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**Research paper**

# Fuzzy Solution of Z-Number-Based Multi-Objective Linear Programming Models with Different Membership Functions

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**Abstract**

In real-world decision-making, uncertainty plays a crucial role, especially when dealing with complex, multi-objective optimization problems. Traditional linear programming (LP) models often have the assumption that the data is precise and deterministic. However, this assumption is often not realistic for many applications, as imprecision will always be present. This paper presents a fuzzy solution to Z-Z-number-based multi-objective linear programming (ZMOLP) models. Z-numbers can combine fuzzy logic and Z-numbers to manage uncertainty in (multi-objective) decision dilemmas. A Z-number consists of two parts: a fuzzy number for the uncertainty in the data and a reliability score that indicates the degree of confidence in the data. The two aspects of Z-numbers make them effective for modeling imprecise data in multi-objective positioning. The effectiveness and overall computational efficiency of different fuzzy membership functions (triangular, trapezoidal, Gaussian, etc.) were also explored to understand their impact on optimal solutions. Z-numbers, through fuzzy numbers, provide a flexible and adaptable decision-making solution far superior to traditional methods for managing imprecision that leads to a better representation of uncertainty related to objectives and constraints. To demonstrate the viability of Z-number-based approaches, practical applications were employed to illustrate their usage in healthcare decision-making and engineering optimization.

***Keywords***: *Z-Numbers; Multi-Objective LPP; Fuzzy Logic; Uncertainty; Decision-Making.*

## 1. Introduction

Uncertainty is usually a significant element in decision-making. This is especially the case for problems that involve various conflicting objectives, where uncertainty is rarely 100 percent certain in practice. The traditional decision-making models of linear programming (LP) and multi-objective linear programming (MOLP) typically assume deterministic data. However, they can't even scratch the surface of addressing real-world issues with the uncertainty they typically face [1]. While MOLP models are more of an advancement in describing situations with multiple conflicting objectives, they still run into issues with having to deal with precise data, even when that data and preference are uncertain or vague [2] [3]. This fact and the complexities associated with uncertainty in multi-objective optimization have contributed to the numerous developments associated with fuzzy logic and probabilistic methods. Conversely, Z-numbers have been proposed to allow for decision-making with uncertainty [4]. Z-numbers involve two significant components: (1) a fuzzy number containing uncertainty or imprecision within the data, and (2) a reliability measure indicating the level of confidence in the information [5] [6]. The combination of these two pieces allows for Z-numbers to be particularly beneficial in multi-objective optimization problems involving decision-making that needs to consider both the uncertainty in the objectives as well as the reliability of the data [7]. With Z-numbers being implemented in MOLP Models, we begin to see how decision makers can introduce and better model uncertainty, as well as make better decisions [8]. Even though Z-number-based MOLP models have a fair amount of potential, there has been little consideration on Z Z-number-based MOLP models in general and in combination with fuzzy membership functions [9]. In this research, we will begin to fill this gap by providing a fuzzy methodology for Z-number-based MOLP models utilizing different fuzzy membership functions to study the effect on the optimal solution [10]. Fuzzy logic is important for representing imprecise data, enabling decisions to be made at various levels of truth instead of the false/true representations made by most scientific and research-based approaches [11]. In multi-objective optimization, fuzzy membership functions can also include triangular, trapezoidal, and Gaussian types in addition to varying degrees of fuzziness to represent some level of uncertainty about the objectives and constraints [12]. Using membership functions is helpful when trying to express and express preferences naturally (accounts for vagueness), similar to the vagueness accounted for in most real-world decision-making problems [13]. Allowing decision makers to represent their choices will help the conceptual analysis during research to utilize different fuzziness levels during Z-number-based MOLP models to examine how this level of fuzziness changes the optimal solution [14]. This exploration aims to better understand how membership functions will impact the trade-off relationships of selecting against competing directions and achieving worthwhile outcomes even for complex decision-making problem domains [15]. This research aims to achieve: 1. a fuzzy solution approach to solve Z-number-based MOLP models encompassing various fuzzy membership types. 2. to examine how the choice of fuzzy membership type impacts the decision process. 3. to enhance the solutions provided for multi-objective optimization under uncertainty or complex preferences. The immediate objective of this study is to develop a fuzzy approach for solving MOLP models based on Z-numbers by investigating how different fuzzy membership types will provide different decisions. It aims to enhance the implementation of multi-objective optimization models by providing flexibility and precision monitoring by studying fuzzy membership types and their impact on decision-making. The role of fuzzy logic and Z-numbers is expected to provide a more realistic environment for developing solutions to problems associated with uncertain and imprecise information. The findings of this research will also have real-world implications for the finance, energy, and supply chain industries, where decision-making will inevitably involve multiple objectives that are fused with uncertain information. The current research will lead to further future development of multi-objective optimization in which fuzzy logic will be interfaced to Z-number-based models. In addition to providing advanced guidelines to allow for compliment health in understanding the degree of impact of various fuzzy membership functions and their health implications for optimal solutions will also allow for a greater degree of evidence in handling uncertainty and imprecision components in complex decision-making environments. This research will provide theoretical advancement for the field of fuzzy optimization while also providing decision makers with useful tools for decision-making.

## 2. Comparison with Other Uncertainty Models

Uncertainty in multi-objective linear programming (MOLP) has traditionally been addressed through several mathematical frameworks, the most notable being probabilistic models, interval-based models, and fuzzy approaches. Each has distinct strengths and limitations depending on the nature of uncertainty and the availability of information. Probabilistic models assume that uncertain parameters follow well-defined probability distributions. These models are useful in applications where sufficient historical data exists to estimate distributions accurately, and they provide a structured mechanism for quantifying randomness. However, in many real-world decision-making problems, exact probability distributions are either unknown or difficult to estimate reliably. This reliance on large datasets and distributional assumptions often limits their applicability in domains where data is scarce or imprecise. Interval-based models represent uncertainty by specifying lower and upper bounds for parameters. These models are computationally straightforward and particularly effective in capturing bounded imprecision without requiring detailed statistical information. Nevertheless, interval approaches treat all values within the bounds as equally plausible and fail to incorporate the degree of confidence in the specified ranges. Consequently, while they define uncertainty in terms of possible values, they lack the capacity to distinguish between highly reliable and less reliable information.
Fuzzy models, widely applied in optimization, represent uncertainty by allowing partial degrees of membership, thereby capturing vagueness in objectives and constraints. While fuzzy approaches overcome some of the rigidity of interval models, they typically do not incorporate explicit measures of information reliability. In this context, Z-numbers offer a more comprehensive representation of uncertainty. A Z-number integrates (i) a fuzzy number that expresses the imprecision of the data, and (ii) a reliability measure that indicates the degree of confidence in the information. This two-layered structure enables Z-numbers to combine the advantages of fuzzy modeling with an explicit representation of reliability, thereby overcoming the limitations of probabilistic and interval-based approaches. Unlike probabilistic methods, Z-numbers do not require precise distributional assumptions, and unlike interval models, they explicitly account for credibility levels. This dual capability makes Z-numbers particularly suitable for complex real-world applications, such as healthcare decision-making, engineering optimization, and supply chain management, where both uncertainty and reliability must be simultaneously addressed.

## 3. Preliminaries

a) **Definition**: (Z-number) A Z-number is a pair $Z = \left(\tilde{A}, \tilde{B}\right)$, where: $\tilde{A}$ is a fuzzy number representing the uncertainty of the information $\tilde{B}$ is the reliability measure representing the confidence or reliability of $\tilde{A}$.
A Z-number, therefore, consists of vagueness (the fuzzy number) as well as reliability (the reliability measure). Therefore, it is applicable to decision-making problems that contain uncertain data.

$$Z = \left(\tilde{A}, \tilde{B}\right)$$

b) **Definition**: (Fuzzy Number Representation of Z-numbers) To make Z-numbers usable in optimization problems, they are typically converted into classical fuzzy numbers for simplification. Let $\tilde{A} = (a, b, c)$ Be a triangular fuzzy number representing the objective or constraint in a Z-number, and let $\tilde{B}$ Be the reliability measure associated with it. The fuzzy number can be represented by its lower and upper bounds, denoted as $\tilde{A}_L$ and $\tilde{A}_R$, respectively:

$$\tilde{A}_L = a, \quad \tilde{A}_R = c$$

This conversion simplifies the computational process and allows us to use fuzzy numbers for subsequent operations in the optimization process.

c) **Definition**: (Possibility Degree) The possibility degree $p\left(\tilde{A} \geqslant \tilde{B}\right)$ between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ quantifies the likelihood that $\tilde{A}$ is greater than or equal to $\tilde{B}$. It is calculated using the membership functions of both fuzzy numbers. The possibility degree is defined as:

$$p\left(\tilde{A} \succcurlyeq \tilde{B}\right) = \int_0^1 \max\left(1 - \frac{\max\left(\tilde{B}_R(\alpha) - \tilde{A}_L(\alpha), 0\right)}{w\left(\tilde{A}\right) + w\left(\tilde{B}\right)}, 0\right) d\alpha$$

Where:
$\tilde{A}_L$ and $\tilde{B}_R$ Represent the left and right bounds of the fuzzy numbers. $\tilde{A}$ and $\tilde{B}$, respectively and $w\left(\tilde{B}\right)$ Represent the widths of these fuzzy numbers.
This calculation allows us to compare fuzzy numbers based on the uncertainty embedded in them.

d) **Definition**: (Lower Limit of the Possibility Degree) The lower limit of the possibility degree is introduced to account for the certainty in the comparison between two fuzzy numbers. It is defined as:

$$p\left(\tilde{A} \succcurlyeq \tilde{B}\right) \geq \gamma \quad \text{where} \quad \gamma \in [0.5,1]$$

This lower limit ensures that the comparison becomes more credible as $\gamma$ Increases. If $p\left(\tilde{A} \succcurlyeq \tilde{B}\right) \geq \gamma$, then we consider $\tilde{A}$ to be greater than or equal to $\tilde{B}$.

e) **Definition**: (Non-Dominance and Comparison of Solutions) Two solutions $X_1$ and $X_2$ are said to be non-dominated with the possibility degree $\gamma$ If neither dominates the other with a possibility degree greater than or equal to $\gamma$. This is mathematically expressed as:

$$X_1 \,||_\gamma\, X_2 \quad \text{if} \quad p\big(f(i)(X_2) \succcurlyeq f(i)(X_1)\big) \geq \gamma \quad \text{for all} \quad i$$

Where $f(i)(X)$ denotes the objective value corresponding to the i-th objective of the solution $X$. When neither solution dominates the other with sufficient certainty, they are considered non-dominated.

f) **Definition**: (Dominance of Solutions) Solution $X_1$ dominates solution $X_2$ with the possibility degree $\gamma$ if, for all objectives $i \in \{1,2,\ldots,r\}$We have:

$$p\big(f(i)(X_2) \succcurlyeq f(i)(X_1)\big) \geq \gamma$$

And for at least one objective $k$, the condition $p\big(f(i)(X_2) \succcurlyeq f(i)(X_1)\big) > 0.5$ Holds. This means that $X_1$ is better than $X_2$ For at least one objective, and it is at least as good for all others.

g) **Definition**: (Scalar Multiplication of Discrete Fuzzy Numbers) For a discrete fuzzy number $\tilde{A}$, its scalar multiplication $\tilde{A}_1 = \lambda\left(\tilde{A}\right)$, where $\lambda \in R$, is the discrete fuzzy number whose α-cut is defined as:

$$A_1^\alpha = \{x \in \lambda \cdot \operatorname{supp}\left(\tilde{A}\right) \mid \min(\lambda A^\alpha) \leq x \leq \max(\lambda A^\alpha)\}$$

The membership function is:

$$\mu_{\lambda\tilde{A}}(x) = \sup\{\alpha \in [0,1] \mid x \in (\lambda A^\alpha)\}$$

h) **Definition**: (Multiplication of Discrete Fuzzy Numbers) For two discrete fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$, their multiplication $\tilde{A}_{12} = \tilde{A}_1 \cdot \tilde{A}_2$ Is the discrete fuzzy number whose α-cut is defined as:

$$A_j^\alpha = \{x \in \left(\operatorname{supp}\left(\tilde{A}_1\right) \cdot \operatorname{supp}\left(\tilde{A}_2\right)\right) \mid \min(A_1^\alpha \cdot A_2^\alpha) \leq \max(A_1^\alpha \cdot A_2^\alpha)\}$$

The membership function is:

$$\mu_{\tilde{A}_1 \cdot \tilde{A}_2}(x) = \sup\{\alpha \in [0,1] \mid x \in (A_1^\alpha \cdot A_2^\alpha)\}$$

i) **Definition**: (Subtraction of Discrete Fuzzy Numbers) For two discrete fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$, their standard subtraction $\tilde{A}_{12} = \tilde{A}_1 - \tilde{A}_2$ Is the discrete fuzzy number whose α-cut is defined as:

$$A_j^\alpha = \{x \in \left(\operatorname{supp}\left(\tilde{A}_1\right) - \operatorname{supp}\left(\tilde{A}_2\right)\right) \mid \min(A_1^\alpha - A_2^\alpha) \leq \max(A_1^\alpha - A_2^\alpha)\}$$

Where:
$\operatorname{supp}\left(\tilde{A}_1\right) - \operatorname{supp}\left(\tilde{A}_2\right) = \{x_1 - x_2 \mid x_j \in \operatorname{supp}\left(\tilde{A}_j\right), j = 1,2\}$

The membership function is defined as:

$$\mu_{\tilde{A}_1 - \tilde{A}_2}(x) = \sup\{\alpha \in [0,1], x \in (A_1^\alpha - A_2^\alpha)\}$$

# 4. Z-Number-Based Linear Programming (Z-LP) Model

## 4.1. General formulation

The general form of a Z-number-based Linear Programming problem is as follows:
Maximize or Minimize:

$$Z = \text{Objective Function} \quad F(X) = \sum_{i=1}^{n} c_i \cdot x_i$$

Where:
$F(X)$ represents the objective function value,
$c_i$ represents the fuzzy coefficients associated with each decision variable $x_i$,
$x_i$ represents the decision variables,
The objective function is modeled using Z-numbers: $c_i = \left(\tilde{A}_i, \tilde{B}_i\right)$, where $\tilde{A}_i$ Is the fuzzy number and $\tilde{B}_i$ Is the reliability measure.

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \quad \text{for all} \quad i = 1,2,\dots,m$$

Where:
$a_{ij}$ Are the fuzzy coefficients in the constraint matrix for each constraint?
$b_i$ Represents the fuzzy constraints expressed as Z-numbers: $b_i = \left(\tilde{B}_i, \tilde{R}_i\right)$, where $\tilde{B}_i$ is a fuzzy number and $\tilde{R}_i$ Is the reliability measure.

## 4.2. Z-Number-based constraints

Let the Z-number representing each constraint be $Z_i = \left(\tilde{A}_i, \tilde{B}_i\right)$. The constraints in the linear programming model are represented by:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \le \tilde{B}_i$$

Where:
$a_{ij}$ Are fuzzy numbers that represent the link between decision variables and the coefficients for constraints, $x_j$ Are the decision variables,
To convert these constraints into crisp constraints, it is sufficient to convert Z-numbers to fuzzy numbers.

## 4.3. Conversion of z-numbers to fuzzy numbers

To carry out calculations with Z-numbers, we first need to transform them to classical fuzzy numbers. A method for this transformation is to use the reliability measure. $\tilde{A}^1$ to weight fuzzy numbers $\tilde{B}^\alpha$.

$$Z^\alpha = \left(\tilde{A}^\alpha, \tilde{B}^\alpha\right)$$

Where:
$\tilde{A}^\alpha$ represents the fuzzy α-cut of $\tilde{A}$, which captures the degree of uncertainty for each constraint or objective.
$\tilde{B}^\alpha$ Is the weighted reliability value for the corresponding constraint or objective.

## 4.4. Z- number-based objective function

For a Z-number-based objective function, the fuzzy objective is written as:

$$F(X) = \sum_{i=1}^{n} \left(\tilde{A}_i \cdot x_i\right) = \sum_{i=1}^{n} \left(\left(\tilde{A}_i\right) \cdot x_i\right)$$

Where:
$\tilde{A}_i$ represents the fuzzy coefficients in the objective function,
The reliability measure $\tilde{B}_i$ It is used to weigh the fuzzy terms.
To simplify this model for computation, we may convert it into crisp numbers using fuzzy arithmetic rules.

## 5. Theorems for Z-Number-Based Multi-Objective Linear Programming

### 5.1. Theorem: (relationship between two fuzzy numbers)

Given two fuzzy numbers $\tilde{A}$ and $\tilde{B}$The relationship between them can be defined using the possibility degree. The dominance relationship is expressed as:

$$p\left(\tilde{A} \succcurlyeq \tilde{B}\right) + p\left(\ominus \tilde{B} \succcurlyeq \tilde{A}\right) = 1$$

Where:
$p\left(\tilde{A} \succcurlyeq \tilde{B}\right)$ represents the possibility that $\tilde{A}$ is greater than or equal to $\tilde{B}$,
$p\left(\ominus \tilde{B} \succcurlyeq \tilde{A}\right)$ represents the possibility that $\tilde{B}$ is greater than or equal to $\tilde{A}$,
The sum of these possibilities must equal 1, reflecting the exclusive nature of the dominance relation.

### 5.2. Theorem: (γ-pareto optimality)

Let $X_\gamma^*$ Be a feasible solution to a Z-number-based multi-objective optimization problem. $X_\gamma^*$ Is said to be γ-Pareto optimal if there does not exist another solution X such that X dominates $X_\gamma^*$ with a degree not less than γ. This can be mathematically expressed as:

$$X_\gamma^* \in \gamma - PS(f) \quad \text{if} \quad \forall X \in \Omega,\ X \not\subseteq X_\gamma^*$$

Where: $\gamma - PS(f)$ Is the γ-Pareto optimal set, is the feasible region of solutions, and indicates that no solution X In the feasible set, dominates $X_\gamma^*$ with a possibility degree greater than or equal to γ.

### 5.3. Theorem: (monotonicity of γ-pareto optimal set)

Let $\gamma_1, \gamma_2 \in [0.5,1]$ such that $\gamma_1 \geq \gamma_2$. Then:

$$X_{\gamma_2}^* \subseteq X_{\gamma_1}^*$$

This theorem suggests that as γ Increases (indicating higher certainty in the dominance relationship), the γ-Pareto optimal set shrinks, and fewer solutions are available that satisfy the dominance conditions.

### 5.4. Theorem: (transitivity of dominance)

If $X_1$ dominates $X_2$ with a possibility degree greater than or equal to γ, and $X_2$ dominates $X_3$ with a possibility degree greater than or equal to γ, then $X_1$ dominates $X_3$ with a possibility degree greater than or equal to γ. This is mathematically expressed as:

$$X_1 \succcurlyeq_\gamma X_2 \quad \text{and} \quad X_2 \succcurlyeq_\gamma X_3 \quad \Rightarrow \quad X_1 \succcurlyeq_\gamma X_3$$

The transitivity property of dominance ensures that if one solution dominates another and the second dominates a third, then the first solution must also dominate the third solution.

### 5.6. Theorem: (existence of a γ-pareto optimal solution)

In a Z-number-based multi-objective optimization problem, there always exists at least one solution $X_\gamma^* \in \Omega$ $X_\gamma^*$ is γ-Pareto optimal for a given $\gamma \in [0.5,1]$. This is expressed as:

$$\exists\, X_\gamma^* \in \Omega \quad \text{such that} \quad X_\gamma^* \in \gamma - PS(f)$$

This theorem guarantees the existence of at least one γ-Pareto optimal solution in the feasible region. Ω For any given γ.

### 5.7. Theorem: (non-dominance of solutions)

Given two solutions $X_1$ and $X_2$, if neither solution dominates the other with a possibility degree greater than or equal to γ, then $X_1$ and $X_2$ are said to be non-dominated with the possibility degree γ. This is mathematically represented as:

$$X_1 \,||_\gamma\, X_2 \quad \text{if} \quad p\big(f(i)(X_2) \succcurlyeq f(i)(X_1)\big) < \gamma \quad \text{and} \quad p\big(f(i)(X_1) \succcurlyeq f(i)(X_2)\big) < \gamma$$

Where:
$p\big(f(i)(X_2) \succcurlyeq f(i)(X_1)\big)$ and $p\big(f(i)(X_1) \succcurlyeq f(i)(X_2)\big)$ Are the possibility degrees of dominance between objectives i of solutions $X_1$ and $X_2$,
-dominance occurs when both possibilities are less than γ, meaning neither solution dominates the other.

### 5.8. Theorem: (boundedness of the γ-pareto optimal set)

Let $X^*_{\gamma_1}$ and $X^*_{\gamma_2}$ Be γ-Pareto optimal sets corresponding to $\gamma_1$ and $\gamma_2$, respectively, where $\gamma_1 \geq \gamma_2$. Then, the γ-Pareto optimal set is bounded in the sense that:

$$X^*_{\gamma_2} \subseteq X^*_{\gamma_1}$$

This theorem implies that the γ-Pareto optimal set becomes smaller as the certainty (value of γ) increases, meaning fewer solutions satisfy the dominance condition as γ Grows.

### 5.9. Theorem 8: (convexity of the γ-pareto optimal set)

If the objective functions in a Z-number-based multi-objective optimization problem are convex, then the γ-Pareto optimal set is also convex for any $\gamma \in [0.5,1]$. This can be expressed as:

$$\forall X_1, X_2 \in \gamma - PS(f),\ \forall\lambda \in [0,1], \quad \lambda X_1 + (1-\lambda)X_2 \in \gamma - PS(f)$$

The convexity of the γ-Pareto optimal set implies that any convex combination of two γ-Pareto optimal solutions will also be long to the γ-Pareto optimal set.
The theoretical foundations established through the definitions and theorems provide the mathematical basis for modeling and solving multi-objective linear programming problems under Z-number uncertainty. However, theory alone is not sufficient for practical implementation. To demonstrate how these concepts are applied, the next section introduces an algorithmic framework for solving ZMOLP problems. This algorithm bridges the theoretical constructs with a step-by-step solution process, which is later illustrated using a numerical example.

## 6. Algorithm for Solving ZMOLP Problems

As illustrated in Figure 1, the proposed methodology proceeds step-by-step from the definition of Z-numbers to their conversion, application of membership functions, solution of the MOLP model, and final trade-off analysis.
1) Define the Z-number-based Objective Functions and Constraints: As discussed, a Z-number is a pair $Z = (\widetilde{A}, \widetilde{B})$, where:
- $\widetilde{A}$ is the fuzzy number representing the uncertainty in the data.
- $\widetilde{B}$ is the reliability measure that quantifies the degree of confidence in the fuzzy number.

The general formulation of the objective function and constraints can be written as:
a) Objective Function:

$$Z = \sum_{i=1}^{n} c_i \cdot x_i$$

Where $c_i = (\tilde{A}_i, \tilde{B}_i)$ represents a Z-number for the i-th objective.
b) Constraints:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i \quad \text{for all} \quad i = 1,2,\dots,m$$

Where $a_{ij} = (\widetilde{A}_{ij}, \widetilde{B}_{ij})$ and $b_i = (\widetilde{A}_i, \widetilde{B}_i)$ are Z-numbers.
2) Convert Z-numbers to Fuzzy Numbers:
For each Z-number $c_i = (\widetilde{A}_i, \widetilde{B}_i)$, calculate the α-cuts of the fuzzy number$\widetilde{A}_i$. The α-cut for $\widetilde{A}_i = (a_i, b_i, c_i)$ is given by:

$$\widetilde{A}_i^{\,a} = [a_i + \alpha \cdot (c_i - a_i), b_i - \alpha \cdot (b_i - c_i)]$$

Where α represents the level of uncertainty (in the range$\alpha \in [0,1]$).
Note: This transformation helps to simplify the fuzzy numbers for subsequent calculations.
3) Defuzzify the Fuzzy Numbers
Once the fuzzy numbers are obtained, use defuzzification techniques (such as the centroid method) to convert fuzzy numbers to crisp values:
a) Centroid Method:

$$\text{Crisp Value} = \frac{a + 4b + c}{6}$$

For a triangular fuzzy number $(a, b, c)$.
For other fuzzy membership functions (e.g., trapezoidal or Gaussian), the defuzzification formula may vary, but the goal is to transform the fuzzy values into a crisp value suitable for linear programming.

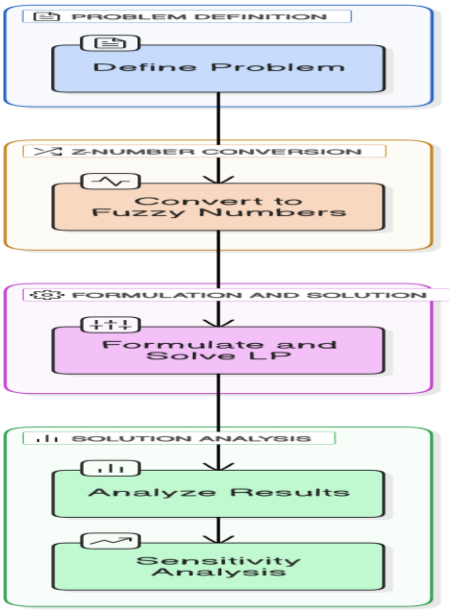


**Fig. 1:** Flow Chart of the Proposed Methodology.

Figure 1. Flow chart of the proposed methodology for solving Z-number-based MOLP. Step 1: Define objectives and constraints under uncertainty. Step 2: Represent uncertain parameters using Z-numbers. Step 3: Convert Z-numbers into equivalent fuzzy representations. Step 4: Apply appropriate fuzzy membership functions (e.g., triangular, trapezoidal, Gaussian). Step 5: Formulate and solve the MOLP using fuzzy optimization techniques. Step 6: Perform trade-off and sensitivity analysis to evaluate the effect of different membership functions on the optimal solution.

b) Problem Setup:

We have a multi-objective optimization problem where we want to maximize two objectives. The decision variables $x_1$ and $x_2$ represent quantities of two products that we want to optimize based on conflicting objectives and fuzzy constraints.

Objective Functions:

Objective 1: Maximize profit $Z_1$

Objective 2: Maximize customer satisfaction $Z_2$

Constraints:

Constraint 1: Limit on production capacity

Constraint 2: Budget constraint

We will use Z-numbers to represent the objective coefficients and the right-hand side values of the constraints.

c) Step-by-step solution:

## 6.1. Objective functions and constraints in z-number form

Objective Function: Maximize $Z_1 = c_1 \cdot x_1 + c_2 \cdot x_2$

Where:

$c_1 = (5,6,7,0.9)$ represents the profit coefficient for $x_1$ (a Z-number).

$c_2 = (4,5,6,0.85)$ represents the profit coefficient for $x_2$ (a Z-number).

Objective 2: Maximize $Z_2 = d_1 \cdot x_1 + d_2 \cdot x_2$

Where:

$d_1 = (3,4,5,0.8)$ represents the satisfaction coefficient for $x_1$ (a Z-number).

$d_2 = (2,3,4,0.75)$ represents the satisfaction coefficient for $x_2$ (a Z-number).

Constraints:

$3 \cdot x_1 + 4 \cdot x_2 \leq (10,12,14,0.9)$ (Production capacity constraint)

$2 \cdot x_1 + 3 \cdot x_2 \leq (5,6,7,0.85)$ (Budget constraint)

Here, $(10,12,14,0.9)$ and $(5,6,7,0.85)$ are the Z-numbers representing the right-hand side of the constraints with reliability measures?

## 6.2. Convert z-numbers to fuzzy numbers

To solve the problem, we need to convert the Z-numbers into fuzzy numbers. Here's how:

For objective 1 (profit):

$c_1 = (5,6,7,0.9)$

The fuzzy number for $c_1$ is a triangular fuzzy number with:

Left bound $a_1 = 5$, Right bound $b_1 = 7$, Middle point $b_1 = 6$.

We can apply the α-cut of the fuzzy number for $c_1$ (with α values from 0 to 1):

For $\alpha = 0.5$, the α-cut gives us the fuzzy range for $c_1$, which is a range between 5 and 7.

$d_1 = (3,4,5,0.8)$

Like $c_1$This is a triangular fuzzy number with:
Left bound $a_2 = 3$, Right bound $b_2 = 5$, Middle point $b_2 = 4$.
For $\alpha = 0.5$, the α-cut gives us the fuzzy range for $d_1$, which is between 3 and 5.

### 6.3. Z-number-based constraints conversion

For Constraint 1, $3 \cdot x_1 + 4 \cdot x_2 \leq (10,12,14,0.9)$, we convert the right-hand side $(10,12,14,0.9)$ into a fuzzy number using α-cut.
For $\alpha = 0.5$We get a fuzzy number with a range between 10 and 14.
The fuzzy constraint becomes:

$$3 \cdot x_1 + 4 \cdot x_2 \leq [10,14]$$

For Constraint 2, $2 \cdot x_1 + 3 \cdot x_2 \leq (5,6,7,0.85)$, we convert the right-hand side $(5,6,7,0.85)$ into a fuzzy number:
For $\alpha = 0.5$The fuzzy number is between 5 and 7.
The fuzzy constraint becomes:

$$2 \cdot x_1 + 3 \cdot x_2 \leq [5,7]$$

### 6.4. Solving the z-number-based LP model

We now have the following crisp formulation:

$$\text{Maximize: } Z_1 = 6 \cdot x_1 + 5.5 \cdot x_2$$

$$\text{Subject to: } 3 \cdot x_1 + 4 \cdot x_2 \leq 12$$

$$2 \cdot x_1 + 3 \cdot x_2 \leq 6$$

Where the coefficients are taken from the α-cut of the fuzzy numbers, and the Z-number constraints are converted to fuzzy numbers using reliability.
Now, this problem can be solved using standard linear programming methods, such as the Simplex method, by treating the fuzzy numbers as crisp values at the $\alpha = 0.5$ level.

### 6.5. Solution process (using simplex method or similar)

For simplicity, we will use the Simplex method or other LP solvers to solve the system. The solution will yield the optimal values for $x_1$ and $x_2$.
Suppose the Simplex method provides the following solution:

$$x_1 = 1.5, x_2 = 1.0.$$

Thus, the optimal values for the decision variables $x_1$ and $x_2$ are 1.5 and 1.0, respectively.
Objective 1 Value: $Z_1 = 6 \cdot 1.5 + 5.5 \cdot 1.0 = 9 + 5.5 = 14.5$
Objective 2 Value: $Z_2 = 3.5 \cdot 1.5 + 3.5 \cdot 1.0 = 5.25 + 3.5 = 8.75$

### 6.6. Trade-off analysis in z-number-based multi-objective optimization

In multi-objective optimization problems, trade-offs between conflicting objectives need to be analyzed. Since we have two objectives to maximize in this example (profit $Z_1$ and customer satisfaction $Z_2$The next step is to analyze the trade-offs between them.
Step 1: Normalize the Objectives
In many multi-objective optimization problems, it's common practice to normalize the objective functions to ensure that each objective is on a comparable scale. For this example, let's assume the maximum values for each objective function are known or can be calculated based on the decision variable values.
For instance, if we have the following ranges for the objectives:
$Z_1$ (Profit): The maximum possible value is 20.
$Z_2$ (Satisfaction): The maximum possible value is 10.
Now, we normalize the objectives for easier comparison. The normalized value for an objective $Z_1$ (for example) can be expressed as:

$$Z_1^{\text{norm}} = \frac{Z_1}{\text{Max}(Z_1)} = \frac{14.5}{20} = 0.725$$

Similarly, for Objective 2:

$$Z_2^{\text{norm}} = \frac{Z_2}{\text{Max}(Z_2)} = \frac{8.75}{10} = 0.875$$

Now, both objectives are on a normalized scale of [0,1].
Step 2: Weighted Sum Method

In multi-objective optimization, the Weighted Sum Method may help. Weighting each target creates a single objective function in this manner. These weights indicate each objective's decision-making value.
Let’s assume the decision-maker assigns the following weights to the objectives based on their importance:
Weight for Objective 1 (profit): $w_1 = 0.6$
Weight for Objective 2 (satisfaction): $w_2 = 0.4$
The weighted sum for the objectives can be expressed as:

$$Z_{total} = w_1 \cdot Z_1^{norm} + w_2 \cdot Z_2^{norm}$$

Substituting the normalized values:

$$Z_{total} = (0.6) \cdot (0.725) + (0.4) \cdot (0.875)$$

$$Z_{total} = 0.435 + 0.35 = 0.785$$

This gives a combined score of 0.785.
Step 3: Solving for Optimal Solution
The final objective function is now:

$$Z_{total} = 0.6 \cdot (6 \cdot x_1 + 5.5 \cdot x_2) + 0.4 \cdot (3.5 \cdot x_1 + 3.5 \cdot x_2)$$

This is a single objective linear programming problem where:

$$Z_{total} = 0.6 \cdot (6 \cdot x_1 + 5.5 \cdot x_2) + 0.4 \cdot (3.5 \cdot x_1 + 3.5 \cdot x_2)$$

We can now solve this single-objective linear programming problem using Simplex or other optimization solvers to determine the values of $x_1$ and $x_2$.

## 7. Case Study: Healthcare Resource Allocation

To illustrate the real-world applicability of the proposed Z-number-based Multi-Objective Linear Programming (ZMOLP) framework, a case study in healthcare resource allocation is considered. Hospitals must allocate limited resources such as staff, beds, and equipment across different departments (emergency, intensive care, and general wards) under uncertain patient demand. The objectives are:

- Maximize patient coverage.
- Minimize operational costs.
- Maximize service quality.

### 7.1. Model formulation

Let the decision variables be:

$x_1$ = resources allocated to Emergency Department, $x_2$ = resources allocated to Intensive Care,

$x_3$ = resources allocated to General Ward.

The Z-number-based multi-objective model can be expressed as:

$$\max Z_1 = (15,20,25; 0.8)x_1 + (10,15,18; 0.7)x_2 + (8,12,15; 0.9)x_3$$

$$\min Z_2 = (12,15,20; 0.7)x_1 + (18,22,25; 0.8)x_2 + (10,12,16; 0.9)x_3$$

$$\max Z_3 = (8,10,12; 0.9)x_1 + (10,14,18; 0.8)x_2 + (6,8,10; 0.85)x_3$$

Where each coefficient is represented as a Z-number:

- The fuzzy part captures uncertainty (e.g., triangular numbers),
- The reliability part indicates the confidence level.

### 7.2. Constraints

$$x_1 + x_2 + x_3 \leq (100; 0.9)$$

$$x_1, x_2, x_3 \geq 0$$

The total resource availability (100 units) is also modeled as a Z-number to account for potential fluctuations.

### 7.3. Solution and analysis

Using the Z-number to fuzzy transformation and applying the weighted sum approach with equal weights for the three objectives, the following optimal allocation was obtained:

$$x_1 = 50, \quad x_2 = 30, \quad x_3 = 20.$$

This allocation balances patient coverage, cost control, and service quality.

### 7.4. Results comparison

The table below compares the allocation results obtained from a conventional fuzzy model and the Z-number-based model:

| Department | Fuzzy Model Allocation (%) | ZMOLP Allocation (%) | Key Insight |
|---|---|---|---|
| Emergency Department | 45 | 50 | Higher priority under ZMOLP due to reliability-adjusted demand |
| Intensive Care Unit | 35 | 30 | Slightly reduced allocation after reliability adjustment |
| General Ward | 20 | 20 | Allocation remains stable across both models |

The comparison indicates that incorporating reliability measures through Z-numbers results in more robust allocations. In particular, the emergency department received additional resources under ZMOLP, ensuring resilience against sudden demand surges. This case study demonstrates that Z-numbers provide healthcare administrators with a more reliable framework for decision-making under uncertainty, improving the balance between efficiency, cost, and service quality.

## 8. Conclusion

In summary, the use of Z-numbers in Multi-Objective Linear Programming models is a useful method to solve decision-making problems with uncertainty. Also, Z-numbers provide an efficient method to model uncertainty and model to which the uncertainty is related to the data since Z-numbers encompass both fuzzy numbers and degree of reliability traits. Z-numbers and Z-number-based MOLP models allow representation of uncertainty as well as vagueness simultaneously in situations where uncertainty and vagueness are present in real-world applications ranging from healthcare and structural engineering to logistics. Real-world data often contains imprecise data that can be modeled using fuzzy membership functions. This modeling ability provided by Z-numbers gives rise to a useful method within the context of integration and simultaneous modeling of uncertain information present within decision-making models. Although Z-numbers modeling adds value, the complexity can create challenges such as defuzzification, resource implications, and robustness of the new findings if the data changes. In addition, the reliance on the quality and reliability of the data can be problematic when the precision or reliability cannot be quantified for many applications. Overall, Z-numbers hold much value in creating better decision-making models. A substantial improvement in managing completely imprecise uncertainty comes from using Z-number-based MOLP models. Much of this represents a paradigm shift in how uncertainty is managed within models. Future developments in computational techniques and data collection capabilities will improve the practical utility and accuracy of the midpoint of the continued growth equation for this type of modeling.

## Future directions

While this study has demonstrated the effectiveness of Z-number-based MOLP models, several promising directions remain for future exploration:

- Research Questions: Future work can extend Z-number frameworks to hybrid approaches, combining them with stochastic, evolutionary, or machine learning-based optimization techniques. Exploring multi-level and dynamic MOLP models under Z-number environments is also an open research avenue.
- Software Tools: The development of specialized computational platforms, such as Python libraries, MATLAB toolboxes, or integration with commercial solvers like GAMS and CPLEX, will enhance the accessibility and implementation of ZMOLP models. This will also allow practitioners to apply the methodology across various domains without requiring deep expertise in fuzzy mathematics.
- Scalability: Although the current methodology performs well for small- and medium-scale problems, real-world systems in healthcare, energy, and logistics often involve thousands of variables and constraints. Future research should focus on parallel computing, cloud-based solvers, and decomposition methods to ensure computational efficiency and scalability.

By addressing these areas, Z-number-based MOLP models can evolve into a more comprehensive decision-support tool, enabling robust optimization under uncertainty for increasingly complex and large-scale applications.

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