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Research paper

On Vertex Eccentricity Labeled Energy of A Graph

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Abstract

In this paper, we introduce a new spectral graph invariant called the Vertex Eccentricity Labeled Energy (VELE). VELE is derived from the eigenvalues of the Vertex Eccentricity Labeled (VEL) matrix, whose off-diagonal entries for a connected graph G are defined as the sum of eccentricities of vertex pairs, and zero otherwise. We explore fundamental algebraic and spectral properties of VELE, including its trace, characteristic polynomial, and energy bounds. Closed-form formulas for VELE are provided for several classical graph families, such as complete graphs, cycles, wheels, and stars. Additionally, the behavior of VELE is studied under common graph operations such as Cartesian product, splitting, and m-splitting, highlighting distinctive spectral patterns related to vertex eccentricity. These results expand the family of spectral descriptors and offer new analytical tools in structural graph analysis.

Keywords: *Eccentricity; Graph Energy; Vertex Eccentricity Labeled Energy.*

1. Introduction

Graph theory is fundamental in modelling complex structures across chemistry, physics, communication, transportation, and various other fields [28], [4]. A graph $G = (V, E)$ comprises a finite vertex set V and an edge set E connecting pairs of vertices. All graphs considered here are finite, simple, undirected, and connected, unless otherwise stated.

Two vertices $u, v \in V(G)$ are adjacent, denoted $u \sim v$, if $\{u, v\} \in E(G)$. The degree $d(v)$ of a vertex v is the number of its neighbors. The distance $d(u, v)$ is the length of the shortest path connecting u and v. The eccentricity $e(v)$ of a vertex v is defined as the maximum distance between v and any other vertex in the graph.

A graph is called self-centered (or d-self-centered) if all its vertices share the same eccentricity d. The maximum eccentricity in a graph is called its diameter, while the minimum is its radius.

Spectral graph theory analyzes structural properties through eigenvalues of graph-associated matrices [28], [11]. The adjacency matrix $A(G) = [a_{ij}]$ of a graph G of order n is defined as: $a_{ij} = 1$ if $v_i \sim v_j$, and 0 otherwise. Since $A(G)$ is symmetric, all its eigenvalues are real.

The concept of graph energy, introduced by Gutman in 1978 [10], models the total π-electron energy in molecular orbital theory [16], [28] and is defined as the sum of the absolute values of the eigenvalues of $A(G)$: $E(G) = \sum |\lambda_i|$. Foundational studies by McClelland and others [6], [22–24] explored bounds and chemical implications. Extremal results by Caporossi, Gutman, Koolen, Moulton, Brualdi, and Solheid [12], [13], as well as hypoenergetic and hyperenergetic graph studies [20], further developed the field.

Inspired by classical graph energy, several related invariants have been introduced, based on matrices such as the Laplacian [21], distance [13], signless Laplacian [19], eccentricity [2], and Randić matrices [26]. These variants emphasize different structural aspects and have broad applicability [28], [4], [26]. Surveys by Gutman, Polansky, Trinajstić, and Bonchev [28], [4], [11] highlight the proliferation of such energy-based measures.

Recent studies have extended these concepts to signless Laplacian energy [5], [9], eccentricity Laplacian energy [15], [25], generalized distance energy [17], and new bounds based on vertex cover and spectral radius [1], [7]. Further constructions such as Laplacian borderenergetic graphs [8] illustrate the growing depth of spectral invariants.

Unlike classical energy variants, which are typically based on degree or adjacency matrices, the Vertex Eccentricity Labeled Energy (VELE) is constructed from global vertex eccentricities. In VELE, the eccentricities appear in the off-diagonal entries of the matrix, contrasting with eccentricity energy, where they appear only on the diagonal. VELE thus captures extremal distance interactions between vertex pairs in a unique spectral framework.

For a connected graph G with vertices $v_1, \dots, v_n$We define the vertex eccentricity labeled matrix $V_{EL}(G) = [m_{ij}]$ by: $m_{ij} = e(v_i) + e(v_j)$ if $i \neq j$; 0 if $i = j$. This matrix is symmetric, and its eigenvalues $\xi_1, \dots, \xi_n$ are real. The Vertex Eccentricity Labeled Energy is defined as: $VELE(G) = \sum |\xi_i|$.

In this paper, we study VELE in detail, derive closed-form expressions for classical graphs, and examine its behavior under graph operations such as Cartesian product, splitting, and m-splitting. These results enrich the theory of energy-like invariants and offer new tools for spectral graph analysis.
The paper is structured as follows: Section 2 formalizes the VEL matrix and VELE. Section 3 presents algebraic results and bounds. Section 4 derives closed-form VELE expressions for classical graphs. Section 5 explores VELE under graph operations. We conclude with a summary and suggestions for future research.

2. Preliminaries

Let $G = (V, E)$ be a simple, undirected, connected graph of order n, with vertex set $V = \{v_1, v_2, \dots, v_n\}$. The eccentricity of a vertex v_i is defined as

$$e(v_i) = \max_{v_j \in V(G)} d(v_i, v_j),$$

where $d(v_i, v_j)$ is the shortest path distance between v_i and v_j.
The vertex eccentricity labeled matrix $V_{EL}(G) = [m_{ij}]$ of a graph G by:

$$m_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if } i \neq j \text{ and there exists a path between } v_i \text{ and } v_j, \\ 0, & \text{if } i = j \text{ or } v_i \text{ and } v_j \text{ are disconnected.} \end{cases}$$

Since G is connected, all pairs of distinct vertices are connected by a path, so the off-diagonal entries are well-defined. The matrix $V_{EL}(G)$ is real and symmetric, ensuring real eigenvalues $\xi_1, \xi_2, \dots, \xi_n$.
The vertex eccentricity labeled energy (VELE) of G is defined as:

$$V_{EL}E(G) = \sum_{i=1}^{n} |\xi_i|.$$

The spectrum of $V_{EL}(G)$ is the multiset of its eigenvalues. the vertex eccentricity labeled spectrum of G as:

$$V_{EL}Spec(G) = \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_r \\ m_1 & m_2 & \dots & m_r \end{pmatrix},$$

Where $\xi_1 > \xi_2 > \dots > \xi_r$ are the distinct eigenvalues and m_i denotes the multiplicity of ξ_i.

Definition 1: *Two graphs G_1 and G_2 are said to be vertex eccentricity labeled cospectral if*

$$V_{EL}Spec(G_1) = V_{EL}Spec(G_2).$$

They are called vertex eccentricity labeled equienergetic if

$$V_{EL}E(G_1) = V_{EL}E(G_2).$$

Illustration 1: Consider the graph G with six vertices shown below. Each vertex has eccentricity 2:

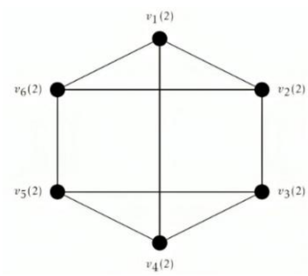

Fig. 1: Graph G.

Since all vertices have eccentricity 2, the VEL matrix is:

$$V_{EL}(G) = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 & 4 \\ 4 & 4 & 4 & 0 & 4 & 4 \\ 4 & 4 & 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 4 & 4 & 0 \end{bmatrix}$$

Its characteristic polynomial is:
$\det(\xi I - V_{EL}) = (\xi - 20)(\xi + 4)^5$

Hence, the VELE spectrum is:

$V_{EL}Spec(G) = \begin{pmatrix} 20 & -4 \\ 1 & 5 \end{pmatrix}$ and $V_{EL}E(G) = |20| + 5 \cdot |-4| = 20 + 20 = 40.$

The following known linear algebra results will be used in later sections.
Proposition 1 [27] If M is an $n \times n$ block matrix of the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then $\det(M) = \det(A) \cdot \det(D - CA^{-1}B)$, provided A is nonsingular.
Proposition 2 [18] Let J be the $n \times n$ matrix of all ones and I the $n \times n$ identity matrix. Then the eigenvalues of $aI + bJ$ are $a + nb$ (with multiplicity 1), and a (with multiplicity $n - 1$).

3. Algebraic and spectral results for vertex eccentricity labeled energy

Theorem 1: Let $\xi_1, \xi_2, \ldots, \xi_n$ be the eigenvalues of the vertex eccentricity labeled matrix $V_{EL}(G)$. Then:

$\sum_{i=1}^{n} \xi_i = 0.$

$\sum_{i=1}^{n} \xi_i^2 = 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 = 2H,$

Where $H = \sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2$.
Proof. Since $V_{EL}(G)$ has all diagonal entries equal to 0, it follows that $\mathrm{trace}(V_{EL}(G)) = 0$. But the trace of a matrix is also the sum of its eigenvalues. Hence,

$\sum_{i=1}^{n} \xi_i = \mathrm{trace}(V_{EL}(G)) = 0.$

By a standard result on eigenvalues,

$\sum_{i=1}^{n} \xi_i^2 = \mathrm{trace}([V_{EL}(G)]^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\, m_{ji},$

Where m_{ij} is the (i, j)-entry of $V_{EL}(G)$. Note that $m_{ii} = 0$ for all i, so the diagonal terms do not contribute. For $i \neq j$, $m_{ij} = e(v_i) + e(v_j)$ and $m_{ji} = m_{ij}$. Thus,

$\sum_{i=1}^{n} \xi_i^2 = \sum_{i\neq j} (m_{ij})^2 = 2\sum_{1\leqslant i<j\leqslant n} (m_{ij})^2 = 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2.$

Set

$H = \sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2.$

It follows immediately that

$\sum_{i=1}^{n} \xi_i^2 = 2H.$

Theorem 2: *Let G be a graph of order n, and let the characteristic polynomial of $V_{EL}(G)$ be*

$P(G, \lambda) = a_0\, \lambda^n + a_1\, \lambda^{n-1} + a_2\, \lambda^{n-2} + \cdots + a_n.$

Then:

$a_0 = 1.$

$a_1 = 0.$

$a_2 = -\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 = -H.$

$a_3 = -2\sum_{1\leqslant i<j<k\leqslant n} [(e(v_i) + e(v_j))\,(e(v_i) + e(v_k))\,(e(v_j) + e(v_k))].$

For $n > 1$, $a_n = (-1)^n \det[V_{EL}(G)]$.

Proof. Characteristic polynomials of $n \times n$ matrices are monic, so $a_0 = 1$. By definition, $a_1 = -\mathrm{trace}(V_{EL}(G))$. Since $V_{EL}(G)$ has 0 on its diagonal, $\mathrm{trace}(V_{EL}(G)) = 0$. Thus $a_1 = 0$.
The coefficient a_2 is (up to sign) the sum of the principal 2×2 minors of $V_{EL}(G)$. Specifically,

$$a_2 = \sum_{1\leqslant i<j\leqslant n} \begin{vmatrix} 0 & m_{ij} \\ m_{ji} & 0 \end{vmatrix} = \sum_{1\leqslant i<j\leqslant n} (0 - m_{ij}\, m_{ji}) = -\sum_{1\leqslant i<j\leqslant n} (m_{ij})^2.$$

Since $m_{ij} = e(v_i) + e(v_j)$ for $i \neq j$, we get
$a_2 = -\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 = -H.$

The coefficient a_3 corresponds to the sum of all 3×3 principal minors of $V_{EL}(G)$. Each 3×3 principal submatrix looks like

$$\begin{bmatrix} 0 & m_{ij} & m_{ik} \\ m_{ji} & 0 & m_{jk} \\ m_{ki} & m_{kj} & 0 \end{bmatrix},$$

Whose determinant is $-2\, m_{ij}\, m_{ik}\, m_{jk}$. Therefore,

$$a_3 = -2\sum_{1\leqslant i<j<k\leqslant n} m_{ij}\, m_{ik}\, m_{jk} = -2\sum_{1\leqslant i<j<k\leqslant n} (e(v_i) + e(v_j))\,(e(v_i) + e(v_k))\,(e(v_j) + e(v_k)).$$

In general, the constant term a_n of the characteristic polynomial of an $n \times n$ matrix M equals $(-1)^n \det(M)$. Thus,

$$a_n = (-1)^n \det[V_{EL}(G)].$$

Theorem 3: *Let G be a graph of order n, and let $V_{EL}E(G)$ denote the vertex eccentricity labeled energy of G. Then*

$$V_{EL}E(G) \leqslant \sqrt{2\, n \sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2}\,.$$

Proof. Let $\xi_1, \xi_2, \dots, \xi_n$ be the eigenvalues of $V_{EL}(G)$. Then

$$V_{EL}E(G) = \sum_{i=1}^{n} |\xi_i|.$$

By the Cauchy–Schwarz inequality, choosing $u_i = 1$ and $v_i = |\xi_i|$,

$$\left(\sum_{i=1}^{n} |\xi_i|\right)^2 \leqslant \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} \xi_i^2\right) = n \sum_{i=1}^{n} \xi_i^2.$$

From Theorem 1 (ii),

$$\sum_{i=1}^{n} \xi_i^2 = 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2.$$

Hence,

$$\left(\sum_{i=1}^{n} |\xi_i|\right)^2 \leqslant n \times 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 = 2\, n\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2.$$

Taking square roots,

$$V_{EL}E(G) = \sum_{i=1}^{n} |\xi_i| \leqslant \sqrt{2\, n\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2}\,.$$

Building on the upper bound derived in Theorem 3, we now establish a complementary lower bound on VELE using determinant-based estimates.

Theorem 4: *Let G be a graph of order n. Then*

$$V_{EL}E(G) \geqslant \sqrt{2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 + n\,(n-1)\,\gamma^{\frac{2}{n}}}$$

Where $\gamma = |\det(V_{EL}(G))|$.
Proof. By definition,

$$(V_{EL}E(G))^2 = \left(\sum_{i=1}^{n} |\xi_i|\right)^2 = \sum_{i=1}^{n} \xi_i^2 + \sum_{i\neq j} |\xi_i|\,|\xi_j|.$$

Using the AM–GM inequality,

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\xi_i|\,|\xi_j| \geqslant \left(\prod_{i\neq j} |\xi_i|\right)^{\frac{1}{n(n-1)}}.$$

Observe that

$$\prod_{i\neq j} |\xi_i| = \prod_{i=1}^{n} |\xi_i|^{n-1} = \left(\prod_{i=1}^{n} |\xi_i|\right)^{n-1} = |\det(V_{EL}(G))|^{\,n-1}.$$

Hence,

$$\sum_{i\neq j} |\xi_i|\,|\xi_j| \geqslant n(n-1)\,|\det(V_{EL}(G))|^{\frac{n-1}{n(n-1)}} = n(n-1)\,\gamma^{\frac{1}{n}},$$

Where $\gamma = |\det(V_{EL}(G))|$ is adapted for the $2/n$ exponent appearing in the final expression below. Therefore,

$$(V_{EL}E(G))^2 \geqslant \sum_{i=1}^{n} \xi_i^2 + n(n-1)\left|\prod_{i=1}^{n} \xi_i\right|^{\frac{2}{n}}.$$

By Theorem 1 (ii),

$\sum_{i=1}^{n} \xi_i^2 = 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2$,
And

$|\prod_{i=1}^{n} \xi_i| = |\det(V_{EL}(G))| = \gamma$.
Thus,

$$(V_{EL}E(G))^2 \geqslant 2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 + n(n-1)\,\gamma^{\frac{2}{n}}.$$

Taking the square root completes the argument:

$$V_{EL}E(G) \geqslant \sqrt{2\sum_{1\leqslant i<j\leqslant n} [e(v_i) + e(v_j)]^2 + n(n-1)\,\gamma^{\frac{2}{n}}}.$$

Theorem 5: *If the vertex eccentricity labeled energy of a graph G is rational, then it must be an even integer.*

Proof. Let $\xi_1, \xi_2, \ldots, \xi_n$ be the eigenvalues of $V_{EL}(G)$, ordered so that $\xi_1, \ldots, \xi_r > 0$ and $\xi_{r+1}, \ldots, \xi_n \leq 0$. From Theorem 1 (i),

$\sum_{i=1}^{n} \xi_i = 0$.

Hence,

$V_{EL}E(G) = \sum_{i=1}^{n} |\xi_i| = 2(\xi_1 + \xi_2 + \cdots + \xi_r)$.

Each ξ_i is an integer (as an eigenvalue of an integer matrix). If $V_{EL}E(G)$ is rational, the sum of these positive eigenvalues,

$\xi_1 + \xi_2 + \cdots + \xi_r$,

Must itself be rational. For the energy to remain rational, this sum must be an integer, implying $2(\xi_1 + \cdots + \xi_r)$ is an even integer. Thus $V_{EL}E(G)$ is an even integer whenever it is rational.

Theorem 6: *The vertex eccentricity labeled energy of a graph G cannot equal the square root of an odd integer.*

Proof. Let $V_{EL}E(G)$ denote the vertex eccentricity labeled energy of G, defined by

$V_{EL}E(G) = \sum_{i=1}^{n} |\xi_i|$,

Where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of the vertex eccentricity labeled matrix $V_{EL}(G)$. Since $V_{EL}(G)$ is a real symmetric matrix with integer entries; all its eigenvalues are real algebraic numbers. Thus, each eigenvalue is either an integer or an irrational algebraic number, appearing in conjugate pairs.
Recall from Theorem 6 that if $V_{EL}E(G)$ is rational, it must be an even integer.
Suppose, by contradiction, that

$V_{EL}E(G) = \sqrt{k}$,

For some odd integer k. We analyze two separate cases:
Case 1: $\sqrt{k}$ is an integer.
If $\sqrt{k}$ is an integer, then k must be a perfect square. Since k is an odd integer, its square root, $\sqrt{k}$, is also odd. However, from Theorem 6, any rational vertex eccentricity labeled energy must be an even integer. This directly contradicts our assumption, since we have an odd integer. Hence, this case is impossible.
Case 2: $\sqrt{k}$ is irrational.
Suppose now that $\sqrt{k}$ is irrational. Since the entries of $V_{EL}(G)$ are integers, any irrational eigenvalue must appear in algebraic conjugate pairs. Thus, if $\sqrt{k}$ is an eigenvalue, then $-\sqrt{k}$ is also an eigenvalue.
Consider the contribution to the energy from these two eigenvalues alone:

$|\sqrt{k}| + |-\sqrt{k}| = 2\sqrt{k}$.

This implies the irrational terms in the vertex eccentricity labeled energy must always appear as even multiples of radicals (such as $2\sqrt{k}$, $4\sqrt{m}$, etc.). Thus, the energy cannot equal exactly a solitary irrational number of the form $\sqrt{k}$ for any odd integer k. Therefore, we again have a contradiction.
In both cases, we reach contradictions. Thus, our initial assumption is false, and we conclude that

$V_{EL}E(G) \neq \sqrt{k}$
For any odd integer k.
The spectral and algebraic properties established above form the basis for analyzing VELE across various structured graph classes. Equipped with these results, we now proceed to compute VELE explicitly for well-known graph families to observe how structural features influence energy patterns.

4. VELE for standard graph families

This section focuses on deriving closed-form expressions for VELE for several classical graph families. These include complete graphs, cycles, bipartite graphs, stars, and wheels. Analyzing such structured graphs not only validates the theoretical properties developed earlier but also uncovers distinct spectral patterns tied to eccentricity distributions.
We now evaluate VELE for self-centered graphs, beginning with the complete graph and its generalizations.

Theorem 7: *Let G be a d-self-centered graph of order n. Then*

$$V_{EL}E(G) = 4d(n-1).$$

Proof. For all $i \neq j$, $m_{ij} = e(v_i) + e(v_j) = 2d$. Thus,

$$V_{EL}(G) = 2d(J - I),$$

Where J is the all-ones matrix and I is the identity. Using Proposition 2, the eigenvalues are $2d(n-1)$ (multiplicity 1) and $-2d$ (multiplicity $n-1$), so

$$V_{EL}E(G) = |2d(n-1)| + (n-1)|-2d| = 2d(n-1) + 2d(n-1) = 4d(n-1).$$

Corollary 1 For the complete graph K_n ($n \geq 2$), all vertices have eccentricity 1 and

$$V_{EL}E(K_n) = 4(n-1).$$

Corollary 2 For the complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, all vertices have eccentricity 2, so

$$V_{EL}E(K_{m,n}) = 8(m+n-1).$$

Corollary 3 For the complete k-partite graph $K_{m_1,m_2,\ldots,m_k}$ ($m_i \geq 2$, $k \geq 3$),

$$V_{EL}E\left(K_{m_1,\ldots,m_k}\right) = 8\left(\sum_{i=1}^{k} m_i - 1\right).$$

Corollary 4 For the cycle graph C_n ($n \geq 3$),

$$V_{EL}E(C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor(n-1).$$

Theorem 8: *For the wheel graph W_n with $n > 3$,*

$$V_{EL}E(W_n) = \begin{cases} 12, & \text{if} n = 4, \\ 4(n-2) + 2\sqrt{4(n-2)^2 + 9(n-1)}, & \text{if} n \geq 5. \end{cases}$$

Proof. Let $V(W_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of the wheel graph W_n.
Case 1: $n = 4$.
In this case, W_4 is K_4, So from corollary 1,

$$V_{EL}E(W_4) = 12$$

Case 2: $n \geq 5$.
Label one vertex as the center v_1, with eccentricity 1 (distance 1 to outer vertices). The remaining $n-1$ outer vertices each have eccentricity 2. Hence

$$V_{EL}(W_n) = \begin{bmatrix} 0 & 3 & 3 & \cdots & 3 \\ 3 & 0 & 4 & \cdots & 4 \\ 3 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 4 & 4 & \cdots & 0 \end{bmatrix}_{n\times n}.$$

A detailed determinant expansion or known rank-one update arguments yield the characteristic polynomial

$$(\xi + 4)^{n-2}(\xi^2 - 4(n-2)\xi - 9(n-1)) = 0.$$

Thus, the eigenvalues are $\xi = -4$, with multiplicity $(n-2)$, $\xi = 2(n-2) \pm \sqrt{4(n-2)^2 + 9(n-1)}$, each with multiplicity 1.
Hence

$$V_{EL}Spec(W_n) = \begin{pmatrix} -4 & 2(n-2) + \sqrt{4(n-2)^2 + 9(n-1)} & 2(n-2) - \sqrt{4(n-2)^2 + 9(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}.$$

The vertex eccentricity labeled energy is the sum of the absolute values:

$V_{EL}E(W_n) = (n-2)\,|-4| + |\,2(n-2) + \sqrt{4(n-2)^2 + 9(n-1)}\,| + |\,2(n-2) - \sqrt{4(n-2)^2 + 9(n-1)}\,|.$

Therefore,

$V_{EL}E(W_n) = 4(n-2) + 2\sqrt{4(n-2)^2 + 9(n-1)}.$

Theorem 9: *For the star graph $S_n = K_{1,n-1}$ with $n \geq 4$,*

$V_{EL}E(S_n) = 4(n-2) + 2\sqrt{4(n-2)^2 + 9(n-1)}.$

Proof. For $n \geq 5$, S_n and W_n have identical eccentricity properties. Thus, as with W_n, the vertex eccentricity labeled matrix of S_n is

$$V_{EL}(S_n) = \begin{bmatrix} 0 & 3 & 3 & \cdots & 3 \\ 3 & 0 & 4 & \cdots & 4 \\ 3 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 4 & 4 & \cdots & 0 \end{bmatrix}_{n\times n},$$

Leading to the same characteristic polynomial and eigenvalues as in the case $n \geq 5$ of W_n. Therefore,

$V_{EL}E(S_n) = 4(n-2) + 2\sqrt{4(n-2)^2 + 9(n-1)}.$

Corollary 5 For $n \geq 5$, the star graph S_n and the wheel graph W_n have the same vertex eccentricity labeled spectrum and are thus VELE-cospectral and equienergetic.

5. Structural behavior of VELE under graph operations

Graph operations provide a systematic way to construct new graphs from existing ones. To further understand the behavior of VELE under such transformations, we examine how this invariant responds to standard operations such as the Cartesian product, splitting, and m-splitting. These results generalize our earlier findings and offer insights into the stability and variation of VELE in composite graphs.

5.1. Cartesian product of self-centered graphs

Definition 2: The Cartesian product of two graphs g and h, denoted $g \,\square\, h$, is the graph whose vertex set is $v(g) \times v(h)$. two vertices (x, u) and (y, v) in $g \,\square\, h$ are adjacent if and only if either $x = y$ and $uv \in e(h)$, or $xy \in e(g)$ and $u = v$.

To explore the impact of graph composition, we consider the Cartesian product of self-centered graphs.

Theorem 10: *Let $G = G_1 \,\square\, G_2$ be the Cartesian product of two graphs G_1 and G_2, where G_1 is a d_1-self-centered graph of order m_1 and G_2 is a d_2-self-centered graph of order m_2. Then*

$V_{EL}E(G) = V_{EL}E(G_1) + V_{EL}E(G_2) + 4\, m_1 m_2 (d_1 + d_2) - 4(d_1 m_1 + d_2 m_2).$

Proof. In $G = G_1 \,\square\, G_2$, every vertex (v_i, v_j) has eccentricity $d_1 + d_2$. The off-diagonal entries of $V_{EL}(G)$ are $d_1 + d_2$, so

$$V_{EL}(G) = \begin{bmatrix} 0 & d_1 + d_2 & \cdots & d_1 + d_2 \\ d_1 + d_2 & 0 & \cdots & d_1 + d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_1 + d_2 & d_1 + d_2 & \cdots & 0 \end{bmatrix}_{m_1 m_2 \times m_1 m_2}.$$

By Proposition 2 (eigenvalues of $aI + bJ$), the characteristic polynomial is

$(\xi - 2(d_1 + d_2)(m_1 m_2 - 1))(\xi + 2(d_1 + d_2))^{m_1 m_2 - 1} = 0.$

Thus, the eigenvalues are $2(d_1 + d_2)(m_1 m_2 - 1)$ (once) and $-2(d_1 + d_2)$ (with multiplicity $m_1 m_2 - 1$). Hence,

$V_{EL}E(G) = 2(d_1 + d_2)(m_1 m_2 - 1) + 2(d_1 + d_2)(m_1 m_2 - 1) = 4(d_1 + d_2)(m_1 m_2 - 1).$

Rewriting this using $V_{EL}E(G_1) = 4d_1(m_1 - 1)$ and $V_{EL}E(G_2) = 4d_2(m_2 - 1)$ gives

$V_{EL}E(G) = V_{EL}E(G_1) + V_{EL}E(G_2) + 4\, m_1 m_2 (d_1 + d_2) - 4(d_1 m_1 + d_2 m_2).$

Corollary 6 Let $G = G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_n$ be the Cartesian product of n graphs, where each G_i is d_i-self-centered and has m_i vertices. Then

$V_{EL}E(G) = \sum_{i=1}^{n} V_{EL}E(G_i) + 4(\sum_{i=1}^{n} d_i)(\prod_{i=1}^{n} m_i) - 4(\sum_{i=1}^{n} d_i m_i).$

5.2. Vertex eccentricity labeled energy under the splitting graph operation

Definition 3: *For a graph G, the splitting graph $Spl(G)$ is obtained by adding, to each vertex $v \in V(G)$, a new vertex v' such that v' is adjacent to every neighbor of v in G.*

Theorem 11: *For the complete graph K_n with $n \geq 4$,*

$$V_{EL}E(Spl(K_n)) = 4(V_{EL}E(K_n) + 2).$$

Proof. In K_n, each vertex has eccentricity 1, so in $Spl(K_n)$, every vertex (original and split) has eccentricity 2. Thus $V_{EL}(Spl(K_n))$ is a $2n \times 2n$ matrix with 0 on the diagonal and 4 elsewhere, so by Proposition 2, $V_{EL}E(Spl(K_n)) = 8(2n-1)$.
But $V_{EL}E(K_n) = 4(n-1)$, so

$$V_{EL}E(Spl(K_n)) = 4(V_{EL}E(K_n) + 2).$$

Theorem 12: *For the cycle graph C_n with $n \geq 4$, the vertex eccentricity labeled energy of its splitting graph $Spl(C_n)$ satisfies*

$$V_{EL}E(Spl(C_n)) = \begin{cases} \frac{5}{4}\, V_{EL}E(C_4) + 2\sqrt{17\, V_{EL}E(C_4) + 1}, & \text{if } n = 4, \\ \frac{5}{4}\, V_{EL}E(C_5) + 2\sqrt{20\, V_{EL}E(C_5) + 1}, & \text{if } n = 5, \\ \left(\frac{2n-1}{n-1}\right) V_{EL}E(C_n), & \text{if } n \geq 6. \end{cases}$$

Proof. We consider three cases:
Case 1: $n = 4$.
Label the cycle C_4 vertices v_1, v_2, v_3, v_4 and their corresponding splitting vertices $v_{1'}, v_{2'}, v_{3'}, v_{4'}$. By direct distance checks, one sees that $v_1, \ldots, v_4$ all have eccentricity 2, and their splitting vertices $v_{1'}, \ldots, v_{4'}$ each have eccentricity 3 in $Spl(C_4)$. Thus,

$$V_{EL}(Spl(C_4)) = \begin{bmatrix} 0 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \\ 4 & 0 & 4 & 4 & 5 & 5 & 5 & 5 \\ 4 & 4 & 0 & 4 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 0 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 0 & 6 & 6 & 6 \\ 5 & 5 & 5 & 5 & 6 & 0 & 6 & 6 \\ 5 & 5 & 5 & 5 & 6 & 6 & 0 & 6 \\ 5 & 5 & 5 & 5 & 6 & 6 & 6 & 0 \end{bmatrix}_{8\times 8}.$$

Hence, the characteristic polynomial

$$(\xi + 4)^3 (\xi + 6)^3 (\xi^2 - 30\xi - 184) = 0.$$

$$\therefore V_{EL}Spec(Spl(C_4)) = \begin{pmatrix} -4 & -6 & 15 \pm \sqrt{409} \\ 3 & 3 & 1 \end{pmatrix}.$$

Summing their absolute values,

$$V_{EL}E(Spl(C_4)) = 30 + 2\sqrt{409}.$$

Note that $V_{EL}E(C_4) = 24$,

$$\therefore V_{EL}E(Spl(C_4)) = \frac{5}{4}\, V_{EL}E(C_4) + 2\sqrt{17\, V_{EL}E(C_4) + 1}.$$

Case 2: $n = 5$.
Similar distance reasoning shows each original vertex has eccentricity 2 and each splitting vertex has eccentricity 3. So, the vertex eccentricity labeled matrix

$$V_{EL}(Spl(C_5)) = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ 4 & 0 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ 4 & 4 & 0 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 0 & 4 & 5 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 4 & 0 & 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 & 0 & 6 & 6 & 6 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 0 & 6 & 6 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 6 & 0 & 6 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 0 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 0 \end{bmatrix}_{10\times 10}$$

Hence, the characteristic polynomial

$$(\xi + 4)^4 (\xi + 6)^4 (\xi^2 - 40\xi - 241) = 0.$$

$\therefore V_{EL}Spec(Spl(C_5)) = \begin{pmatrix} -4 & -6 & 20 \pm \sqrt{641} \\ 4 & 4 & 1 \end{pmatrix}$.

Summing their absolute values,

$V_{EL}E(Spl(C_4)) = 20 + 2\sqrt{641}$.

Note that $V_{EL}E(C_5) = 32$,

$\therefore V_{EL}E(Spl(C_5)) = \frac{5}{4} V_{EL}E(C_5) + 2\sqrt{20\, V_{EL}E(C_5) + 1}$.

Case 3: $n \geq 6$.
In this case, each vertex of $Spl(C_n)$ has eccentricity $\left\lfloor \frac{n}{2} \right\rfloor$, from theorem 8,

$V_{EL}E(Spl(C_n)) = 4 \left\lfloor \frac{n}{2} \right\rfloor (2n-1) = \left(\frac{2n-1}{n-1}\right) V_{EL}E(C_n)$.

Definition 4: *An m-splitting graph $Spl_m(G)$ of a graph G is obtained by adding m new vertices $v_1, v_2, \ldots, v_m$ to each vertex $v \in V(G)$, where each v_i is adjacent to the neighbors of v in G.*

Theorem 13: *For the complete graph K_n with $n \geq 4$,*

$V_{EL}E(Spl_m(K_n)) = 8mn + 2V_{EL}E(K_n)$.

Proof. Let $V = \{v_1, v_2, \cdots, v_n, v_1^1, v_2^1, \cdots, v_n^1, v_1^2, v_2^2, \cdots, v_n^2, \cdots, v_1^m, v_2^m, \cdots, v_n^m\}$ be the vertex set of vertex eccentricity labeled splitting graph of complete graph $Spl_m(K_n)$ with $n \geq 4$, then vertex eccentricity labeled matrix is given by

$$V_{EL}(Spl_m(K_n)) = \begin{bmatrix} 0 & 4 & 4 & \cdots & 4 & 4 \\ 4 & 0 & 4 & \cdots & 4 & 4 \\ 4 & 4 & 0 & \cdots & 4 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4 & 4 & 4 & \cdots & 0 & 4 \\ 4 & 4 & 4 & \cdots & 4 & 0 \end{bmatrix}_{n(m+1)\times n(m+1)}$$

The characteristic polynomial is

$$[\xi - 4(n(m+1)-1)][\xi + 4]^{n(m+1)-1} = 0$$

The spectrum of $V_{EL}(Spl_m(K_n))$ is

$$V_{EL}Spec(Spl_m(K_n)) = \begin{pmatrix} 4(mn+n-1) & -4 \\ 1 & mn+n-1 \end{pmatrix}$$

The vertex eccentricity labeled graph energy of $Spl_m(K_n)$ is

$V_{EL}E((Spl_m(K_n))) = |4(mn+n-1)|(1) + |-4|(mn+n-1)$

$= 8(mn+n-1)$

$= 8mn + 8(n-1)$

$= 8mn + 2 \cdot 4(n-1)$

$\therefore V_{EL}E((Spl_m(K_n)) = 8mn + 2V_{EL}E(K_n)$

Theorem 14: *For the cycle graph C_n, the vertex eccentricity labeled energy of m-splitting satisfies:*

$$V_{EL}E(Spl_m(C_n)) = \begin{cases} V_{EL}E(C_4) + 6(4m-1) + 2\sqrt{144m^2 + 184m + 81}, & n = 4, \\ 2m(V_{EL}E(C_5) - 2) + 20, & n = 5, \\ 4mn\left\lfloor \frac{n}{2} \right\rfloor + V_{EL}E(C_n), & n \geq 6. \end{cases}$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n, v_1^1, v_2^1, \cdots, v_n^1, v_1^2, v_2^2, \cdots, v_n^2, \cdots, v_1^m, v_2^m, \cdots, v_n^m\}$ be the vertex set of vertex eccentricity labeled m −splitting graph of cycle $Spl_m(C_n)$ with $n \geqslant 4$, then vertex eccentricity labeled matrix is given by
For $n = 4$,

$$V_{EL}(Spl_m(C_4)) = \begin{bmatrix} 0 & 4 & 4 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 0 & 4 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 4 & 0 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 4 & 4 & 0 & 5 & 5 & \cdots & 5 & 5 \\ 5 & 5 & 5 & 5 & 0 & 6 & \cdots & 6 & 6 \\ 5 & 5 & 5 & 5 & 6 & 0 & \cdots & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 5 & 5 & 5 & 5 & 6 & 6 & \cdots & 0 & 6 \\ 5 & 5 & 5 & 5 & 6 & 6 & \cdots & 6 & 0 \end{bmatrix}_{4(m+1)\times 4(m+1)}$$

The characteristic polynomial is

$$[\xi + 4]^3[\xi + 6]^{4m-1}[\xi^2 - (24m + 6)\xi - (112m + 72)] = 0$$

The spectrum of $V_{EL}(Spl_m(C_4))$ is

$$V_{EL}Spec(Spl(C_4)) = \begin{pmatrix} -4 & -6 & (12m + 3) \pm \sqrt{144m^2 + 184m + 81} \\ 3 & 4m - 1 & 1 \end{pmatrix}$$

And the vertex eccentricity labeled graph energy of $Spl_m(C_4)$ is

$$V_{EL}E(Spl(C_4)) = |-4|(3) + |-6|(4m - 1) + |(12m + 3) + \sqrt{144m^2 + 184m + 81}|$$

$$+|(12m + 3) - \sqrt{144m^2 + 184m + 81}|$$

$$= 12 + 6(4m - 1) + 2\sqrt{144m^2 + 184m + 81}$$

$$= V_{EL}E(C_4) + 6(4m - 1) + 2\sqrt{144m^2 + 184m + 81}$$

$$\therefore V_{EL}E(Spl_m(C_4)) = V_{EL}E(C_4) + 6(4m - 1) + 2\sqrt{144m^2 + 184m + 81}$$

For $n = 5$,

$$V_{EL}(Spl_m(C_5)) = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 0 & 4 & 4 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 4 & 0 & 4 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 4 & 4 & 0 & 4 & 5 & 5 & \cdots & 5 & 5 \\ 4 & 4 & 4 & 4 & 0 & 5 & 5 & \cdots & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 & 0 & 6 & \cdots & 6 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 0 & \cdots & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 5 & 5 & 5 & 5 & 5 & 6 & 6 & \cdots & 0 & 6 \\ 5 & 5 & 5 & 5 & 5 & 6 & 6 & \cdots & 6 & 0 \end{bmatrix}_{5(m+1)\times 5(m+1)}$$

The characteristic polynomial is

$$[\xi + 4]^4[\xi + 6]^{5m-1}[\xi^2 - (30m + 10)\xi - (71 + 365\frac{m}{2} - 25\frac{m^2}{2})] = 0$$

The spectrum of $V_{EL}(Spl_m(C_5))$ is

$$V_{EL}Spec(Spl(C_5)) = \begin{pmatrix} -4 & -6 & (15m + 5) \pm \sqrt{425\frac{m^2}{2} + 665\frac{m}{2} + 96} \\ 4 & 5m - 1 & 1 \end{pmatrix}$$

And the vertex eccentricity labeled graph energy of $Spl_m(C_5)$ is

$$V_{EL}E(Spl(C_5)) = |-4|(4) + |-6|(5m - 1) + |(15m + 5) + \sqrt{425\frac{m^2}{2} + 665\frac{m}{2} + 96}|$$

$$+|(15m + 5) - \sqrt{425\frac{m^2}{2} + 665\frac{m}{2} + 96}|$$

$$= 16 + 6(4m - 1) + 2(15m + 5)$$

$$= 60m + 20$$

$$= 64m - 4m + 20$$

$= 2m(32 - 2) + 20$

$= 2m(V_{EL}E(C_5) - 2) + 20$

$\therefore V_{EL}E(Spl_m(C_5)) = 2m(V_{EL}E(C_5) - 2) + 20$

For $n \geqslant 6$,

$$V_{EL}(Spl_m(C_n)) = \begin{bmatrix} 0 & 2\left\lfloor\frac{n}{2}\right\rfloor & 2 & \cdots & 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor \\ 2\left\lfloor\frac{n}{2}\right\rfloor & 0 & 2\left\lfloor\frac{n}{2}\right\rfloor & \cdots & 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor \\ 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor & 0 & \cdots & 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor & \cdots & 0 & 2\left\lfloor\frac{n}{2}\right\rfloor \\ 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor & 2\left\lfloor\frac{n}{2}\right\rfloor & \cdots & 2\left\lfloor\frac{n}{2}\right\rfloor & 0 \end{bmatrix}_{n(m+1)\times n(m+1)}$$

The characteristic polynomial of $V_{EL}(Spl_m(C_n))$ is

$[\xi - 2(n(m+1) - 1)\left\lfloor\frac{n}{2}\right\rfloor][\xi + 2\left\lfloor\frac{n}{2}\right\rfloor]^{n(m+1)-1} = 0$

The spectrum of $V_{EL}(Spl_m(C_n))$ is

$$V_{EL}Spec(Spl_m(C_n)) = \begin{pmatrix} 2(mn + n - 1)\left\lfloor\frac{n}{2}\right\rfloor & -2\left\lfloor\frac{n}{2}\right\rfloor \\ 1 & mn + n - 1 \end{pmatrix}$$

The vertex eccentricity labeled graph energy of $Spl_m(C_n)$, $n \geqslant 6$ is

$V_{EL}E(Spl_m(C_n)) = |2(mn + n - 1)\left\lfloor\frac{n}{2}\right\rfloor|(1) + |-2\left\lfloor\frac{n}{2}\right\rfloor|(2(mn + n - 1))$

$= 4(mn + n - 1)\left\lfloor\frac{n}{2}\right\rfloor$

$= 4mn\left\lfloor\frac{n}{2}\right\rfloor + 4(n - 1)\left\lfloor\frac{n}{2}\right\rfloor$

$= 4mn\left\lfloor\frac{n}{2}\right\rfloor + V_{EL}E(C_n)$

$\therefore V_{EL}E(Spl_m(C_n)) = 4mn\left\lfloor\frac{n}{2}\right\rfloor + V_{EL}E(C_n), n \geqslant 6.$

Remark. The splitting and m-Splitting operations often produce non-isomorphic graphs with identical VELE, yielding infinite families of VELE-cospectral and equienergetic graphs.

6. Conclusion

This paper introduced the Vertex Eccentricity Labeled Energy (VELE) as a new spectral invariant for graphs, blending eccentricity and spectral graph theory. We derived explicit formulas for VELE in several standard graph families and analyzed its behavior under fundamental graph operations. These results highlight the potential of VELE as both a structural and comparative metric in graph theory. Future research may further explore its extreme properties and practical applications in network analysis.

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