1,\theta_2)}$ is a subgroup of G.
Proof: We have $(u_m^1, u_r^2)^c\ (t_1t_2{}^{-1}) \leq \max\{(u_m^1, u_r^2)^c\ (t_1), (u_m^1, u_r^2)^c\ (t_2)\}$

$= (\theta_1, \theta_2)\ ;\ \forall\ t_1, t_2 \in \overline{(u_m^1, u_r^2)^c}_{\ (\theta_1,\theta_2)}$

Again, since $t_1, t_2 \in \overline{(u_m^1, u_r^2)^c}_{\ (\theta_1,\theta_2)}$ so $(u_m^1, u_r^2)^c\ (t_1) \leq (\theta_1, \theta_2)$ and $(u_m^1, u_r^2)^c\ (t_2) \leq (\theta_1, \theta_2)$.
Proposition 5.2: Let G be a group and (u_m^1, u_r^2) be an IS$_b$ of G such that $\overline{(u_m^1, u_r^2)}_{\ (\theta_1,\theta_2)}$ is a subgroup of G for all $(\theta_1, \theta_2) \in [0,1] \times [0,1]$ with $(\theta_1, \theta_2) \geq (u_m^1, u_r^2)(e)$. Then $(u_m^1, u_r^2)^c$ is an anti-IS$_G$ of G.
Proof: Since $(u_m^1, u_r^2)^c(t_1t_2{}^{-1}) \leq \max\{(u_m^1, u_r^2)\ (t_1), (u_m^1, u_r^2)\ (t_2)\} = (\theta_1, \theta_2)$
Let $t_1, t_2 \in G$ and $(u_m^1, u_r^2)^c(t_1) = (\theta_1, \theta_2), (u_m^1, u_r^2)^c\ (t_2) = (\theta_3, \theta_4)$.
Suppose that $(\theta_1, \theta_2) \subseteq (\theta_3, \theta_4)$. Then since, $(u_m^1, u_r^2)^c\ (t_1t_2{}^{-1}) \leq (\theta_3, \theta_4)$

$= \max\{(\theta_1, \theta_2), (\theta_3, \theta_4)\}$

$= \max\{(u_m^1, u_r^2)^c(t_1), (u_m^1, u_r^2)^c(t_2)\}$

$\Rightarrow (u_m^1, u_r^2)^c(t_1t_2{}^{-1}) \leq \max\{(u_m^1, u_r^2)^c(t_1), (u_m^1, u_r^2)^c(t_2)\}$

Thus, $(u_m^1, u_r^2)^c$ is an anti-IS$_G$ of G.

Definition 5.2: $(u_m^1, u_r^2)^c$ *be an anti-IS$_G$ of G. The subgroup* $\overline{(u_m^1, u_r^2)^c}_{\ (\theta_1,\theta_2)}$, *with* $(\theta_1, \theta_2) \geq (u_m^1, u_r^2)^c(e)$, *is called lower level subgroup of* $(u_m^1, u_r^2)^c$.

6. α −ISbs

Definition 6.1: *Let* (u_m^1, u_r^2) *be an ISb of* G. *Let* $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [1,0]$. *Then the ISb* $(u_m^1, u_r^2)^\alpha$ *of* G *is called the* α *−ISb of* G *(with respect to IS* (u_m^1, u_r^2) *) and is denoted as* $(u_m^1, u_r^2)^\alpha(t_1) = min\{(u_m^1, u_r^2)(t_1), (\alpha_1, \alpha_2)\}, \forall\, t_1 \in G$.

Remark 1: *Clearly,* $(u_m^1, u_r^2)^{(1,0)} = (u_m^1, u_r^2)$ *and* $(u_m^1, u_r^2)^{(0,0)} = (0,0)$

Result 1: Let (u_m^1, u_r^2) and (u_m^3, u_r^4) be two ISbs of X. Then, $((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha = (u_m^1, u_r^2)^\alpha \cap (u_m^3, u_r^4)^\alpha$.

Proof: $((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha = \min\{((u_m^1, u_r^2) \cap (u_m^3, u_r^4)), \alpha\}$

$= \min\{\min((u_m^1, u_r^2), (u_m^3, u_r^4)), \alpha\}$

$= \min\{\min((u_m^1, u_r^2), \alpha), \max((u_m^3, u_r^4), \alpha)\}$

$= \min\{(u_m^1, u_r^2)^\alpha, (u_m^3, u_r^4)^\alpha\}$

$= (u_m^1, u_r^2)^\alpha \cap (u_m^3, u_r^4)^\alpha$

Hence, $((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha = (u_m^1, u_r^2)^\alpha \cap (u_m^3, u_r^4)^\alpha$.

Definition 6.2: *Let* (u_m^1, u_r^2) *be an ISb of* G. *Let* $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [1,0]$. *Then* (u_m^1, u_r^2) *is called an* α *−ISG of* G *if* $(u_m^1, u_r^2)^\alpha$ *is an ISG of* G *i.e., if the following conditions hold:*

i) $(u_m^1, u_r^2)^\alpha(t_1 t_2) \geq \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\}$

ii) $(u_m^1, u_r^2)^\alpha({t_1}^{-1}) = (u_m^1, u_r^2)^\alpha(t_1); \forall\, t_1, t_2 \in G$.

Proposition 6.1: If $(u_m^1, u_r^2) : G \to [0,1]$ is an α −ISG of G, then

i) $(u_m^1, u_r^2)^\alpha(t_1) \leq (u_m^1, u_r^2)^\alpha(e); \forall\, t_1 \in G$, where e is the identity of G

ii) $(u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1}) = (u_m^1, u_r^2)^\alpha(e) \Rightarrow (u_m^1, u_r^2)^\alpha(t_1) = (u_m^1, u_r^2)^\alpha(t_2); \forall\, t_1, t_2 \in G$

Proof: (i) $(u_m^1, u_r^2)^\alpha(e) = (u_m^1, u_r^2)^\alpha(t_1 {t_1}^{-1})$

$\geq \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha({t_1}^{-1})\}$

$= \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_1)\}$

$= (u_m^1, u_r^2)^\alpha(t_1)$

Therefore, $(u_m^1, u_r^2)^\alpha(t_1) \leq (u_m^1, u_r^2)^\alpha(e); \forall\, t_1 \in G$

i) $(u_m^1, u_r^2)^\alpha(t_1) = (u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1} t_2)$

$\geq \min\{(u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1}), (u_m^1, u_r^2)^\alpha(t_2)\}$

$= \min\{(u_m^1, u_r^2)^\alpha(e), (u_m^1, u_r^2)^\alpha(t_2)\}$; [since, $(u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1}) = (u_m^1, u_r^2)^\alpha(e)$]

$= (u_m^1, u_r^2)^\alpha(t_2)$

Therefore, $(u_m^1, u_r^2)^\alpha(t_1) \geq (u_m^1, u_r^2)^\alpha(t_2); \forall\, t_1, t_2 \in G$ (*)

And

$(u_m^1, u_r^2)^\alpha(t_2) = (u_m^1, u_r^2)^\alpha(t_2 {t_1}^{-1} t_1)$

$\geq \min\{(u_m^1, u_r^2)^\alpha(t_2 {t_1}^{-1}), (u_m^1, u_r^2)^\alpha(t_1)\}$

$\geq \min\{(u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1}), (u_m^1, u_r^2)^\alpha(t_1)\}$

$= \min\{(u_m^1, u_r^2)^\alpha(e), (u_m^1, u_r^2)^\alpha(t_1)\}$; [since, $(u_m^1, u_r^2)^\alpha(t_1 {t_2}^{-1}) = (u_m^1, u_r^2)^\alpha(e)$]

$= (u_m^1, u_r^2)^\alpha(t_1)$

Therefore, $(u_m^1, u_r^2)^\alpha(t_2) \geq (u_m^1, u_r^2)^\alpha(t_1); \forall\, t_1, t_2 \in G$. (**)
Thus, from (*) and (**), we have, $(u_m^1, u_r^2)^\alpha(t_1) = (u_m^1, u_r^2)^\alpha(t_2); \forall\, t_1, t_2 \in G$.

Proposition 6.2: If (u_m^1, u_r^2) be an IS_G of G, then (u_m^1, u_r^2) is also an α −ISG of G.
Proof: Let $t_1, t_2 \in G$ be any elements of the group G.

$(u_m^1, u_r^2)^\alpha(t_1 t_2) = \min\{(u_m^1, u_r^2)(t_1 t_2), \alpha\}$

$\geq \min\{\min\{(u_m^1, u_r^2)(t_1), (u_m^1, u_r^2)(t_2)\}, \alpha\}$

$= \min\{\min\{(u_m^1, u_r^2)(t_1), \alpha\}, \min\{(u_m^1, u_r^2)(t_2), \alpha\}\}$

$= \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\}$

Therefore, $(u_m^1, u_r^2)^\alpha(t_1 t_2) \geq \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\}$
Also, $(u_m^1, u_r^2)^\alpha({t_1}^{-1}) = \min\{(u_m^1, u_r^2)({t_1}^{-1}), \alpha\}$

$= \min\{(u_m^1, u_r^2)(t_1), \alpha\}$

$= (u_m^1, u_r^2)^\alpha(t_1)$

Hence (u_m^1, u_r^2) is an $\alpha -$ ISG of G.

Remark 2: *The Converse of the Above Proposition 6.2 Need Not Be True.*

Example 2: Let $G = \{e, a, b, ab\}$ be a multiplicative group.
Let $(u_m^1, u_r^2) : G \to [0,1]$ be defined as

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (0.9,0.1); t_1 = e \\ (0.7,0.2);\ t_1 = a, b \\ (0.6,0.2);\ t_1 = ab \end{cases} \quad (3)$$

Then (u_m^1, u_r^2) is not an ISG as $(u_m^1, u_r^2)(a \cdot b) = (0.6,0.2)$
And

$\min\{(u_m^1, u_r^2)(a), (u_m^1, u_r^2)(b)\} = \min\{(0.7,0.2), (0.7,0.2)\}$

$= (0.7,0.2)$

Therefore, $(u_m^1, u_r^2)(ab) \ngeq \min\{(u_m^1, u_r^2)(a), (u_m^1, u_r^2)(b)\}$
Let us take $\alpha = (\alpha_1, \alpha_2) = (0.4,0.3)$
Then $(u_m^1, u_r^2)(t_1) > \alpha;\ \forall\, t_1 \in G$
Thus, $(u_m^1, u_r^2)^\alpha(t_1) = \min\{(u_m^1, u_r^2)(t_1), \alpha\}$

$= \alpha;\ \forall\, t_1 \in G$

Therefore, $(u_m^1, u_r^2)^\alpha(t_1\, t_2) \geq \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\};\ \forall\, t_1, t_2 \in G$
Further, clearly, $(u_m^1, u_r^2)^\alpha({t_1}^{-1}) = (u_m^1, u_r^2)^\alpha(t_1)$ as $a - 1 = a, b - 1 = b, (ab) - 1 = ab$
Hence (u_m^1, u_r^2) is an $\alpha -$ ISG of G.
Proposition 6.3: Let (u_m^1, u_r^2) be an ISb of G such that $(u_m^1, u_r^2)({t_1}^{-1}) = (u_m^1, u_r^2)(t_1)$ hold for all $t_1 \in G$. Let $\alpha \leq p$, where $p = \inf\{(u_m^1, u_r^2)(t_1) : t_1 \in G\}$. Then (u_m^1, u_r^2) is an α −ISG of G.
Proof: Since $\alpha \leq p \Rightarrow p \geq \alpha$
i.e., $\inf\{(u_m^1, u_r^2)(t_1) : t_1 \in G\} \geq \alpha$
$\Rightarrow (u_m^1, u_r^2)(t_1) \geq \alpha$
And so $\min\{(u_m^1, u_r^2)(t_1), \alpha\} = \alpha;\ \forall\, t_1 \in G$
i.e., $(u_m^1, u_r^2)^\alpha(t_1) = \alpha;\ \forall\, t_1 \in G$
Thus, $(u_m^1, u_r^2)^\alpha(t_1\, t_2) \geq \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\};\ \forall\ t_1, t_2 \in G$
Further, $(u_m^1, u_r^2)({t_1}^{-1}) = (u_m^1, u_r^2)(t_1);\ \forall\ t_1 \in G$
implies that $(u_m^1, u_r^2)^\alpha({t_1}^{-1}) = (u_m^1, u_r^2)^\alpha(t_1)$
Hence (u_m^1, u_r^2) is an α −ISG of G.
Proposition 6.4: Intersection of two α −ISGs of G is also an $\alpha -$ ISG of G.
Proof: Let (u_m^1, u_r^2) and (u_m^3, u_r^4) be two $\alpha -$ ISGs of G.
Let $t_1, t_2 \in G$ be any element, then

$((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1\, t_2) = ((u_m^1, u_r^2)^\alpha \cap (u_m^3, u_r^4)^\alpha)(t_1\, t_2)$ by Result 1

$= \min\{(u_m^1, u_r^2)^\alpha(t_1\, t_2), (u_m^3, u_r^4)^\alpha(t_1\, t_2)\}$

$\geq \min\{\min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^1, u_r^2)^\alpha(t_2)\}, \min\{(u_m^3, u_r^4)^\alpha(t_1), (u_m^3, u_r^4)^\alpha(t_2)\}\}$

$= \min\{\min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^3, u_r^4)^\alpha(t_1)\}, \min\{(u_m^1, u_r^2)^\alpha(t_2), (u_m^3, u_r^4)^\alpha(t_2)\}\}$

$= \min\{((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1), ((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_2)\}$

Thus,

$((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1 t_2) \geq \min\{((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1), ((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_2)\}$

Also, $((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1^{-1}) = ((u_m^1, u_r^2)^\alpha \cap (u_m^3, u_r^4)^\alpha)(t_1^{-1})$ by Result 1

$= \min\{(u_m^1, u_r^2)^\alpha(t_1^{-1}), (u_m^3, u_r^4)^\alpha(t_1^{-1})\}$

$= \min\{(u_m^1, u_r^2)^\alpha(t_1), (u_m^3, u_r^4)^\alpha(t_1)\}$

$= ((u_m^1, u_r^2) \cap (u_m^3, u_r^4))^\alpha(t_1)$

Hence, $(u_m^1, u_r^2) \cap (u_m^3, u_r^4)$ is an $\alpha -$ ISG of G.
Corollary 1: Intersection of a family of $\alpha -$ISGs of G is again an $\alpha -$ISG of G.

Remark 3: *Union of two $\alpha -$ ISGs of G need not be an $\alpha -$ISG of G.*

Example 3: Let $G = Z$, the group of integers under ordinary addition of integers.
Define the two ISbs (u_m^1, u_r^2) and (u_m^3, u_r^4) by

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (0.9,0); \text{if } t_1 = 3Z \\ (0.7,0.1); \text{otherwise} \end{cases} \tag{4}$$

And

$$(u_m^3, u_r^4)(t_1) = \begin{cases} (0.8,0); \text{if } t_1 = 2Z \\ (0.6,0.2); \ t_1 = \text{otherwise} \end{cases} \tag{5}$$

Then clearly (u_m^1, u_r^2) and (u_m^3, u_r^4) are ISGs of Z.
Also, for $\alpha = (\alpha_1, \alpha_2) = (0.9,0)$ both (u_m^1, u_r^2) and (u_m^3, u_r^4) are $\alpha -$ ISGs of Z
But

$$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1) = \begin{cases} (0.9,0); \text{if } t_1 = 3Z \\ (0.8,0); \text{ if } t_1 = 2Z - 3Z\text{otherwise} \\ (0.7,0.1); \text{if } t_1 \notin 2Z \ t_1 \notin 3Z \end{cases} \tag{6}$$

Is not $(0.9,0) -$ ISG of Z as for $t_1 = 9, t_2 = 2, t_1 + t_2 = 11$

$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1 + t_2) = ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(11) = (0.7,0.1)$

And

$\min\{(0.9,0), (0.8,0)\} = (0.8,0) > t_1 + t_2(t_1 + t_2)$

Therefore, $((u_m^1, u_r^2) \cup (u_m^3, u_r^4))$ is not an ISG and hence not a $(0.9,0) -$ISG of Z.
Example 4: Let $G = Z$, be the group of integers under ordinary addition of integers.
We define the following two ISGs by

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (0.9,0.1); \text{if } t_1 = 2Z \\ (0.7,0.3); \text{otherwise} \end{cases} \tag{7}$$

And

$$(u_m^3, u_r^4)(t_1) = \begin{cases} (1,0); \text{if } t_1 = 2Z \\ (0.3,0.3); \ t_1 = \text{otherwise} \end{cases} \tag{8}$$

Then,

$$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1) = \begin{cases} (1,0); \text{if } t_1 = 2Z \\ (0.7,0.3); \ t_1 = \text{otherwise} \end{cases} \tag{9}$$

Is an ISG of Z.
As for $t_1 = 2, t_2 = 3, t_1 + t_2 = 5$

$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1 + t_2) = ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(5) = (0.7,0.3)$

$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1) = (1,0), ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_2) = (0.7,0.3)$

Therefore,

$$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1 + t_2) = \min\{((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1), ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_2)\}$$

Similarly, for $t_1 = 2,\ t_2 = -3,\ t_1 - t_2 = -1$
for $t_1 = 3,\ t_2 = 5,\ t_1 + t_2 = 8$
for $t_1 = 3,\ t_2 = 5,\ t_1 - t_2 = -2$

$$((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1 + t_2) \geq \min\{((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1), ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_2)\}$$

Also, $((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(-t_1) = ((u_m^1, u_r^2) \cup (u_m^3, u_r^4))(t_1)\ ;\forall\, t_1 \in Z$
For $\alpha = (0.9,0)$, it is easy to verify that $(u_m^1, u_r^2), (u_m^3, u_r^4)$ and $((u_m^1, u_r^2) \cup (u_m^3, u_r^4))$ are $\alpha -$ ISGs of Z.

Definition 6.3: *Let (u_m^1, u_r^2) be an $\alpha -$ ISG of G, where $\alpha \in [0,1]$. For any $t_1 \in G$, define an IS $(u_m^1, u_r^2)^\alpha$ of G, called an $\alpha -$ imprecise right coset of (u_m^1, u_r^2) in G as follows-*

$$(u_m^1, u_r^2)^\alpha a(t_1) = \min\{(u_m^1, u_r^2)(t_1 a^{-1}), \alpha\};\ \forall\, t_1, a \in G.$$

Similarly, we define the $\alpha -$ imprecise left coset $a(u_m^1, u_r^2)^\alpha$ of (u_m^1, u_r^2) in G as follows $a(u_m^1, u_r^2)^\alpha(t_1) = \min\{(u_m^1, u_r^2)(a^{-1} t_1), \alpha\}$; $\forall\, t_1, a \in G$.

Definition 6.4: *Let (u_m^1, u_r^2) be an $\alpha -$ISG of G, where $\alpha \in [0,1]$. Then (u_m^1, u_r^2) is called an $\alpha -$ normal ISG of G if and only if $a(u_m^1, u_r^2)^\alpha = (u_m^1, u_r^2)^\alpha a;\ \forall\, a \in G$.*

Note: (i) Clearly, $(1,0) -$ ISG of G is original normal ISG of G.
ii) $(u_m^1, u_r^2)^\alpha a(t_1) = (u_m^1, u_r^2)^\alpha(t_1 a^{-1})$ and $a(u_m^1, u_r^2)^\alpha(t_1) = (u_m^1, u_r^2)^\alpha(a^{-1} t_1);\ \forall\, t_1 \in G$.

Remark 4: *If (u_m^1, u_r^2) is a normal ISG of G, then (u_m^1, u_r^2) is also an $\alpha -$ normal ISG of G.*

Proof: Let (u_m^1, u_r^2) be a normal ISG of G Then for any $t_1 \in G$, we have $a(u_m^1, u_r^2) = (u_m^1, u_r^2)a$
Therefore, for any $t_1 \in G$, we have $(a(u_m^1, u_r^2))(t_1) = ((u_m^1, u_r^2)a)(t_1)$
i.e., $(u_m^1, u_r^2)(a^{-1} t_1) = (u_m^1, u_r^2)(t_1 a^{-1})$
So, $\min\{(u_m^1, u_r^2)(a^{-1} t_1), \alpha\} = \min\{(u_m^1, u_r^2)(t_1 a^{-1}), \alpha\}$
i.e., $a(u_m^1, u_r^2)^\alpha(t_1) = (u_m^1, u_r^2)^\alpha a(t_1)$
So, we have, $a(u_m^1, u_r^2)^\alpha = (u_m^1, u_r^2)^\alpha a;\ \forall\, a \in G$.
Hence (u_m^1, u_r^2) is an $\alpha -$ normal ISG of G.
The converse of the above result need not be true.
Example 5: Let us consider a dihedral group

$$D_3 = \begin{cases} a, a^2 = e = b^3 \\ \quad ab, ab^2 \\ \quad b, b^2 \end{cases} \tag{10}$$

Such that $b^3 = a^2 = e$ and $ab = b^2 a$ and define it in imprecise form by

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (0.9,0.1);\ \text{if } t_1 = a, a^2 = e = b^3 \\ (0.7,0.3);\ \text{if } t_1 = b, ab, ab^2, b^2 \end{cases} \tag{11}$$

Then (u_m^1, u_r^2) is not a normal ISG as

$$(u_m^1, u_r^2)\ (b^2(ab)) = (u_m^1, u_r^2)(b^3 a)$$

$$= (u_m^1, u_r^2)(ea)$$

$$= (u_m^1, u_r^2)(a)$$

$$= (0.9,0.01)$$

$$(u_m^1, u_r^2)(ab(b^2)) = (u_m^1, u_r^2)(b^2 ab^2)$$

$$= (u_m^1, u_r^2)(b^3 ab)$$

$$= (u_m^1, u_r^2)(eab)$$

$$= (u_m^1, u_r^2)(ab)$$

$$= (0.7,0.03)$$

Therefore, $(u_m^1, u_r^2)\ (b^2(ab)) \neq (0.7,0.03) = (u_m^1, u_r^2)(ab(b^2))$

Now we take, $\alpha = (\alpha_1, \alpha_2) = (0.6, 0.04)$
Then $a(u_m^1, u_r^2)^\alpha(t_1) = \min\{(u_m^1, u_r^2)(a^{-1}t_1), \alpha\}$

$= (0.6, 0.04)$

$= \min\{(u_m^1, u_r^2)(t_1 a^{-1}), \alpha\}$

$= (u_m^1, u_r^2)^\alpha a(t_1);\ \forall a, t_1 \in G$

Hence, (u_m^1, u_r^2) is an $\alpha -$ normal ISG of G.

7. α −anti-ISbs

Definition 7.1: *Let (u_m^1, u_r^2) be an ISb of G. Let $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Then the ISb $(u_m^1, u_r^2)_\alpha^c$ of G is called an $\alpha -$anti-ISb of G (with respect to IS (u_m^1, u_r^2)) and is defined as $(u_m^1, u_r^2)_{(\alpha_1,\alpha_2)}^c(t_1) = max\{(u_m^1, u_r^2)^c(t_1), (\alpha_1, \alpha_2)^c\}$.*

Clearly, (i) $(u_m^1, u_r^2)_{(1,0)}^c(t_1) = (u_m^1, u_r^2)^c(t_1)$
As, $(u_m^1, u_r^2)_{(1,0)}^c(t_1) = \max\{(u_m^1, u_r^2)^c(t_1), (1,0)^c\}$

$= \max\{(1, u_m^1)(t_1), (1,1)\}$

$= (1, u_m^1)(t_1)$

$= (u_m^1, u_r^2)^c(t_1)$

i) $(u_m^1, u_r^2)_{(0,0)}^c(t_1) = (1,0)$

As, $(u_m^1, u_r^2)_{(0,0)}^c(t_1) = \max\{(u_m^1, u_r^2)^c(t_1), (0,0)^c\}$

$= \max\{(1, u_m^1)(t_1), (1,0)\}$

$= (1,0)$

Result 2: $((u_m^1, u_r^2)^c \cup (u_m^3, u_r^4)^c)_\alpha = (u_m^1, u_r^2)_\alpha^c \cup (u_m^3, u_r^4)_\alpha^c$
Proof: $((u_m^1, u_r^2)^c \cup (u_m^3, u_r^4)^c)_\alpha = \max\{((1, u_m^1) \cup (1, u_m^3))(t_1), (1, \alpha_1)\}$

$= \max\{\max((1, u_m^1)(t_1), (1, u_m^3)(t_1)), (1, \alpha_1)\}$

$= \max\{\max((1, u_m^1)(t_1), (1, \alpha_1)), \max((1, u_m^3)(t_1), (1, \alpha_1))\}$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^3, u_r^4)_\alpha^c(t_1)\}$

$= (u_m^1, u_r^2)_\alpha^c \cup (u_m^3, u_r^4)_\alpha^c$

Definition 7.2: *Let (u_m^1, u_r^2) be an ISb of G. Let $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Then $(u_m^1, u_r^2)^c$ is called an $\alpha -$anti-ISG of G if $(u_m^1, u_r^2)_\alpha^c$ is an anti-ISG of G i.e., if the following conditions hold:*

i) $(u_m^1, u_r^2)_\alpha^c(t_1 t_2) \le \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^1, u_r^2)_\alpha^c(t_2)\}$

ii) $(u_m^1, u_r^2)_\alpha^c({t_1}^{-1}) = (u_m^1, u_r^2)_\alpha^c(t_1);\ \forall\ t_1, t_2 \in G$

Example 6: Let us consider a multiplicative group $G = \{1, -1, i, -i\}$. Then we define an ISG over G by

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (0.9,0);\ \text{if } t_1 = 1 \\ (0.4,0.4);\ \text{if } t_1 = -1 \\ (0.3,0.3);\ \text{if } t_1 = i, -i \end{cases} \quad (12)$$

Now,

$$(u_m^1, u_r^2)^c(t_1) = \begin{cases} (1,0.9);\ \text{if } t_1 = 1 \\ (1,0.4);\ \text{if } t_1 = -1 \\ (1,0.3);\ \text{if } t_1 = i, -i \end{cases} \quad (13)$$

Is clearly an anti-ISG of G.
Also, it can be easily verified that $(u_m^1, u_r^2)^c$ is a $(1,0.05) -$anti-ISG of G.
Proposition 7.1: If $(u_m^1, u_r^2)^c : G \to [0,1]$ is an $\alpha -$anti-ISG of G, then
i) $(u_m^1, u_r^2)_\alpha^c(t_1) \ge (u_m^1, u_r^2)_\alpha^c(e);\ \forall\ t_1 \in G$, where e is the identity element of G

ii) $(u_m^1, u_r^2)_\alpha^c(t_1 t_2^{-1}) = (u_m^1, u_r^2)_\alpha^c(e)$

$\Rightarrow (u_m^1, u_r^2)_\alpha^c(t_1) = (u_m^1, u_r^2)_\alpha^c(t_2);\ \forall\, t_1, t_2 \in G.$

Proof: (i) $(u_m^1, u_r^2)_\alpha^c(e) = (u_m^1, u_r^2)_\alpha^c(t_1 t_1^{-1})$

$\leq \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^1, u_r^2)_\alpha^c(t_1^{-1})\}$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^1, u_r^2)_\alpha^c(t_1)\}$

$= (u_m^1, u_r^2)_\alpha^c(t_1)$

iii) $(u_m^1, u_r^2)_\alpha^c(t_1) = (u_m^1, u_r^2)_\alpha^c(t_1 t_2^{-1} t_2)$

$\leq \max\{(u_m^1, u_r^2)_\alpha^c(t_1 t_2^{-1}), (u_m^1, u_r^2)_\alpha^c(t_2)\}$

$= \max\{(u_m^1, u_r^2)_\alpha^c(e), (u_m^1, u_r^2)_\alpha^c(t_2)\}$

$= (u_m^1, u_r^2)_\alpha^c(t_2)$

Thus $(u_m^1, u_r^2)_\alpha^c(t_1) = (u_m^1, u_r^2)_\alpha^c(t_2);\ \forall\, t_1, t_2 \in G.$
Proposition 7.2: If $(u_m^1, u_r^2)^c$ be an anti- IS$_G$ of the group G, then $(u_m^1, u_r^2)^c$ is also an α −anti-IS$_G$ of G.
Proof: Let $t_1, t_2 \in G$. be any elements of the group G.

$(u_m^1, u_r^2)_\alpha^c(t_1 t_2) = \max\{(u_m^1, u_r^2)^c(t_1 t_2), (1, \alpha_1)\}$

$\leq \max\{\max\{(u_m^1, u_r^2)^c(t_1), (u_m^1, u_r^2)^c(t_2)\}, (1, \alpha_1)\}$

$= \max\{\max\{(u_m^1, u_r^2)^c(t_1), (1, \alpha_1)\}, \max\{(u_m^1, u_r^2)^c(t_2), (1, \alpha_1)\}\}$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^1, u_r^2)_\alpha^c(t_2)\}$

Therefore, $(u_m^1, u_r^2)_\alpha^c(t_1 t_2) \leq \max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^1, u_r^2)_\alpha^c(t_2)\}$
Also, $(u_m^1, u_r^2)_\alpha^c(t_1^{-1}) = \max\{(u_m^1, u_r^2)^c(t_1^{-1}), (1, \alpha_1)\}$

$= \max\{(u_m^1, u_r^2)^c(t_1), (1, \alpha_1)\}$

$= (u_m^1, u_r^2)_\alpha^c(t_1)$

Hence, $(u_m^1, u_r^2)^c$ is α −anti-IS$_G$ of G.

Remark 5: *The converse of above proposition 7.2 need not be true.*

Example 7: Let us consider a multiplicative group $V_4 = \{e, a, b, c\}$
Let $(u_m^1, u_r^2) : V_4 \to [0,1]$ be defined as

$$(u_m^1, u_r^2)(t_1) = \begin{cases} (1,0.6); \text{if } t_1 = e \\ (1,0.5); \text{if } t_1 = a, b \\ (1,0.4); \text{if } t_1 = c \end{cases} \tag{14}$$

Then clearly, (u_m^1, u_r^2) is not an anti-IS$_G$ as for $t_1 = a, t_2 = b, t_1 t_2 = c$
We have, $(u_m^1, u_r^2)(t_1 t_2) \nleq \max\{(u_m^1, u_r^2)(t_1), (u_m^1, u_r^2)(t_2)\}$
Now, let us take $\alpha = (1,0.3)$. Then $(u_m^1, u_r^2)(t_1) < (1,0.3); \forall\, t_1 \in V_4$
so that $(u_m^1, u_r^2)_\alpha(t_1) = \max\{(u_m^1, u_r^2)(t_1), (1,0.3)\} = (1,0.3); \forall\, t_1 \in V_4$
Therefore, $(u_m^1, u_r^2)_\alpha(t_1 t_2) \leq \max\{(u_m^1, u_r^2)_\alpha(t_1), (u_m^1, u_r^2)_\alpha(t_2)\}; \forall\, t_1, t_2 \in V_4$.
Further, $e^{-1} = e, a^{-1} = a, b^{-1} = b, c^{-1} = c$ so, $(u_m^1, u_r^2)_\alpha(t_1^{-1}) = (u_m^1, u_r^2)_\alpha(t_1)$ hold.
Hence, (u_m^1, u_r^2) is an α −IS$_G$ of G.

Remark 6: *Suppose for* $u_m^1 \neq 0,1; u_r^2 \neq 0,1$ *then* $(u_m^1, u_r^2)^c$ *is an anti-IS$_G$ then its complement* (u_m^1, u_r^2) *is again an IS$_G$.*

Proposition 7.3: Let (u_m^1, u_r^2) be an IS$_b$ of G such that
$(u_m^1, u_r^2)^c(t_1^{-1}) = (u_m^1, u_r^2)^c(t_1); \forall\, t_1 \in G$. Let $\alpha^c \leq (q_1, q_2)^c$, where
$(q_1, q_2)^c = \sup\{(u_m^1, u_r^2)^c(t_1) : t_1 \in G\}$. Then $(u_m^1, u_r^2)^c$ is an α −anti-IS$_G$ of G.
Proof: Since $\alpha^c = (\alpha_1, \alpha_2)^c \leq (q_1, q_2)^c$

$\Rightarrow (1, \alpha_1) \leq (1, q_1)$

$\Rightarrow (1, \alpha_1) \leq \sup\{(u_m^1, u_r^2)^c(t_1) : t_1 \in G\}$

$\Rightarrow (1,\alpha_1) \le \sup\{(1,u_m^1)(t_1) : t_1 \in G\}$

$(1,\alpha_1) \le (1,u_m^1)(t_1)$ i.e.; $\alpha_1 \le u_m^1$ and
So, $\max\{(1,u_m^1)(t_1),(1,\alpha_1)\}(1,\alpha_1)$; $\forall t_1 \in G$ i.e.; $(u_m^1,u_r^2)_\alpha^c(t_1) = (1,\alpha_1)$; $\forall t_1 \in G$
Thus, $(u_m^1,u_r^2)_\alpha^c(t_1t_2) \le \max\{(u_m^1,u_r^2)_\alpha^c(t_1),(u_m^1,u_r^2)_\alpha^c(t_2)\}$; $\forall t_1,t_2 \in G$.
Further, $(u_m^1,u_r^2)^c(t_1^{-1}) = (u_m^1,u_r^2)^c(t_1)$; $\forall t_1 \in G$ hold.
So, $(u_m^1,u_r^2)_\alpha^c(t_1^{-1}) = (u_m^1,u_r^2)_\alpha^c(t_1)$
Hence, $(u_m^1,u_r^2)^c$ is α −anti-ISG of G.
Example 8: $G = \{e,a,b,ab\}$

$$(u_m^1,u_r^2)^c(t_1) = \begin{cases}(1,0.6); \text{if } t_1 = e,ab\\ (1,0.3); \text{if } t_1 = a,b\end{cases} \tag{15}$$

Such that $(u_m^1,u_r^2)^c(t_1^{-1}) = (u_m^1,u_r^2)^c(t_1)$; $\forall\ t_1 \in G$
Let $\alpha^c = (1,0.65) < (1,0.3) = (q_1,q_2)^c$. Then it can be easily verified that $(u_m^1,u_r^2)^c$ is an α − anti-ISG of G.
Proposition 7.4: Union of two α − anti- ISGs of G is an α − anti-ISG of G.
Proof: Let $(u_m^1,u_r^2)^c$ and $(u_m^3,u_r^4)^c$ be two α − anti-ISGs of G.
Let $t_1,t_2 \in G$ be any element, then

$$\left((u_m^1,u_r^2)\cup(u_m^3,u_r^4)\right)_\alpha^c(t_1t_2) = ((u_m^1,u_r^2)_\alpha^c \cup (u_m^3,u_r^4)_\alpha^c)(t_1t_2) \text{ by Result 1}$$

$$= \max\{(u_m^1,u_r^2)_\alpha^c(t_1t_2),(u_m^3,u_r^4)_\alpha^c(t_1t_2)\}$$

$$\le \max\{\max\{(u_m^1,u_r^2)_\alpha^c(t_1),(u_m^1,u_r^2)_\alpha^c(t_2)\},\max\{(u_m^3,u_r^4)_\alpha^c(t_1),(u_m^3,u_r^4)_\alpha^c(t_2)\}\}$$

$$= \max\{\max\{(u_m^1,u_r^2)_\alpha^c(t_1),(u_m^3,u_r^4)_\alpha^c(t_1)\},\max\{(u_m^1,u_r^2)_\alpha^c(t_2),(u_m^3,u_r^4)_\alpha^c(t_2)\}\}$$

$$= \max\{((u_m^1,u_r^2)_\alpha^c \cup (u_m^3,u_r^4)_\alpha^c)(t_1),((u_m^1,u_r^2)_\alpha^c \cup (u_m^3,u_r^4)_\alpha^c)(t_2)\}$$

$$= \max\{((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_1),((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_2)\}$$

Thus,

$$((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_1t_2) \le \max\{((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_1),((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_2)\}$$

Also, $((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_1^{-1}) = ((u_m^1,u_r^2)_\alpha^c \cup (u_m^3,u_r^4)_\alpha^c)(t_1^{-1})$ by Result 1

$$= \max\{(u_m^1,u_r^2)_\alpha^c(t_1^{-1}),(u_m^3,u_r^4)_\alpha^c(t_1^{-1})\}$$

$$= \max\{(u_m^1,u_r^2)_\alpha^c(t_1),(u_m^3,u_r^4)_\alpha^c(t_1)\}$$

$$= ((u_m^1,u_r^2)\cup(u_m^3,u_r^4))_\alpha^c(t_1)$$

Hence, $(u_m^1,u_r^2)^c \cup (u_m^3,u_r^4)^c$ is an α −anti-ISG of G.
Corollary 2: Union of an arbitrary collection of α −anti-ISG of G is again an α −anti-ISG of G.

Remark 7: *Intersection of two α −anti-ISGs of G need not be an α −anti-ISG of G.*

Example 9: Let us consider a multiplicative group $G = \{e,a,b,ab\}$
Define the two ISs $(u_m^1,u_r^2)^c$ and $(u_m^3,u_r^4)^c$ by

$$(u_m^1,u_r^2)^c(t_1) = \begin{cases}(1,0.8); \text{if } t_1 = e,ab\\ (1,0.5); \text{if } t_1 = a,b\end{cases} \tag{16}$$

And

$$(u_m^3,u_r^4)^c(t_1) = \begin{cases}(1,0.7); \text{if } t_1 = e,ab\\ (1,0.6); \text{if } t_1 = a\\ (1,0.3); \text{if } t_1 = b\end{cases} \tag{17}$$

Then clearly both $(u_m^1,u_r^2)^c$ and $(u_m^3,u_r^4)^c$ are $\alpha = (1,0.2)$ − anti-ISGs of G.
Now $((u_m^1,u_r^2)^c \cap (u_m^3,u_r^4)^c)(t_1) = \min\{(u_m^1,u_r^2)^c,(u_m^3,u_r^4)^c\}$
Therefore,

$$((u_m^1,u_r^2)^c \cap (u_m^3,u_r^4)^c)(t_1) = \begin{cases}(1,0.8); \text{if } t_1 = e,ab\\ (1,0.6); \text{if } t_1 = a\\ (1,0.5); \text{if } t_1 = b\end{cases} \tag{18}$$

Is not $(1,0.2)-$ anti-IS$_G$ of G as for $t_1 = ab, t_2 = a, t_1t_2 = b$

$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1t_2) = (1,0.5)$

And

$\max\{(1,0.8),(1,0.6)\} = (1,0.6)$

Therefore, $(1,0.5) \nless (1,0.6)$

i.e.; $((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(ab.a) \nleq \max\{((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(ab), ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(a)\}$

Example 10: Let G be the multiplicative group. We define the following two anti-IS$_{GS}$ by

$$(u_m^1, u_r^2)^c(t_1) = \begin{cases}(1,0.8); \text{if } t_1 = e, ab \\ (1,0.5); \text{if } t_1 = a, b\end{cases} \tag{19}$$

And

$$(u_m^3, u_r^4)^c(t_1) = \begin{cases}(1,0.7); \text{if } t_1 = e, ab \\ (1,0.6); \text{if } t_1 = a, b\end{cases} \tag{20}$$

Then,

$$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1) = \begin{cases}(1,0.8); \text{if } t_1 = e, ab \\ (1,0.6); \text{if } t_1 = a, b\end{cases} \tag{21}$$

Is an anti-IS$_G$ of G.
As for $t_1 = e$, $t_2 = a$, $t_1 t_2 = a$

$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(a) = (1,0.6)$

$= \max\{(1,0.8),(1,0.6)\}$

$= \max\{((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(e), ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(a)\}$

For $t_1 = ab$, $t_2 = b$, $t_1 t_2 = a$

$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(a) = (1,0.6)$

$= \max\{(1,0.8),(1,0.6)\}$

$= \max\{((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(ab), ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(b)\}$

For $t_1 = a$, $t_2 = b$, $t_1 t_2 = ab$

$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(ab) = (1,0.8)$

$< (1,0.6)$

$= \max\{(1,0.6),(1,0.6)\}$

$= \max\{((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(a), ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(b)\}$

Therefore,

$((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1 t_2) \le \max\{((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1), ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_2)\}; \forall t_1, t_2 \in G.$

Also, $((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1^{-1}) = ((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)(t_1)\,; \forall t_1 \in G$
For $\alpha = (1,0.2)$, it is easy to verify that $(u_m^1, u_r^2)^c$, $(u_m^3, u_r^4)^c$ and $((u_m^1, u_r^2)^c \cap (u_m^3, u_r^4)^c)$ are $\alpha-$anti-IS$_{GS}$ of G.

Definition 7.3: *Let $(u_m^1, u_r^2)^c$ be an $\alpha-$ anti-IS$_G$ of G, where $\alpha \in [0,1]$. For any $t_1 \in G$, define an IS $(u_m^1, u_r^2)^c_\alpha a$ of G, called an $\alpha-$anti imprecise right coset of $(u_m^1, u_r^2)^c$ of G as follows*

$(u_m^1, u_r^2)^c_\alpha a(t_1) = \max\{(u_m^1, u_r^2)^c(t_1 a^{-1}), \alpha^c\}; \forall\ t_1, a \in G.$

Similarly, we define the $\alpha-$anti imprecise left coset $a(u_m^1, u_r^2)^c_\alpha$ of $(u_m^1, u_r^2)^c$ in G as follows:

$a(u_m^1, u_r^2)^c_\alpha(t_1) = \max\{(u_m^1, u_r^2)^c(a^{-1} t_1), \alpha^c\}; \forall\ t_1, a \in G.$

Definition 7.4: *Let $(u_m^1, u_r^2)^c$ be an $\alpha -$ anti-ISG of G, where $\alpha \in [0,1]$. Then $(u_m^1, u_r^2)^c$ is called an $\alpha -$anti normal ISG of G if and only if $a(u_m^1, u_r^2)_\alpha^c = (u_m^1, u_r^2)_\alpha^c a; \forall a \in G$.*

Example 11: $G = \{1, \omega, \omega^2\}$

$$(u_m^1, u_r^2)^c(t_1) = \begin{cases} (1,0.6); \text{for } t_1 = 1 \\ (1,0.4); \text{for } t_1 = \omega, \omega^2 \end{cases} \tag{22}$$

For $t_1 = 1$, $t_2 = \omega$, $t_1 t_2 = \omega$

$(u_m^1, u_r^2)^c(\omega) = (1,0.4)$

$= \max\{(1,0.6), (1,0.4)\}$

$= \max\{(u_m^1, u_r^2)^c(1), (u_m^1, u_r^2)^c(\omega)\}$

For $t_1 = \omega$, $t_2 = \omega^2$, $t_1 t_2 = \omega^3 = 1$

$(u_m^1, u_r^2)^c(1) = (1,0.6)$

$< (1,0.4)$

$= \max\{(1,0.4), (1,0.4)\}$

$= \max\{(u_m^1, u_r^2)^c(\omega), (u_m^1, u_r^2)^c(\omega^2)\}$

Also, $(u_m^1, u_r^2)^c(t_1^{-1}) = (u_m^1, u_r^2)^c(t_1); \forall t_1 \in G$.
Therefore, $(u_m^1, u_r^2)^c$ is an anti-ISG of G.
Now, for $\alpha = (1,0)^c = (1,1)$

$$(u_m^1, u_r^2)_\alpha^c(t_1) = \begin{cases} \max\{(1,0.6), (1,1)\}; \text{for } t_1 = 1 \\ \max\{(1,0.4), (1,1)\}; \text{for } t_1 = \omega, \omega^2 \end{cases} \tag{23}$$

$$= \begin{cases} (1,0.6); \text{for } t_1 = 1 \\ (1,0.4); \text{for } t_1 = \omega, \omega^2 \end{cases} \tag{24}$$

Therefore, $(u_m^1, u_r^2)^c$ is $(1,1) -$ anti-ISG of G.
Note:(i) Clearly, $(1,1) -$ anti-normal ISG is ordinary an anti-normal ISG of G.

iii) $(u_m^1, u_r^2)_\alpha^c a(t_1) = (u_m^1, u_r^2)_\alpha^c(t_1 a^{-1})$

And

$a(u_m^1, u_r^2)_\alpha^c(t_1) = (u_m^1, u_r^2)_\alpha^c(a^{-1} t_1); \forall t_1 \in G$.

Theorem 7.1: *If $(u_m^1, u_r^2)^c$ is an anti-normal ISG of G, then $(u_m^1, u_r^2)^c$ is also an $\alpha -$anti-normal ISG of G.*

Proof: Let $(u_m^1, u_r^2)^c$ be an anti-normal ISG of G, then for any $t_1 \in G$,
We have

$a(u_m^1, u_r^2)^c = (u_m^1, u_r^2)^c a$

Therefore, for any $t_1 \in G$, we have

$(a(u_m^1, u_r^2)^c)(t_1) = ((u_m^1, u_r^2)^c a)(t_1)$

i.e.; $(u_m^1, u_r^2)^c(a^{-1} t_1) = (u_m^1, u_r^2)^c(t_1 a^{-1})$

So, $\max\{(u_m^1, u_r^2)^c(a^{-1} t_1), \alpha^c\} = \max\{(u_m^1, u_r^2)^c(t_1 a^{-1}), \alpha^c\}$

i.e.; $a(u_m^1, u_r^2)_\alpha^c(t_1) = (u_m^1, u_r^2)_\alpha^c a(t_1); \forall a \in G$.

Hence, $(u_m^1, u_r^2)^c$ is an $\alpha -$ anti-normal ISG of G.
The converse of the above result need not be true.
Example 12: Let us consider a dihedral group

$$D_3 = \begin{cases} a, a^2 = e = b^3 \\ ab, ab^2 \\ b, b^2 \end{cases} \tag{25}$$

Such that $b^3 = a^2 = e$ and $ab = b^2a$ and define it in imprecise form by

$$(u_m^1, u_r^2)^c(t_1) = \begin{cases} (1{,}0.9);\ \text{if } t_1 = a, a^2 = e = b^3 \\ (1{,}0.4);\ \text{if } t_1 = b, ab, ab^2, b^2 \end{cases} \tag{26}$$

Then $(u_m^1, u_r^2)^c$ is not an anti-normal ISG as

$(u_m^1, u_r^2)^c(b^2(ab)) = (u_m^1, u_r^2)^c(b^3a)$

$= (u_m^1, u_r^2)^c(ea)$

$= (u_m^1, u_r^2)^c(a)$

$= (1{,}0.9)$

$(u_m^1, u_r^2)^c(ab(b^2)) = (u_m^1, u_r^2)^c(b^2ab^2)$

$= (u_m^1, u_r^2)^c(b^3ab)$

$= (u_m^1, u_r^2)^c(eab)$

$= (u_m^1, u_r^2)^c(ab)$

$= (1{,}0.4)$

Therefore, $(u_m^1, u_r^2)^c(b^2(ab)) \neq (u_m^1, u_r^2)^c(ab(b^2))$
Now we take, $\alpha^c = (\alpha_1, \alpha_2)^c = (1{,}0.3)$
Then $(a(u_m^1, u_r^2)_\alpha^c)(t_1) = (1{,}0.3)$

$= ((u_m^1, u_r^2)_\alpha^c a)(t_1); \forall\ a,\ t_1 \in G$

Hence, $(u_m^1, u_r^2)^c$ is $(1{,}0.3) -$ anti-normal ISG of G.

Definition 7.5: *Let $(u_m^1, u_r^2)_\alpha^c$ and $(u_m^3, u_r^4)_\alpha^c$ be two $\alpha -$ anti-ISGs of groups G_1 and G_2, respectively. Then, product of $(u_m^1, u_r^2)_\alpha^c$ and $(u_m^3, u_r^4)_\alpha^c$ is defined as $((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_1 t_2) = max\{(u_m^1, u_r^2)_\alpha^c(t_1), (u_m^3, u_r^4)_\alpha^c(t_2)\}$; $\forall\ t_1 \in G_1,\ t_2 \in G_2$.*

Theorem 7.2: *Let $(u_m^1, u_r^2)_\alpha^c$ and $(u_m^3, u_r^4)_\alpha^c$ be two $\alpha -$ anti-ISGs of groups G_1 and G_2, respectively. Then $(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c$ is an $\alpha -$anti-ISG of $G_1 \circ G_2$.*

Proof: Let $t_{1_1}, t_{1_2} \in G_1$ and $t_{2_1}, t_{2_2} \in G_2$ then $(t_{1_1}, t_{2_1}), (t_{1_2}, t_{2_2}) \in G_1 \circ G_2$.

$[(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c]\left((t_{1_1}, t_{2_1})(t_{1_2}, t_{2_2})^{-1}\right) = [(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c]((t_{1_1}, t_{2_1})(t_{1_2}^{-1}, t_{2_2}^{-1}))$

$= [(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c](t_{1_1}t_{1_2}^{-1}, t_{2_1}t_{2_2}^{-1})$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}t_{1_2}^{-1}), (u_m^3, u_r^4)_\alpha^c(t_{2_1}t_{2_2}^{-1})\}$

$\leq \max\{\max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}), (u_m^1, u_r^2)_\alpha^c(t_{1_2}^{-1})\}, \max\{(u_m^3, u_r^4)_\alpha^c(t_{2_1}), (u_m^3, u_r^4)_\alpha^c(t_{2_2}^{-1})\}\}$

$\leq \max\{\max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}), (u_m^1, u_r^2)_\alpha^c(t_{1_2})\}, \max\{(u_m^3, u_r^4)_\alpha^c(t_{2_1}), (u_m^3, u_r^4)_\alpha^c(t_{2_2})\}\}$

$= \max\{\max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}), (u_m^3, u_r^4)_\alpha^c(t_{2_1})\}, \max\{(u_m^1, u_r^2)_\alpha^c(t_{1_2}), (u_m^3, u_r^4)_\alpha^c(t_{2_2})\}\}$

$= \max\{((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_{1_1}t_{2_1}), ((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_{1_2}t_{2_2})\}$

Therefore,

$[(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c]\left((t_{1_1}, t_{2_1})(t_{1_2}, t_{2_2})^{-1}\right) \leq \max\{((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_{1_1}t_{2_1}), ((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_{1_2}t_{2_2})\}$

Also,

$[(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c](t_{1_1}, t_{2_1})^{-1} = [(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c](t_{1_1}^{-1}, t_{2_1}^{-1})$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}^{-1}), (u_m^3, u_r^4)_\alpha^c(t_{2_1}^{-1})\}$

$= \max\{(u_m^1, u_r^2)_\alpha^c(t_{1_1}), (u_m^3, u_r^4)_\alpha^c(t_{2_1})\}$

$= ((u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c)(t_{1_1}, t_{2_1});\ \forall\ (t_{1_1}, t_{2_1}) \in G_1 \circ G_2$

Thus, $(u_m^1, u_r^2)_\alpha^c \circ (u_m^3, u_r^4)_\alpha^c$ is an α–anti-IS$_G$ of $G_1 \circ G_2$.

8. Application of an anti-IG in Networking Problem

Group theory has several applications in different branches of mathematics. Geometric concepts such as symmetry and certain types of transformations are expressed through groups. Here such transformation of symmetric figures is expressed using the concept of anti-IS$_G$. When one symmetric figure is transformed into another complete symmetric figure, it covers a journey of time. The uncovered journey of time, or the fraction of untransformed figures in between the transformation of one symmetric figure to another, is defined by the concept of anti-IS. Furthermore, in terms of the composition operation, the entire collection of partially transformed figures embodies the idea of anti-IS$_G$. Next, this anti-IS$_G$ symbolizes a closed network figure that can be used to solve a variety of networking issues.
Algorithmically, a Symmetric Anti-IG with respect to anti-IS can be determined as follows:
1) Identification of anti-IG operation:
List every potential transformation that could leave a given geometric element unaltered. These consist of:

- Anti-imprecise Identity Operation: Consider the anti-imprecise operation that keeps the element entirely unaltered after the imprecise operation.
- Anti-imprecise Rotational Operations: Consider the anti-imprecise rotations after the imprecise rotations around an axis by $360^\circ/n$, where $n \in \mathbb{Z}$.
- Anti-imprecise Reflection Operations: Consider the anti-imprecise planer reflection after imprecise reflection through a plane.
- Anti-imprecise Inversion Operation: Consider the anti-imprecise inversion after the imprecise inversion of all atoms through the centre.
- Anti-imprecise Rotation-Reflection Operations: Consider the anti-imprecise rotation after the imprecise rotation by $360^\circ/n$, then anti-imprecise reflection after the imprecise reflection.

2) Test of Each Anti-Imprecise Operation:
Observation of each geometric element after applying each operation.
3) Verification of Invariance:
For every anti-imprecise operation after the imprecise operation, check to see if the geometric element looks exactly the same.
Construction of the Anti-IG:
Gather every anti-imprecise operation that rendered the geometric element unaltered, then confirm that they constitute an anti-IG by examining the anti-IG axioms:
1) $(u_m^1, u_r^2)^c\,(t_1, t_2)\ \le\ \max\{(u_m^1, u_r^2)^c\,(t_1), (u_m^1, u_r^2)^c\,(t_2)\}\ ;\ \forall\, t_1, t_2 \in G$

2) $(u_m^1, u_r^2)^c\,(t_1{}^{-1})\ \le (u_m^1, u_r^2)^c(t_1)$

3) $(u_m^1, u_r^2)(e) \le (u_m^1, u_r^2)(t_1);\ \forall\, t_1 \in G$

Example 13: Consider a symmetric molecule NH_3 which is studied using anti-IS$_G$. The elements in $\mathfrak{T}_3 = NH_3$ are the transformed figures of NH_3 due to different compositions, given by $\mathfrak{T}_3 = \{\mathfrak{E}, \mathfrak{C}_3^1, \mathfrak{C}_3^2, \sigma_a, \sigma_b, \sigma_c\}$ where $\mathfrak{C}_3^1$ is one time rotation by an angle 120°, $\mathfrak{C}_3^2$ is two times rotation by an angle 120° while $\mathfrak{C}_3^3$ is the identity denoted by $\mathfrak{E}$ (after $\mathfrak{C}_3^3$, original configuration is obtained) and $\sigma_a, \sigma_b, \sigma_c$ are three planes of symmetry passing vertically through the rotational axis and each H atom $(\mathfrak{H}_a, \mathfrak{H}_b, \mathfrak{H}_c)$. Then the grades of MF and RF are assigned to each of the element of $\mathfrak{T}_3$. The Cayley table of $\mathfrak{T}_3$ is given in Table 1.

Table 1: Cayley Table of $\mathfrak{T}_3$

*	$\mathfrak{E}$	$\mathfrak{C}_3^1$	$\mathfrak{C}_3^2$	σ_a	σ_b	σ_c
$\mathfrak{E}$	$\mathfrak{E}$	$\mathfrak{C}_3^1$	$\mathfrak{C}_3^2$	σ_a	σ_b	σ_c
$\mathfrak{C}_3^1$	$\mathfrak{C}_3^1$	$\mathfrak{C}_3^2$	$\mathfrak{E}$	σ_c	σ_a	σ_b
$\mathfrak{C}_3^2$	$\mathfrak{C}_3^2$	$\mathfrak{E}$	$\mathfrak{C}_3^1$	σ_b	σ_c	σ_a
σ_a	σ_a	σ_c	σ_b	$\mathfrak{E}$	$\mathfrak{C}_3^1$	$\mathfrak{C}_3^2$
σ_b	σ_b	σ_a	σ_c	$\mathfrak{C}_3^2$	$\mathfrak{E}$	$\mathfrak{C}_3^1$
σ_c	σ_c	σ_b	σ_a	$\mathfrak{C}_3^1$	$\mathfrak{C}_3^2$	$\mathfrak{E}$

Then the grades of MF and RF are assigned to each of the element of $\mathfrak{T}_3$ given in equation 27.

$$(u_m^1, u_r^2)^c(\,t_1) = \begin{cases} (0.9,0);\ \text{if}\ t_1 = \mathfrak{E}, \mathfrak{C}_3^1, \mathfrak{C}_3^2 \\ (0.7,0.3);\ \text{if}\ t_1 = \sigma_a, \sigma_b, \sigma_c \end{cases} \tag{27}$$

Is an IS$_G$.
Here the group is non-abelian but the IS$_G$ is abelian.
Now,

$$(u_m^1, u_r^2)^c(\,t_1) = \begin{cases} (1,0.9);\ \text{if}\ t_1 = \mathfrak{E}, \mathfrak{C}_3^1, \mathfrak{C}_3^2 \\ (1,0.7);\ \text{if}\ t_1 = \sigma_a, \sigma_b, \sigma_c \end{cases} \tag{28}$$

Is an anti-IS$_G$ of equation 27.
The elements of the precise group can be used as nodes in the network Fig. 1, with the imprecise value of the same elements serving as the uncertain path between two elements indicated by the connecting lines. These connecting lines are given by

$(u_m^1, u_r^2)(\mathfrak{C}_3^1\mathfrak{C}_3^1) =\ (u_m^1, u_r^2)(\mathfrak{C}_3^2) =\ (0.9,0);\ (u_m^1, u_r^2)^c(\mathfrak{C}_3^2) =\ (1,0.9)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^1\mathfrak{C}_3^2) = (u_m^1, u_r^2)(\mathfrak{E}) = (0.9,0); (u_m^1, u_r^2)^c(\mathfrak{E}) = (1,0.9)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^1\sigma_a) = (u_m^1, u_r^2)(\sigma_c) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_c) = (1,0.7)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^1\sigma_b) = (u_m^1, u_r^2)(\sigma_a) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_a) = (1,0.7)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^1\sigma_c) = (u_m^1, u_r^2)(\sigma_b) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_b) = (1,0.7)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^2\sigma_a) = (u_m^1, u_r^2)(\sigma_b) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_b) = (1,0.7)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^2\sigma_b) = (u_m^1, u_r^2)(\sigma_c) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_c) = (1,0.7)$

$(u_m^1, u_r^2)(\mathfrak{C}_3^2\sigma_c) = (u_m^1, u_r^2)(\sigma_a) = (0.7,0.3); (u_m^1, u_r^2)^c(\sigma_a) = (1,0.7)$

$(u_m^1, u_r^2)(\sigma_a\sigma_b) = (u_m^1, u_r^2)(\mathfrak{C}_3^1) = (0.9,0); (u_m^1, u_r^2)^c(\mathfrak{C}_3^1) = (1,0.9)$

$(u_m^1, u_r^2)(\sigma_a\sigma_c) = (u_m^1, u_r^2)(\mathfrak{C}_3^2) = (0.9,0); (u_m^1, u_r^2)^c(\mathfrak{C}_3^2) = (1,0.9)$

$(u_m^1, u_r^2)(\sigma_b\sigma_c) = (u_m^1, u_r^2)(\mathfrak{C}_3^1) = (0.9,0); (u_m^1, u_r^2)^c(\mathfrak{C}_3^1) = (1,0.9)$

The nodes for the Fig. 1 model are represented by the elements of $\mathfrak{T}_3$, and the path that connects two elements is an imprecise representation of the process of converting one element figure to another because of composition, ultimately creating a closed network in Fig. 1. Additionally, the non-completion path from one element to the next is represented by the anti-imprecise value of each element.

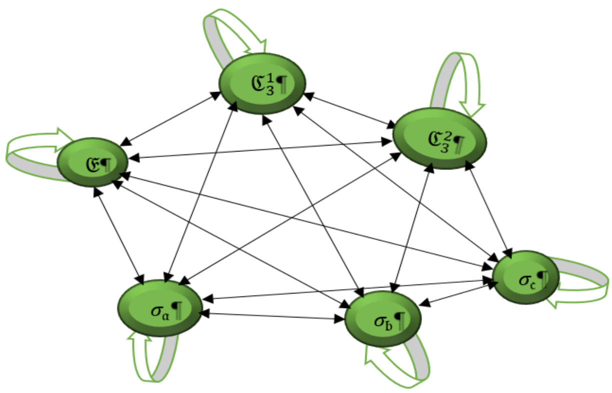

Fig. 1: Network Figure of Equation 27 and 28.

Since every element in $\mathfrak{T}_3$ under the composition of symmetries with the identity element $\mathfrak{E}$ yields the self-element, a self-loop is formed around every element. In addition, the combination of symmetries between two distinct elements results in an imprecise path of either (0.9,0) or (0.7,0.3), from which an anti-imprecise value of (1,0.9) or (1,0.7) is obtained. The non-completion of a journey from one node to another is represented by each anti-imprecise value. Moreover, the loops and connecting lines combine to create an anti-IG, which can be visualized as a closed network pattern. Depending on different anti-IGs, the number of nodes and connecting lines may change. According to the number of elements in the group, we can therefore create a variety of network structures.

9. Conclusion

In this article, we attempted the definition of anti-IS_G based on the definition of IS and proved some propositions. Further, we extend our work to α −IS_G, α −anti-IS_G, α −anti-normal IS_G, and α −anti-imprecise cosets. It is found that their properties of classical group theory hold good using this new concept. In this article, it is proved that an IS_G (normal IS_G) is again an α −IS_G (α −normal IS_G), but the converse is not true, for which some examples are cited. Further, after exploring several interesting characteristics of anti-IS_G, it is found that an anti-IS_G (anti-normal IS_G) is again an α −anti-IS_G (α −anti-normal IS_G), but the converse is not true, for which some examples are cited. Again, it is proved that the union of an arbitrary collection of α −anti-IS_Gs is again an α −anti-IS_G, but the intersection of any two α −anti-IS_Gs is not necessarily an α −anti-IS_G, for which an example is cited. Additionally, a closed network diagram that is covered in the last section of the article can also be constructed from the anti-IS_bs from the anti-IG. The properties of an anti-IG enable the creation of a wide range of fascinating network diagrams that could have practical uses in the domains of invention and research.

10. Future Scope

The advantage of this research is that we can generate imprecise and anti-imprecise algebraic structures by applying group theory to imprecision. Like fuzzy algebra, IG and anti-IG theories have a wide range of applications in the real world, particularly in cases involving ambiguity, imprecision, and uncertainty. Decision-making, evaluating design concepts, and assessing important engineering features could all benefit from its use. This research is particularly beneficial in domains such as medicine, where choices regarding diagnosis and treatment often involve a certain amount of uncertainty. Consequently, these applications can be extended using IS theory. As a result, we will be able to solve real-world issues and accomplish a great deal of work in the future with imprecise theoretical sets.

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