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Research paper

Numerical Solutions of Third Kind Volterra Integral Equations Using Simpson's Rule

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Abstract

Volterra integral equations appear when we convert an initial value problem to an integral equation. The solution of the Volterra integral equation is much easier than the original initial value problem. Many problems of thermodynamics, biology, chemistry, medical sciences, physics, heat flow, neutron diffusion problem, electric circuit problem, transform problem mechanics, and engineering can be represented mathematically in terms of Volterra integral equations of the first and second kind. Numerical Method for solving Third Kind Volterra Integral Equations (VIEs) using Simpson's Rule. By approximating the unknown function using Lagrange Polynomial interpolation and applying Simpson's rule for numerical integration, we obtain a system of linear algebraic equations. The solution of this system provides an approximation to the original integral equation.

Keywords: *Volterra Integral equation; Simpson's 1/3 rd Rule; Simpson's 3/8 th Rule.*

1. Introduction

Volterra was the first to recognize the importance of the theory and studied it very systematically [12]. Volterra integral equations arise in many scientific applications, such as population dynamics, the spread of epidemics, and semiconductor devices. It is known that many problems of the natural sciences are reduced to the solving of integral equations of variable boundaries, which are called integral equations of Volterra type [3]. Vito Volterra (proud Italian) fundamentally investigated these equations and also reduced the mathematical models of many problems of the natural sciences to solve these integral equations. It was also shown that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908 [3]. Wazwaz proposed the Laplace transform method and the Successive approximation method for solving the Volterra integral equation of the first and second kinds [2]. The advantage of that approach is that it reduces the problem; equations make the method efficient and easy to implement [12]. For the two-dimensional Volterra integral equations, there are only a few introductions [20]. Some of the most well-known numerical techniques used to approximate solutions of integral equations are multistep methods, spectral methods, product integration method, Picard's method, Adomain decomposition method, etc. [16]. In 1911, the Third kind of Volterra integral equations with various types of kernel singularities were studied by Evans [17]. The Simpson's 3/8 rule is used to solve the nonlinear Volterra integral equations system. Using this rule, the system is converted to a nonlinear block system, and then by solving this nonlinear system, we find an approximate solution of the nonlinear Volterra integral equations system [2]. Hafez and Youssri used spectral relationships in the form of Legendre-Chebyshev to discuss the numerical solution of nonlinear VIE with a stable kernel [10]. The solution of the integral equation using the Trapezoidal Rule and Simpson's Rule [11].

Volterra Integral Equation –
A linear integral equation of the form

$$n(x)m(x) = f(x) + \lambda \int_a^x k(x,t)m(t)dt \qquad (1)$$

Where the upper limit of the integral is variable, $n(x), f(x), k(x,t)$ are known functions and $m(x)$ It is an unknown function, said to be a Volterra integral equation of the third kind. If λ is a real or complex parameter, and the function $k(x,t)$ Is the kernel of the integral equation [5].

First Kind of Volterra Integral Equation-
If we set $n(x) = 0$ In equation (1),

$$f(x) + \lambda \int_a^x k(x,t)m(t)dt = 0 \quad (2)$$

Then it is called the Volterra Integral equation of the first kind.

Second kind of Volterra Integral Equation-
Volterra integral equations of the second kind, the unknown function m(x) appears inside and outside the integral sign.

$$m(x) = f(x) + \lambda \int_a^x k(x,t)m(t)dt \quad (3)$$

It is called a homogeneous Volterra integral equation of the second kind.

Third Kind of Volterra Integral Equation-
Consider the Volterra integral equation of the third kind,

$$x^\beta m(x) = f(x) + \int_0^x (x-t)^{-\alpha} K(x,t)m(t)dt \quad (4)$$

Where $x\epsilon[0,T], \alpha\epsilon[0,1], \beta > 0, \propto +\beta > 0$ and f(x) Represents a continuous function on I. That is $f(x) = x^\beta g(x)$, m(x) is an unknown function that is to be determined. Now, $k(x,t)$ is continuous on $\Delta= \{k(x,t)/0 \le t \le x \le T\}$ and defined by $k(x,t) = x^{\propto+\beta-1}k_1(x,t)$. Where k_1 It is a continuous function.

Simpson's $\frac{1}{3}$rd Rule –
Let the interval [a, b] be divided into n even subintervals. That is $h = \frac{b-a}{n}$
Define the nodes $x_0 = a, x_1 = h, x_2 = 2h, \ldots, x_n = b$
Then,

$$\int_0^x f(t)dt \approx \frac{h}{3}\left[f(t_0) + 4\sum_{odd\ i=1}^{n-1} f(t_i) + \sum_{even\ i=2}^{n-2} f(t_i) + f(t_n)\right] \quad (5)$$

But we define $f(t) = (x-t)^{-\propto}k(x,t)m(t)$
From equation (5),

$$\int_0^x (x-t)^{-\propto}k(x,t)m(t)dt \approx \frac{h}{3}\left[f(t_0) + 4\sum_{odd\ i=1}^{n-1} f(t_i) + \sum_{even\ i=2}^{n-2} f(t_i) + f(t_n)\right] \quad (6)$$

To find the value of the integral is appropriate to substitute equation (5) into equation (6), then.

$$x^\beta m(x) \approx f(x) + \frac{h}{3}\left[f(t_0) + 4\sum_{odd\ i=1}^{n-1} f(t_i) + \sum_{even\ i=2}^{n-2} f(t_i) + f(t_n)\right]$$

$$m(x) \approx \frac{f(x)}{x^\beta} + \frac{h}{3x^\beta}\left[f(t_0) + 4\sum_{odd\ i=1}^{n-1} f(t_i) + \sum_{even\ i=2}^{n-2} f(t_i) + f(t_n)\right] \quad (7)$$

To find the value of m(x).

The Convergence Theorem (Error bound) –
The convergence theorems of Simpson's $\frac{1}{3}$ Rd. Rule,

$$\text{Error} = -\frac{(b-a)}{180} h^4 f^4(\xi), \text{for some } \xi \in (a,b) \quad (8)$$

Assuming that the fourth $f^4(x)$ exists and is continuous on [a , b]
The error is proportional to h^4. So, the rule converges very quickly as you decrease h.

1) Simpson's rule is a fourth-order method.
2) If the fourth derivative of the function is small or zero. Simpson's rule can give exact results.

Interpretation of Convergences –
If $f \in C^4[a,b]$ The fourth derivative exists and is continuous.
Then $\lim_{n\to\infty}$ Simpson′s approximation $= \int_a^b f(x)dx$
This is a convergence part of the theorem as $n \to \infty$, $h \to 0$ And the approximation convergence to the exact value of the integral.

Example I) –

The Volterra integral equation of the Third kind,

$$x\,u(x) = \frac{6}{7}x^3\sqrt{x} + \int_0^x \frac{1}{2}u(t)dt, x \in [0,2]$$

Where the exact solution is $u(x) = x^{\frac{5}{2}}$. By using Simpson's $\frac{1}{3}$rd Rule for $n = 15$.

Put $x = 0, 0.14, 0.29, \dots, 2$ to obtain a solution of the exact equation $u(x) = x^{\frac{5}{2}}$ then
$u(0) = 0$
$u(0.14) = 0.007714$
$u(0.29) = 0.043634$ and so on up to $u(2) = 5.656854$.
Using MATLAB or Python to find the Numerical solution of Simpson's $\frac{1}{3}$ Rd Rule.

Table 1: Absolute errors of $u(x)$ at $n = 15$ for Example I

X	Simpson's $\frac{1}{3}$ rd Numerical Solution u(x)	Exact Solution u(x)	Error
0	0	0	0.00E+00
0.14	0.008815	0.007714	1.10E-03
0.29	0.045263	0.043634	1.63E-03
0.43	0.122267	0.120243	2.02E-03
0.57	0.249189	0.246834	2.36E-03
0.71	0.433836	0.431201	2.63E-03
0.86	0.6831	0.680194	2.91E-03
1	1.003181	1	3.18E-03
1.14	1.397349	1.396304	1.05E-03
1.29	1.877875	1.874395	3.48E-03
1.43	2.442939	2.439242	3.70E-03
1.57	3.09943	3.095541	3.89E-03
1.71	3.851795	3.84776	4.03E-03
1.86	4.704282	4.700167	4.11E-03
2	5.661259	5.656854	4.40E-03

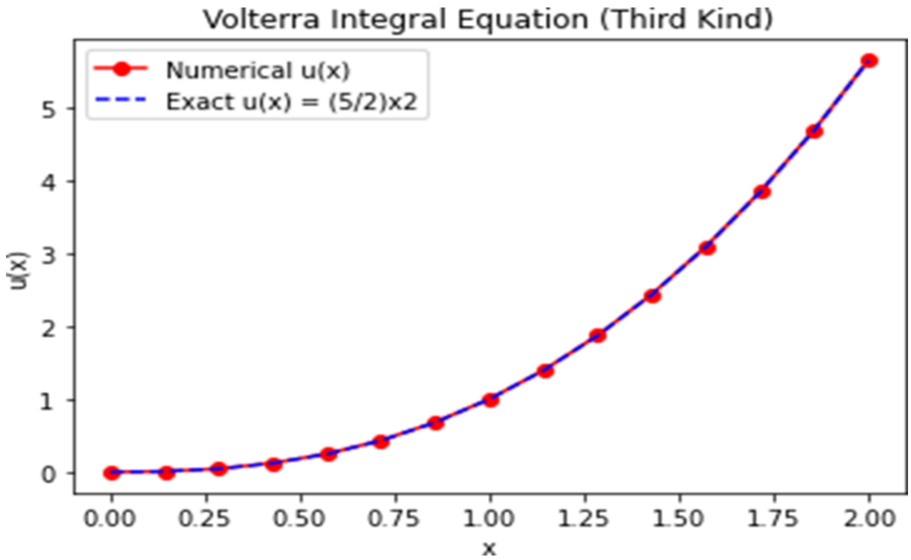

Fig. 1: Comparison of the exact solution and numerical solutions for Example I

Above Figure 1 shows the comparison of the exact solution and the numerical solution of Simpson's $\frac{1}{3}$ Rd Rule.

Example II) –

The Volterra integral equation of the Third kind,

$$x\,u(x) = x^2\left(1 - \frac{x}{3}\right) + \int_0^x t\,u(t)dt, x \in [0,2]$$

Where the exact solution is $u(x) = x$. By using Simpson's $\frac{1}{3}$ rd Rule for $n = 15$.
Put $x = 0, 0.14, 0.29, \dots, 2$ to obtain a solution of the exact equation, then
$u(0) = 0$
$u(0.14) = 0.142857$
$u(0.29) = 0.285714$ and so on up to $u(2) = 2$.
Using MATLAB or Python to find the Numerical solution of Simpson's $\frac{1}{3}$ Rd Rule.

Table 2: Absolute errors of $u(x)$ at $n = 15$ For example II

X	Simpson's one-third solution u(x)	Exact Solution u(x)	Absolute Error
0	0	0	0
0.14	0.142857	0.142857	0
0.29	0.285714	0.285714	0
0.43	0.428571	0.428571	0
0.57	0.571429	0.571429	0
0.71	0.714286	0.714286	0
0.86	0.857143	0.857143	0

1	1	1	0
1.14	1.142857	1.142857	0
1.29	1.285714	1.285714	0
1.43	1.428571	1.428571	0
1.57	1.571429	1.571429	0
1.71	1.714286	1.714286	0
1.86	1.857143	1.857143	0
2	2	2	0

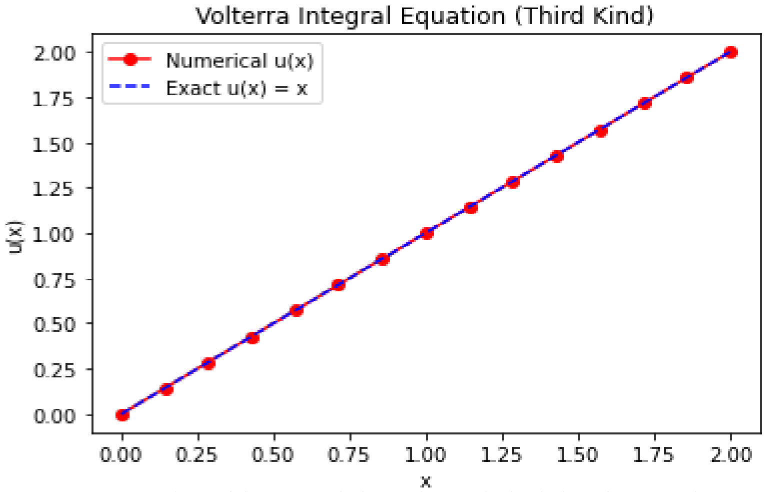

Fig. 2: Comparison of the exact solution and numerical solutions for Example II

Above Figure 2 shows the comparison of the exact solution and the numerical solution of Simpson's $\frac{1}{3}$ Rd Rule.

Simpson's $\frac{3}{8}$ th Rule –

Let the interval [a, b] be divided into n even subintervals. That is $h = \frac{b-a}{n}$ And n must be a multiple of 3, i.e. $n = 3m$

Define the nodes $x_0 = a, x_1 = h, x_2 = 2h, \ldots, x_n = b$

Then,

$$\int_0^x f(t)dt \approx \frac{3h}{8}\left[f(t_0) + 3\sum_{i=1}^{n-1} f(t_i) + f(t_n)\right] \quad (17)$$

But we define $f(t) = (x-t)^{-\alpha} k(x,t) m(t)$

From equation (17),

$$\int_0^x (x-t)^{-\alpha} k(x,t) m(t) dt \approx \frac{3h}{8}\left[f(t_0) + 3\sum_{i=1}^{n-1} f(t_i) + f(t_n)\right] \quad (18)$$

To find the value of the integral is appropriate to use equation (18) in equation (17).

$$x^{\beta} m(x) \approx f(x) + \frac{3h}{8}\left[f(t_0) + 3\sum_{i=1}^{n-1} f(t_i) + f(t_n)\right]$$

$$m(x) \approx \frac{f(x)}{x^{\beta}} + \frac{3h}{8x^{\beta}}\left[f(t_0) + 3\sum_{i=1}^{n-1} f(t_i) + f(t_n)\right] \quad (19)$$

To find the value of $m(x)$.

The Convergence Theorem (Error bound) –

The convergence theorems of Simpson's $\frac{3}{8}$ Th. Rule

$$\text{Error} = -\frac{(b-a)}{80} h^4 f^4(\xi), \text{ for some } \xi \in (a,b) \quad (20)$$

Assuming that the fourth $f^4(x)$ exists and is continuous on [a , b]

Example – III)

To find the Volterra integral equation of the Third Kind

$$x\, m(x) = x^2\left(1 - \frac{x}{3}\right) + \int_0^x t\, m(t)\, dt, x \in [0,1]$$

And the exact solution is $m(x) = x$. Use $n = 15$ Subintervals. By using Simpson's $\frac{3}{8}$ The Rule.

Put $x = 0, 0.071, 0.143, \ldots, 1$ to obtain a solution of the exact equation, then

$m(0) = 0$

$m(0.071) = 0.071429$
$m(0.143) = 0.142857$ and so on up to $m(1) = 1$.
Using MATLAB or Python to find the Numerical solution of Simpson's $\frac{3}{8}$ The Rule.

Table 3: Absolute errors of $m(x)$ at $n = 15$ For example III

x	Simpson's Three-Eighths solution m(x)	Exact Solution m(x)	Absolute Error
0	0	0	0.00E+00
0.071	0.071367	0.071429	6.16E-05
0.143	0.142607	0.142857	2.50E-04
0.214	0.213714	0.214286	5.72E-04
0.286	0.28468	0.285714	1.03E-03
0.357	0.355497	0.357143	1.65E-03
0.429	0.426159	0.428571	2.41E-03
0.5	0.496656	0.5	3.34E-03
0.571	0.566983	0.571429	4.45E-03
0.643	0.63712	0.642857	5.74E-03
0.714	0.707069	0.714286	7.22E-03
0.786	0.776813	0.785714	8.90E-03
0.857	0.846345	0.857143	1.08E-02
0.929	0.915648	0.928571	1.29E-02
1	0.984715	1	1.53E-02

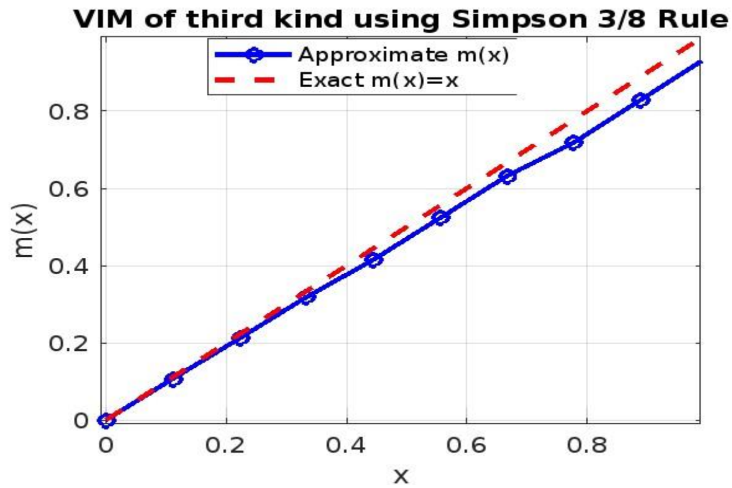

Fig. 3: Comparison of the exact solution and numerical solutions for Example III

Above Figure 3 shows the comparison of the exact solution and the numerical solution of Simpson's $\frac{3}{8}$ The Rule.

Example – IV

The Volterra integral equation of the Third kind,

$$x^{\frac{3}{2}}\, m(x) = f(x) + \int_0^x \frac{\sqrt{2}}{2\pi}\, t\,(x-t)^{-\frac{1}{2}}\, m(t)dt, x \in [0,1]$$

Where $f(x) = x^{\frac{33}{10}}\left(1 - \frac{\Gamma(\frac{19}{5})}{\sqrt{2\pi}\,\Gamma\frac{43}{10}}\right)$ And the exact solution is $m(x) = x^{\frac{9}{5}}$. Use $n = 15$
Subintervals. By using Simpson's $\frac{3}{8}$ The Rule.
Put $x = 0, 0.066667, 0.0.133333, \ldots, 1$ to obtain a solution of the exact equation, then
$m(0) = 0$
$m(0.066667) = 0.007639$
$m(0.133333) = 0.0266$ and so on up to $m(1) = 1$.
Using MATLAB or Python to find the Numerical solution of Simpson's $\frac{3}{8}$ th Rule

Table 4: Absolute errors of $m(x)$ at $n = 15$ For example IV

X	Numerical Solution m(x)	Exact Solution m(x)	Absolute Error
0	0	0	0
0.066667	0.006024	0.007639	0.001615
0.133333	0.020975	0.0266	0.005625
0.2	0.04577	0.055189	0.009419
0.266667	0.076986	0.092628	0.015642
0.333333	0.1182	0.138415	0.020214
0.4	0.166791	0.19218	0.025389
0.466667	0.21818	0.253637	0.035456
0.533333	0.282925	0.32255	0.039625
0.6	0.353952	0.398724	0.044772
0.666667	0.423145	0.481987	0.058842

0.733333	0.510237	0.572193	0.061956
0.8	0.602585	0.669209	0.066625
0.866667	0.687997	0.772919	0.084922
0.933333	0.796648	0.883215	0.086566
1	0.909534	1	0.090466

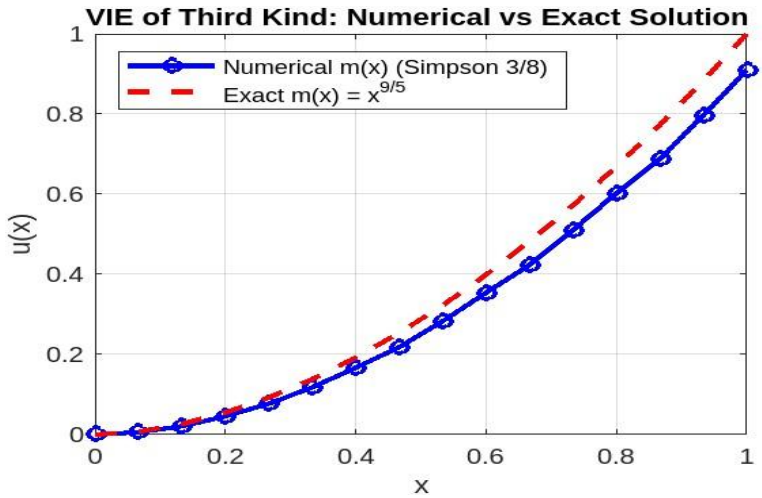

Fig. 4: Comparison of the exact solution and numerical solutions for Example IV

Above Figure 4 shows the comparison of the exact solution and the numerical solution of Simpson's $\frac{3}{8}$ The Rule.

2. Conclusion

In this paper, we solved successfully the Volterra integral equation of the third kind by Simpson's 1/3rd Rule and Simpson's 3/8th Rule. The numerical solution of Volterra integral equations of the third kind using Simpson's Rule provides an effective and reliable approximation technique. By discretizing the integral operator and applying Simpson's 1/3 rule, Simpson's 3/8th Rule, the problem is reduced to a system of algebraic equations that can be solved step by step. The method ensures higher-order accuracy compared to simple quadrature rules such as the trapezoidal method, provided that the integrand is sufficiently smooth. The results obtained demonstrate that Simpson's Rule achieves good agreement with the exact solution, and the error decreases significantly as the number of subintervals increases. The Simpson's rule-based method provides an efficient and accurate numerical solution for third-kind Volterra integral equations, making it a valuable tool in applied mathematics, physics, and engineering applications. The methods' convergence and error estimation ensure reliable results.

3. Future Scope

Future scope on solving Volterra integral equations of the third kind can focus on adaptive Simpson's methods, hybrid techniques, and high-performance computing to improve accuracy and efficiency. Parallel computing and extensions to multi-dimensional problems can broaden applicability. Expanding applications in science and engineering will further enhance the method's practical relevance.

4. Limitations

Simpson's rule is simple and effective for smooth problems; its limitations include the need for even subdivisions, error accumulation in iterative integral equations, inefficiency for oscillatory or singular kernels, high computational cost, and possible instability in stiff third-kind Volterra equations. Simpson's Rule is widely taught in numerical analysis, making it intuitively familiar and easy to implement in computational environments like MATLAB or Python.

References

[1] Abdul Majid Wazwaz, *"Linear and Non-linear integral equations Method and Applications,"* Higher Education Press, Beijing, and Springer – Verlag Berlin Heidelberg 2011.

[2] Adem Kılıc¸man, L. Kargaran Dehkordi, M. Tavassoli Kajani "*Numerical Solution of Nonlinear Volterra Integral Equations System Using Simpson's 3/8 Rule"* Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2012, Article ID 603463, 16 pages https://doi.org/10.1155/2012/603463.

[3] D.C. Sharma and M.C. Goyal, "*Integral Equations"* PHI Learning Private Limited, Delhi 110092, 2017.

[4] Danna Wu, Weishan Zheng, and Yanfeng Chen. *"Error Estimates of a Symmetric Spectral Method for a Linear Volterra Integral Equation"* Symmetry 2023, 15, 60. https://doi.org/10.3390/sym15010060.

[5] Dr. Subhamay Dutta, "*The Method of Successive Approximation (Neumann's Series) of Volterra Integral Equation of the second kind, "Pure* and Applied Mathematical Journal Volume 5 Issue 6 Pages 211-219. https://doi.org/10.11648/j.pamj.20160506.16.

[6] G.A. Afrouzi, D. D. Ganji, H. Hosseinzadeh, R.A. Talarposhti *"Fourth order Volterra integro-differential equations using modified Homotopy-perturbation method"* The Journal of Mathematics and Computer Science Volume 3 No 2 (2011) 179-191. https://doi.org/10.22436/jmcs.03.02.10.

[7] H. Tajadodi *"Spectral Method for Third-Kind Volterra Integral Equation"* Journal of Mathematical Extension Vol. 15, SI-NTFCA, (2021) (29)1-17.

[8] Imtiyaz Ahmad Bhat and Lakshmi Narayan Mishra *"Numerical Solutions of Volterra Integral Equations of Third Kind and Its Convergence Analysis"* Symmetry 2022, 14, 2600. https://doi.org/10.3390/sym14122600.

[9] I.G. Burova, A.G. Doronina, D.E. Zhilin *"Splines of the Fourth Order Approximation and the Volterra Integral Equations"* Wseas Transactions on Mathematics, Volume 20, 2021 https://doi.org/10.37394/23206.2021.20.2.

[10] M. A. Elsayed *"Some Different Methods via the Solution of Volterra Integral Equation"* Applied Mathematics & Information Sciences. 16, No. 6, 973-981 (2022) https://doi.org/10.18576/amis/160614.

[11] M. A. Abdou, A. A. Abbas *"New technique of two numerical methods for solving integral equation of the second kind"* IOSR Journal of Engineering (IOSRJEN) Vol. 08, Issue 9 (September 2018), Pages 34-40.

[12] M.Rahman, "*Integral equations and their applications,"* WIT Press.

[13] Mustapha Yahaya_1 And Sirajo Lawan Bichi "*Application of Simpson's method for solving singular Volterra integral equation"* Malaya Journal of Matematik 9(04) (2021), 206–215. https://doi.org/10.26637/mjm904/004.

[14] P. Jami1, E. Hashemizadeh *"A Method for Numerical Solution of Third-Kind Volterra Integral Equations Using Krall-Laguerre Polynomials" Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*

[15] Poonam Jagtap, Avinash Khambayat "*Coparision of Volterra Integral Equation by Successive Approximation and Adomain Decomposition Methods"* Journal of Emerging Technologies and Innovative Research (JETIR) Volume 11, Issue 8 August 2024.

[16] Sonia Seyed Allaei, Zhan-Wen Yang and Hermann Brunner *"Existence, Uniqueness and Regularity of Solutions to A Class of Third-Kind Volterra Integral Equations"* Journal of Integral Equations and Applications Volume 27, Number 3, Fall 2015. https://doi.org/10.1216/JIE-2015-27-3-325.

[17] Sudhanshu Aggarwal, Aakansha Vyas "*Laplace Transform for the solution of Non-Linear Volterra Integral equation of second kind"* Journal of Advanced Research in Applied Mathematics and Statistics Volume 8 Issue 3& 4 Page no 18-25 https://doi.org/10.24321/2455.7021.202304.

[18] Sahar Muhar Jaabar, Ahmed Hadi Hussain "*A Move Recent Review of the Integral Equations and their applications"* Journal of Physics: conference seric 1818 (2021) 012170. https://doi.org/10.1088/1742-6596/1818/1/012170.

[19] Y. Talaeia, P. M. Limab "An efficient spectral method for solving third-kind Volterra integral equations with non-smooth solutions" *July 18, 2022. https://doi.org/10.1007/s40314-023-02333-7.*

[20] Zana Mohammed Hassan, Sadeq Taha Abdulazeez, Sarbast Kamal Rasheed "*Investigating Solutions of Volterra Integral Equations Using the Successive Approximations"* Mathematical Statistician and Engineering Applications Volume 72 Issue 2 2023 Pages 75-82.