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***Research paper***

# Analytical Solution of Linear Second-Order Non-Homogeneous Fuzzy Partial Differential Equations Using The Fuzzy Sumudu Transform Under Generalized Hukuhara Differentiability

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**Abstract**

In this paper, a linear second-order non-homogeneous fuzzy partial differential equation (FPDE) is constructed, and the Fuzzy Sumudu Transform (FST) method is applied to solve FPDEs within the context of generalized Hukuhara(gH) differentiability technique. The use of FST, a potent integral transform renowned for its scale-invariant and unit-preserving characteristics, to the fuzzy setting is expanded. FPDEs are solved analytically by transforming them into more straightforward algebraic differential equations in the transform domain, utilizing recent advances in the gH-differentiability technique. Initially, the basic characteristics of linear second-order non-homogeneous FPDEs are presented. To highlight the capabilities, a numerical example is provided.

***Keywords***: *Strongly Generalized Hukuhara Differentiability; Fuzzy Sumudu Transform; Initial and Boundary Conditions; Non-Linear Differential Equation; Non-Homogeneous Fuzzy Partial Differential Equation.*

## 1. Introduction

Mathematical modelling plays a crucial role in describing, analyzing, and predicting the behaviour of dynamic systems in diverse fields such as engineering, physics, biology, economics, and decision sciences. Traditionally, such models assume precise knowledge of system parameters, boundary conditions, and initial values. However, in real-world applications, this assumption rarely holds true. Measurements may be imprecise, data may be incomplete, and environmental fluctuations can introduce significant uncertainty. As a result, classical models based on ordinary and partial differential equations (ODEs and PDEs) often fail to accurately capture the underlying dynamics when uncertainty is present. To address this shortcoming, fuzzy set theory, introduced by Zadeh [1], offers a rigorous mathematical framework for dealing with vagueness and imprecision. Building upon this foundation, researchers developed fuzzy differential equations (FDEs), which generalize classical differential equations to the fuzzy context, allowing model parameters and solutions to be represented as fuzzy quantities. FDEs are particularly useful in systems where uncertainty is inherent and cannot be described by probabilistic means alone, such as in human decision processes, biological systems, or linguistic control rules. Solving FDEs analytically is a challenging task due to the nonlinearity and set-valued nature of fuzzy functions. To overcome these difficulties, researchers have proposed several analytical and semi-analytical techniques, including fixed-point methods, variational iteration methods, domain decomposition, and integral transform techniques. Among these, fuzzy integral transforms, notably the fuzzy Laplace and fuzzy Sumudu transforms (FST) have proven to be highly effective tools. The Sumudu transform, originally introduced by Watugala [7], was later generalized to the fuzzy context by several researchers [22], [23]. Compared to the Laplace transform, the FST retains physical units of the original function, facilitates the handling of fuzzy initial conditions, and provides simpler operational rules for derivatives and convolution. These features make it especially attractive for solving fuzzy partial differential equations (FPDEs), including higher-order and multidimensional models. Applications of FST have been successfully demonstrated in linear, nonlinear, and fractional fuzzy models [25], [26], [30]. To solve FPDEs analytically, researchers have extended classical integral transform methods such as the Laplace [2], Fourier, and Sumudu transforms [7], [8], [14], [16] into the fuzzy setting. Among these, the FST has gained increasing interest due to its structural advantages: it preserves initial conditions, avoids the need for convolution theorems in many cases, and yields solutions that often retain the functional structure of the original equations [22 - 25]. Yet, the use of FST in the context of FPDEs remains relatively undeveloped compared to fuzzy Laplace transforms [20], [21]. This study makes both theoretical and computational contributions to the field of fuzzy mathematics. Theoretically, it extends the scope of fuzzy integral transforms by incorporating gH-differentiability. Computationally, it demonstrates the practical applicability and accuracy of the FST in solving complex fuzzy PDEs. In this work, we focus on solving a second-order linear non-homogeneous fuzzy partial differential equation of the form:

$$a\xi^{\alpha}_{\eta\eta}(\eta,\zeta) = b\xi^{\alpha}_{\zeta\zeta}(\eta,\zeta) + c\xi^{\alpha}_{\zeta}(\eta,\zeta) + d\xi^{\alpha}_{\eta}(\eta,\zeta) + f^{\alpha}(\eta,\zeta,\xi(\eta,\zeta))$$

Where $\xi(\eta,\zeta)$ and $f^{\alpha}(\eta,\zeta,\xi(\eta,\zeta))$ are fuzzy valued functions, and a, b, c, and d are real constants with $\eta,\zeta \geq 0$ under appropriate fuzzy initial and boundary conditions. The FST is employed to derive exact symbolic solutions under various gH-differentiability cases (cases 1–4). The derivation involves the application of several theorems tailored for each differentiability assumption, extending ideas from [23, 25, 26]. To ensure mathematical soundness, we validate the fuzzy nature of the solution by checking fuzzy validity conditions on both the solution and its derivatives, following practices like those in [12], [17]. Where possible, symbolic verification is carried out to confirm that the obtained fuzzy solution satisfies the original FPDEs under the prescribed conditions. The remaining section is categorized as, section 2 provides essential preliminaries on fuzzy numbers, α-cuts, and generalized Hukuhara differentiability, section 3 introduces the FST and outlines its key properties and operational theorems, section 4 presents the main results, with derivations for each differentiability case and symbolic verification, section 5 includes a discussion on the comparative advantages of FST, limitations, and suggestions for future work, section 6 concludes the study and outlines potential extensions. such as applying the method to nonlinear FPDEs or incorporating numerical solvers.

## 2. Basic concepts

**Definition 2.1:** *[10], [9] In a parametric form of functions* $\underline{v}^{\alpha}$ *and* $\overline{v}^{\alpha}$, *an arbitrary FN v is expressed in an ordered pair as* $[\underline{v}^{\alpha},\overline{v}^{\alpha}]$, $\forall\alpha \in [0,1]$ *the following conditions are met:*

i) $\underline{v}^{\alpha}$ and $\overline{v}^{\alpha}$ must be the BLCM increasing and decreasing function in [0,1] and
ii) $\underline{v}^{\alpha}$ must be less than or equal to $\overline{v}^{\alpha}$

**Definition 2.2:** *[15], [17] Let* $f:(\acute{a},\acute{b}) \to \mathfrak{R}_g$ *be FF and* $x_0 \in [\acute{a},\acute{b}]$. *We say that f is SGH-differentiable on* $x_0$, *if there exists an element* $f'(x_0) \in \mathfrak{R}_g$, *such that:*

i) for $\acute{h} > 0$ sufficiently small, $g(x_0+\acute{h}) - g(x_0)$ and $g(x_0) - g(x_0-\acute{h})$ and the limits

$$\lim_{\acute{h}\to 0}\frac{g(x_0+\acute{h})-g(x_0)}{\acute{h}} = \lim_{\acute{h}\to 0}\frac{g(x_0)-g(x_0-\acute{h})}{\acute{h}} = g'(x_0)$$

Or
ii) for $\acute{h} > 0$ sufficiently small, $g(x_0+\acute{h}) - g(x_0)$ and $g(x_0) - g(x_0-\acute{h})$ and the limits

$$\lim_{\acute{h}\to 0}\frac{g(x_0)-g(x_0+\acute{h})}{-\acute{h}} = \lim_{\acute{h}\to 0}\frac{g(x_0-\acute{h})-g(x_0)}{-\acute{h}} = g'(x_0)$$

The type of differentiability in definition 2.2 is referred to as (i)-differentiable and (ii)-differentiable, respectively.

**Definition 2.3:** *[22] Let* $f:\mathfrak{R} \to \mathfrak{R}_g$ *be a continuous fuzzy function. Suppose that* $f(q\eta)e^{-\eta}$ *is an improper fuzzy Riemann-integrable (IFRI) on* $[0,\infty[$, *then* $\int_0^{\infty} f(q\eta)e^{-\eta}d\eta$ *is called FST and is denoted by*

$$S[f(\eta)](q) = \int_0^{\infty} f(q\eta)e^{-\eta}d\eta, q \in [-\rho_1,\rho_2],$$

Where q is used to factor the variable η and $\rho_1,\rho_2 \geq 0$.
In parametric form, we can write it as:

$$S[f(\eta)](q) = [s[\underline{f}^{\alpha}](q), s[\overline{f}^{\alpha}](q)]$$

**Definition 2.4:** *[22] A triangular fuzzy number (TFN)* $\tilde{A}$ *is a fuzzy number represented by a triplet of real numbers:*
$\tilde{A} = (a_1,a_2,a_3)$, *where* $a_1 < a_2 < a_3$

This triplet corresponds to the piecewise linear membership function $\mu_{\tilde{A}}(x)$, defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, x \leq a_1, \\ \frac{x-a_1}{a_2-a_1}, a_1 < x \leq a_2, \\ \frac{a_3-x}{a_3-a_2},\ a_2 < x < a_3, \\ 0, x \geq a_3, \end{cases}$$

The α-level set (or α- cut) of the triangular fuzzy number $\widetilde{A} = (a_1,a_2,a_3)$ for $\alpha \in [0,1]$ is given by,

$$\widetilde{A} = [a^L_{\alpha}, a^R_{\alpha}]$$

**Theorem 2.1:** *[22] Let* $\xi$ *be a continuous FF of the form* $[0,\infty[ \times [0,\infty[\to \mathfrak{R}_g$ . *If* $e^{-\zeta}\xi_{\eta}(\eta,q\zeta)$ *is an IFRI on* $[0,\infty[$, *then*

$S_t[\xi_\eta(\eta,\zeta)](q) = \frac{\partial}{\partial\eta}(S_t[\xi(\eta,\zeta)](q))$,

Where $S_t[\xi(\eta,\zeta)](q)$ represents the FST of ξ w.r.t η.

**Theorem 2.2:** *[23] Suppose* $\xi$: $[0,\infty[\times[0,\infty[\to \Re_g$ *be a continuous FF. If* $\xi_\eta$ *the partial differential of* $\xi$ *w.r.t* $\eta$*. Assume that* $e^{-\zeta}\xi(\eta,q\zeta)$ *and* $e^{-\zeta}\xi_\eta(\eta,q\zeta)$ *are IFRI on* $[0,\infty[$*, then*

i) If $\xi(\eta,\zeta)$ is (i)-differentiable w.r.t ζ, then

$S_t[\xi_\zeta(\eta,\zeta)](q) = \frac{S_t[\xi(\eta,\zeta)](q)-\xi(\eta,0)}{q}$

ii) And, if $\xi(\eta,\zeta)$ is (ii)-differentiable w.r.t ζ, then$S_t[\xi_\zeta(\eta,\zeta)](q) = \frac{-\xi(\eta,0)-(-S_t[\xi(\eta,\zeta)](q))}{q}$,

Where $S_t[\xi(\eta,\zeta)](q)$ indicates the FST of ξ w.r.t ζ.

**Theorem 2.3:** *[19] Let* $\xi$ *be a continuous FF with the form* $[0,\infty[\times[0,\infty[\to \Re_g$*. When* $e^{-\zeta}\xi_\eta(\eta,q\zeta)$ *is assumed to be IFRI on* $[0,\infty[$*, then* $S_t[\xi_{\eta\eta}(\eta,\zeta)](q) = \frac{\partial^2}{\partial\eta^2}(S_t[\xi(\eta,\zeta)](q))$*,*

Where $S_t[\xi(\eta,\zeta)](q)$ indicates the FST of ξ w.r.t ζ.

**Theorem 2.4:** *[19] Consider* $\xi$: $[0,\infty[\times[0,\infty[\to \Re_g$ *be a continuous FF. If* $\xi_\eta$ *the partial differential of* $\xi$ *w.r.t* $\eta$*. Assume that* $e^{-\zeta}\xi(\eta,q\zeta)$ *and* $e^{-\zeta}\xi_\eta(\eta,q\zeta)$ *are IFRI on* $[0,\infty[$*, then*

i) If $\xi(\eta,\zeta)$ is (i)-differentiable w.r.t ζ, then

$S_t[\xi_{\zeta\zeta}(\eta,\zeta)](q) = \frac{S_t[\xi(\eta,\zeta)](q)-\xi(\eta,0)}{q^2} - \frac{\xi'^{\alpha}(\eta,0)}{q}$,

ii) And, if $\xi(\eta,\zeta)$ is (ii)-differentiable w.r.t ζ, then

$S_t[\xi_{\zeta\zeta}(\eta,\zeta)](q) = \frac{-\xi(\eta,0)-(-S_t[\xi(\eta,\zeta)](q))}{q} - \frac{\xi'^{\alpha}(\eta,0)}{q}$

Where $S_t[\xi(\eta,\zeta)](q)$ indicates the FST of ξ w.r.t ζ.

**Theorem 2.5:** *[3] The Sumudu transform amplifies the coefficients of the power series function,* $f(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n$*, by sending it to the power series function,*

$G(\xi) = \sum_{n=0}^{\infty} n!\, a_n \xi^n$

**Theorem 2.6:** *[3] The inverse discrete Sumudu transform,* $f(\zeta)$ *of the power series* $G(\xi) = \sum_{n=0}^{\infty} b_n\xi^n$*, is given by*

$S^{-1}[G(\xi)] = f(\zeta) = \sum_{n=0}^{\infty}(\frac{1}{n!})b_n\zeta^n$.

## 3. Fuzzy Sumudu transform (FST) for linear second-order non-homogeneous fuzzy partial differential equations (FPDEs)

Let us suppose the following linear second-order non-homogeneous FPDEs,

$$a\xi^{\alpha}_{\eta\eta}(\eta,\zeta) = b\xi^{\alpha}_{\zeta\zeta}(\eta,\zeta) + c\xi^{\alpha}_{\zeta}(\eta,\zeta) + d\xi^{\alpha}_{\eta}(\eta,\zeta) + f^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)) \quad (1)$$

Subject to the initial and boundary conditions given by,

$$\begin{cases} \xi^{\alpha}(\eta,0) = \grave{k}^{\alpha}(\eta) = \left[\underline{\grave{k}}^{\alpha}(\eta), \overline{\grave{k}}^{\alpha}(\eta)\right]; \\ \xi^{\alpha}(0,\zeta) = \grave{l}^{\alpha}(\zeta) = \left[\underline{\grave{l}}^{\alpha}(\zeta), \overline{\grave{l}}^{\alpha}(\zeta)\right]; \\ \xi^{\alpha}(\eta,0) = \xi^{\alpha}(0,\zeta) = e \end{cases}$$

Where $\xi$: $[0,\infty[\times[0,\infty[\to \Re_g$ is a fuzzy valued function, and a, b, c, d, and e are real constants with $\eta,\zeta \geq 0$. The function $f^{\alpha}(\eta,\zeta,\xi(\eta,\zeta))$ is a fuzzy valued, while $\grave{k}^{\alpha}(\eta)$ and $\grave{l}^{\alpha}(\zeta)$ define the initial conditions. Such that $\underline{\grave{k}}^{\alpha}(\eta) - \overline{\grave{k}}^{\alpha}(\eta) \geq 0$ and $\underline{\grave{l}}^{\alpha}(\zeta) - \overline{\grave{l}}^{\alpha}(\zeta) \geq 0$.
Applying the FST to both sides of equation (1), we yield:

$S_t[a\xi^{\alpha}_{\eta\eta}(\eta,\zeta)] = S_t[b\xi^{\alpha}_{\zeta\zeta}(\eta,\zeta)] + S_t[c\xi^{\alpha}_{\zeta}(\eta,\zeta)] + S_t[d\xi^{\alpha}_{\eta}(\eta,\zeta)] + S_t[f^{\alpha}(\eta,\zeta,\xi(\eta,\zeta))]$

Four different cases arise for the solutions of equation (1), Case 1: If the function ξ is (i)-differentiable with respect to both η and ζ we yield

$$\begin{cases} S_t\left[a\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] \\ \qquad + S_t[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)] \\ S_t\left[a\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] \\ \qquad + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{2}$$

Applying the theorems to equation (2), we yield the following equations:

$$\begin{cases} a\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - c\left[\underline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad + d\frac{\partial}{\partial n}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)] \\ a\frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - c\left[\overline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad + d\frac{\partial}{\partial n}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{3}$$

Solving Eq. (3), we get, $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ and $S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]$, which meet the initial condition $\xi^{\alpha}(0,\zeta) = \grave{l}^{\alpha}(\zeta) = [\underline{\grave{l}}^{\alpha}(\zeta), \overline{\grave{l}}^{\alpha}(\zeta)]$

$$\begin{cases} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = U^1 \\ S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = V^1 \end{cases} \tag{4}$$

Applying the inverse of FST, we obtain:

$$\begin{cases} \underline{\xi}^{\alpha}(\eta,\zeta) = S_t^{-1}[U^1] \\ \overline{\xi}^{\alpha}(\eta,\zeta) = S_t^{-1}[V^1] \end{cases} \tag{5}$$

The solutions in equation (5) hold valid only if they satisfy the fuzzy boundary conditions for Case 1, if,

$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$;

$\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$

Case 2: Next, considering that ξ is (i)-differentiable w.r.t η and (ii)-differentiable w.r.t ζThe following system is attained:

$$\begin{cases} S_t\left[a\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)] \\ S_t\left[a\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{6}$$

Applying the theorems to equation (6), we yield the following equations:

$$\begin{cases} a\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - c\left[\overline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad + d\frac{\partial}{\partial n}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)] \\ a\frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - c\left[\underline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad + d\frac{\partial}{\partial n}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{7}$$

Following an analogous procedure to Case 1, the solutions to this system are determined. For these solutions to be valid in the fuzzy context, the following conditions must hold:

$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$;

$\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$

Case 3: Next, considering that ξ is (ii)-differentiable w.r.t η and (i)-differentiable w.r.t ζThe following system is attained:

$$\begin{cases} S_t\left[a\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \\ S_t\left[a\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[+c\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)] \end{cases} \tag{8}$$

Using the theorems, the system amplifies to:

$$\begin{cases} a\frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-c\left[\underline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad +d\frac{\partial}{\partial n}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \\ a\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-c\left[\overline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad +d\frac{\partial}{\partial n}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{9}$$

Again, following the same procedure as in the above cases, the solution to equation (9) can be obtained. To ensure fuzzy validity, the following conditions must be satisfied:

$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$;

$\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$

Case 4: If the function ξ is (ii)-differentiable with respect to both η and ζ we yield

$$\begin{cases} S_t\left[a\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \\ S_t\left[a\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[b\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[c\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[d\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t\left[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{10}$$

By applying the theorems on both sides of the equation, we obtain:

$$\begin{cases} a\frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-c\left[\overline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad +d\frac{\partial}{\partial n}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\overline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \\ a\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{b}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{b}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{b}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \frac{cS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-c\left[\underline{\xi}^{\alpha}(\eta,0)\right]}{q} \\ \qquad +d\frac{\partial}{\partial n}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + S_t\left[\underline{f}^{\alpha}(\eta,\zeta,\xi(\eta,\zeta)\right] \end{cases} \tag{11}$$

Following the same procedure as in Case 1, we derive the solutions to Eq. (11). For the solution to be valid in the fuzzy context, the following conditions must hold:

$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$;

$\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$

With the solution to the linear second-order non-homogeneous FPDEs now obtained, the process is complete. The following section will demonstrate the application of the FST technique on linear second-order non-homogeneous FPDEs.

## 4. Numerical example

Example 1. Let us consider the following linear second-order non-homogeneous FPDE:

$$\xi^{\alpha}_{\eta\eta}(\eta,\zeta) = \xi^{\alpha}_{\zeta\zeta}(\eta,\zeta) + \zeta\xi^{\alpha}_{\zeta}(\eta,\zeta) + \eta\xi^{\alpha}_{\eta}(\eta,\zeta) + \eta^2 + \zeta \tag{12}$$

With boundary and initial conditions

$$\begin{cases} \xi^{\alpha}(\eta,0) = 3\eta[\alpha-1,1-\alpha] + \frac{\eta^2}{2}; \\ \xi^{\alpha}(0,\zeta) = \zeta[\alpha-1,1-\alpha]; \\ \xi^{\alpha}_{\eta}(\eta,0) = \xi^{\alpha}_{\zeta}(0,\zeta) = 0 \end{cases}$$

Here $\xi$: $[0,\infty[\times[0,\infty[\to\mathfrak{R}_g$ is a fuzzy valued function, $\eta,\zeta \geq 0$ and $\alpha \in [0,1]$
Applying the FST to both sides of equation (1), we yield:

$$S_t[\xi^{\alpha}_{\eta\eta}(\eta,\zeta)] = S_t[\xi^{\alpha}_{\zeta\zeta}(\eta,\zeta)] + S_t[\zeta\xi^{\alpha}_{\zeta}(\eta,\zeta)] + S_t[\eta\xi^{\alpha}_{\eta}(\eta,\zeta)] + S_t[\eta^2] + S_t[\zeta]$$

Four scenarios can be distinguished from the solutions of equation (12).
Case 1: If the function ξ is (i)-differentiable with respect to both η and ζWe yield:

$$\begin{cases} S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \\ S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \end{cases} \tag{13}$$

Applying the theorems, $S_t[\eta^2] = \eta^2$ and $S_t[\zeta] = q$, to equation (13), we yield the following equations:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\underline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\overline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(\alpha-1) + \frac{\eta^2}{2}\right]\left(-q - \frac{1}{q^2}\right) + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(1-\alpha) + \frac{\eta^2}{2}\right]\left(-q - \frac{1}{q^2}\right) + \eta^2 + q \end{cases} \tag{14}$$

Solving eq. (14), we yield $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ and $S_t[\overline{\xi}^{\alpha}(\eta,\zeta)]$ satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1, 1-\alpha]$, where

$S_t[\xi^{\alpha}(0,\zeta)] = \zeta[\alpha-1, 1-\alpha]$

and the C.F. and P. I for $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ is given by:

$$\text{C.F} = \frac{q}{2}(\alpha-1)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right]$$

And,

$$\text{P.I} = 3\eta(\alpha-1)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right]$$

Similarly, for $S_t[\overline{\xi}^{\alpha}(\eta,\zeta)]$

$$\text{C.F} = \frac{q}{2}(1-\alpha)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right]$$

And,

$$\text{P.I} = 3\eta(1-\alpha)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right]$$

We then get the solutions of the equation. (14) by putting the values of C.F. and P.I., as follows:

$$\begin{cases} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(\alpha-1)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right] + \\ \qquad \frac{q}{2}(\alpha-1)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \\ S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(1-\alpha)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right] + \\ \qquad \frac{q}{2}(1-\alpha)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \end{cases} \tag{15}$$

Since the FST amplifies the coefficient of the power series function, then, using the inverse FST of the $n^{th}$ coefficient of Taylor's expansion of all the above terms of the problem (15), we obtain the $n^{th}$ term of

$$T_n = e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} = 1 + \frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2} + \frac{\left(\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}\right)^2}{2!} + \cdots = \sum_{n=0}^{\infty}\frac{\left(\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\frac{n}{2}}2^n\left[1+\frac{4n\left(\frac{1}{q^2}+q\right)}{2\eta^2}+\frac{4^2n(n-2)\left(\frac{1}{q^2}+q\right)^2}{2^3\eta^4}+\cdots+\cdots\right]$$

There is no term apparent of $p^n$in the expansion, and so we neglect the 2nd term onwards.

Therefore, $T_n$ of $e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} = \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}$

Similarly, $T_n$ of $e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} = \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\frac{n}{2}}2^n(-1)^{\frac{3n}{2}}$

And $T_n$ of $\frac{1}{(q^2-q+1)} = \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}$

And $T_n$ of $\frac{q}{(q^2-q+1)} = \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{(n+2)}{2}}$

i.e., the coefficient of the nth power is $(-1)^{\frac{n}{2}}2^n\eta^{\frac{n}{2}}$.

At last, we yield $\underline{\xi}^{\alpha}(\eta,\zeta)$ and $\overline{\xi}^{\alpha}(\eta,\zeta)$ by applying the inverse of FST on both sides of the equation. (15) as follows:

$$\begin{cases}\underline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(\alpha-1)\left(1-\frac{\zeta^2}{2}\right)+\frac{1}{3}e^{-\zeta}(\eta^2-1)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(\alpha-1)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\frac{1}{3}e^{-\zeta}(\eta^2-1)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\end{cases} \tag{16}$$

The requirements,

$$\overline{\xi}^{\alpha}(\eta,\zeta)\ge\underline{\xi}^{\alpha}(\eta,\zeta),\ \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\ge\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta),\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\ge\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta),\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\ge\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta),\ \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\ge\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$$

Must be met for the solution in Eq. (16) to be considered fuzzy. To find out if the solutions satisfy the requirements as mentioned earlier, the lengths of $\xi$, $\xi_\eta$, $\xi_\zeta$, $\xi_{\eta\eta}$, $\xi_{\zeta\zeta}$ are first computed as follows:

$$\begin{cases}\overline{\xi}^{\alpha}(\eta,\zeta)-\underline{\xi}^{\alpha}(\eta,\zeta) = 6\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) = 6\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)2^{n-1}\eta^{\frac{n+2}{2}}\zeta e^{\frac{\eta^2}{2}}[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}] \\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) = 6\eta(1-\alpha)\zeta+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) = \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)(n+4)2^{n-2}\eta^{\frac{n+4}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) = 6\eta(\alpha-1)\end{cases}$$

From the above calculation, it can be concluded that the solutions are not (ii)-differentiable, as the solutions are not defined for $\alpha\in[0,1]$ The solutions obtained are reduced to crisp solutions, as it is only defined when $\alpha = 1$ Thus, we can say that in this case, no fuzzy solution exists.

Case 2: Next, considering that $\xi$ is (i)-differentiable w.r.t $\eta$ and (ii)-differentiable w.r.t $\zeta$The following system is attained:

$$\begin{cases}S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right]+S_t\left[\zeta\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right]+S_t\left[\eta\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right]+S_t[\eta^2]+S_t[\zeta] \\ S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right]+S_t\left[\zeta\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right]+S_t\left[\eta\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right]+S_t[\eta^2]+S_t[\zeta]\end{cases} \tag{17}$$

Applying the theorems, $S_t[\eta^2] = \eta^2$ and $S_t[\zeta] = q$, to equation (17), we yield the following equations:

$$\begin{cases}\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]=\frac{1}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right]-\frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right]+qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-\\ \qquad qS_t\left[\overline{\xi}^{\alpha}(\eta,0)\right]+\eta\frac{\partial}{\partial\eta}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]+\eta^2+q\\ \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]=\frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right]-\frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right]+qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\\ \qquad qS_t\left[\underline{\xi}^{\alpha}(\eta,0)\right]+\eta\frac{\partial}{\partial\eta}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]+\eta^2+q\end{cases}$$

Rearranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases}\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]+\frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\eta\frac{\partial}{\partial\eta}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]+qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]\\ \qquad =\left[3\eta(\alpha-1)+\frac{\eta^2}{2}\right]\left(-q-\frac{1}{q^2}\right)+\eta^2+q\\ \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-\eta\frac{\partial}{\partial\eta}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]\\ \qquad =\left[3\eta(1-\alpha)+\frac{\eta^2}{2}\right]\left(-q-\frac{1}{q^2}\right)+\eta^2+q\end{cases}\tag{18}$$

Solving eq. (18), we yield $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ and $S_t[\overline{\xi}^{\alpha}(\eta,\zeta)]$ satisfying the initial condition $\xi^{\alpha}(0,\zeta)=\zeta[\alpha-1,1-\alpha]$, where

$S_t[\xi^{\alpha}(0,\zeta)]=\zeta[\alpha-1,1-\alpha]$

We then obtain the complete solution of equation (18) by following the same procedure as in case 1, which is as follows:

$$\begin{cases}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]=3\eta(\alpha-1)(-1-q^2)-\frac{\eta^2}{2}\left[\frac{2}{3(q+1)}+\frac{2q+1}{3(q^2-q+1)}\right]-\left[\frac{1}{3(q^2-q+1)}-\frac{1}{(q+1)}\right]+\\ \qquad \frac{q}{2}(\alpha-1)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}+e^{\frac{-\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}\right]\\ S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]=3\eta(1-\alpha)(-1-q^2)+\frac{\eta^2}{2}\left[\frac{2}{3(q+1)}-\frac{2q-1}{3(q^2-q+1)}\right]-\left[\frac{q-1}{(q^2-q+1)}-\frac{1}{(q+1)}\right]+\\ \qquad \frac{q}{2}(1-\alpha)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}+e^{\frac{-\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}\right]\end{cases}\tag{19}$$

Following the procedure the same as in case 1, we yield $\underline{\xi}^{\alpha}(\eta,\zeta)$ and $\overline{\xi}^{\alpha}(\eta,\zeta)$ by applying the inverse of FST on both sides of eq. (19) as follows:

$$\begin{cases}\underline{\xi}^{\alpha}(\eta,\zeta)=3\eta(\alpha-1)\left(-1-\frac{\zeta^2}{2}\right)-e^{-\zeta}\left(\frac{1}{3}\eta^2+1\right)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}-\\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+\\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^{n}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(\alpha-1)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\\ \overline{\xi}^{\alpha}(\eta,\zeta)=3\eta(1-\alpha)\left(-1-\frac{\zeta^2}{2}\right)-e^{-\zeta}\left(\frac{1}{3}\eta^2+1\right)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}-\\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+\\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^{n}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\end{cases}\tag{20}$$

The requirements,

$\overline{\xi}^{\alpha}(\eta,\zeta)\geq\underline{\xi}^{\alpha}(\eta,\zeta)$, $\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\geq\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$, $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\geq\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$, $\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\geq\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$, $\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\geq\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$ must be met for the solution in Eq. (20) to be considered a fuzzy solution. To find out if the solutions satisfy the requirements as mentioned earlier, the lengths of $\xi$, $\xi_{\eta}$, $\xi_{\zeta}$, $\xi_{\eta\eta}$, $\xi_{\zeta\zeta}$ are first computed as follows:

$$\begin{cases}\overline{\xi}^{\alpha}(\eta,\zeta)-\underline{\xi}^{\alpha}(\eta,\zeta)=6\eta(1-\alpha)\left(1+\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\\ \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)=6\eta(1-\alpha)\left(1+\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)2^{n-1}\eta^{\frac{n+2}{2}}\zeta e^{\frac{\eta^2}{2}}[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}]\\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)=6\eta(1-\alpha)\zeta+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n}\eta^{\frac{n}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\\ \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)=\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)(n+4)2^{n-2}\eta^{\frac{n+4}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right]\\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)-\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)=6\eta(1-\alpha)\end{cases}$$

We can see from the calculation above that the following conditions are satisfied. Therefore, it can be concluded that Case 2 is a fuzzy solution.
Case 3: Assuming that the function ξ is (ii)-differentiable with respect to η and (i)-differentiable w.r.t ζWe yield:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \\ S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \end{cases} \quad (21)$$

Applying the theorems, $S_t[\eta^2] = \eta^2$ and $S_t[\zeta] = q$, to equation (21), we yield the following equations:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\underline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\underline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \end{cases}$$

Rearranging and positioning the boundary conditions in the above equations we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(1-\alpha) + \frac{\eta^2}{2}\right]\left(-q - \frac{1}{q^2}\right) + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(\alpha-1) + \frac{\eta^2}{2}\right]\left(-q - \frac{1}{q^2}\right) + \eta^2 + q \end{cases} \quad (22)$$

Solving eq. (22), we yield $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ and $S_t[\overline{\xi}^{\alpha}(\eta,\zeta)]$ satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1, 1-\alpha]$, where

$S_t[\xi^{\alpha}(0,\zeta)] = \zeta[\alpha-1, 1-\alpha]$

We then obtain the complete solution of equation (22) by following the same procedure as in case 1, which is as follows:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(1-\alpha)(-1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{q-1}{(q^2-q+1)} - \frac{1}{(q+1)}\right] + \\ \qquad \frac{q}{2}(1-\alpha)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \\ S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(\alpha-1)(-1-q^2) - \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} + \frac{2q+1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q^2-q+1)} - \frac{1}{(q+1)}\right] + \\ \qquad \frac{q}{2}(\alpha-1)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2-4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \end{cases} \quad (23)$$

Following the procedure the same as in case 1, we yield $\underline{\xi}^{\alpha}(\eta,\zeta)$ and $\overline{\xi}^{\alpha}(\eta,\zeta)$ by applying the inverse of FST on both sides of eq. (23) as follows:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(1-\alpha)\left(-1-\frac{\zeta^2}{2}\right) - e^{-\zeta}\left(\frac{1}{3}\eta^2+1\right) - \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}} - \\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}} - \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}} + \\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}} + \sum_{n=0}^{\infty}\frac{(\alpha-1)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}} + (-1)^{\frac{3n}{2}}\right] \\ \underline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(\alpha-1)\left(-1-\frac{\zeta^2}{2}\right) - e^{-\zeta}\left(\frac{1}{3}\eta^2+1\right) - \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}} - \\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}} - \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}} + \\ \qquad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}} + \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}} + (-1)^{\frac{3n}{2}}\right] \end{cases} \quad (24)$$

The requirements,
$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$, $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$, $\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$, $\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$, $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$ must be met in order for the solution in Eq. (24) to be considered fuzzy. To find out if the solutions satisfy the aforementioned requirements, the lengths of $\xi$, $\xi_{\eta}$, $\xi_{\zeta}$, $\xi_{\eta\eta}$, $\xi_{\zeta\zeta}$ are first computed as follows:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) - \underline{\xi}^{\alpha}(\eta,\zeta) = 6\eta(1-\alpha)\left(1+\frac{\zeta^2}{2}\right) + \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) = 6\eta(1-\alpha)\left(1+\frac{\zeta^2}{2}\right) + \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)2^{n-1}\eta^{\frac{n+2}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) = 6\eta(1-\alpha)\zeta + \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) = \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)(n+4)2^{n-2}\eta^{\frac{n+4}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) = 6\eta(1-\alpha) \end{cases}$$

From the above calculation, it can be concluded that the solutions are not (ii)-differentiable as the solutions are not defined for $\alpha \in [0,1]$ The solutions obtained are reduced to crisp solutions as it is only defined when $\alpha = 1$ Thus, we can say that in this case, no fuzzy solution exists.

Case 4: Assuming that the function $\xi$ is (ii)-differentiable with respect to both $\eta$ and $\zeta$ we yield

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \\ S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t\left[\zeta\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)\right] + S_t\left[\eta\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta)\right] + S_t[\eta^2] + S_t[\zeta] \end{cases} \tag{25}$$

Applying the theorems, $S_t[\eta^2] = \eta^2$ and $S_t[\zeta] = q$, to equation (25), we yield the following equations:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\overline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \\ \qquad qS_t\left[\underline{\xi}^{\alpha}(\eta,0)\right] + \eta\frac{\partial}{\partial\eta}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + \eta^2 + q \end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + \frac{1}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] + qS_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(1-\alpha) + \frac{\eta^2}{2}\right]\left(-q-\frac{1}{q^2}\right) + \eta^2 + q \\ \frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + \frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \eta\frac{\partial}{\partial\eta}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] + qS_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] \\ \qquad = \left[3\eta(\alpha-1) + \frac{\eta^2}{2}\right]\left(-q-\frac{1}{q^2}\right) + \eta^2 + q \end{cases} \tag{26}$$

Solving eq. (26), we yield $S_t[\underline{\xi}^{\alpha}(\eta,\zeta)]$ and $S_t[\overline{\xi}^{\alpha}(\eta,\zeta)]$ satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1, 1-\alpha]$, where

$S_t[\xi^{\alpha}(0,\zeta)] = \zeta[\alpha-1, 1-\alpha]$

We then obtain the complete solution of equation (26) by following the same procedure as in case 1, which is as follows:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(1-\alpha)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right] + \\ \qquad \frac{q}{2}(1-\alpha)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \\ S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = 3\eta(\alpha-1)(1-q^2) + \frac{\eta^2}{2}\left[\frac{2}{3(q+1)} - \frac{2q-1}{3(q^2-q+1)}\right] - \left[\frac{1}{3(q+1)} - \frac{q-1}{3(q^2-q+1)}\right] + \\ \qquad \frac{q}{2}(\alpha-1)e^{\frac{\eta^2}{2}}\left[e^{\frac{\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}} + e^{\frac{-\eta\sqrt{\eta^2+4\left(\frac{1}{q^2}+q\right)}}{2}}\right] \end{cases} \tag{27}$$

Following the procedure the same as in case 1, we yield $\underline{\xi}^{\alpha}(\eta,\zeta)$ and $\overline{\xi}^{\alpha}(\eta,\zeta)$ by applying the inverse of FST on both sides of the equation. (27) as follows:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\frac{1}{3}e^{-\zeta}(\eta^2-1)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(\alpha-1)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \underline{\xi}^{\alpha}(\eta,\zeta) = 3\eta(\alpha-1)\left(1-\frac{\zeta^2}{2}\right)+\frac{1}{3}e^{-\zeta}(\eta^2-1)-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n-1}(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}-\frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n+4}{2}}2^{n+1}(-1)^{\frac{n}{2}}\zeta^{\frac{n+2}{2}}+ \\ \quad \frac{1}{3}\sum_{n=0}^{\infty}\eta^{\frac{n}{2}}2^n(-1)^{\frac{n}{2}}\zeta^{\frac{n}{2}}+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^{n-1}\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \end{cases} \tag{28}$$

The requirements,
$\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$, $\overline{\xi}_{\eta}^{\alpha}(\eta,\zeta) \geq \underline{\xi}_{\eta}^{\alpha}(\eta,\zeta)$, $\overline{\xi}_{\zeta}^{\alpha}(\eta,\zeta) \geq \underline{\xi}_{\zeta}^{\alpha}(\eta,\zeta)$, $\overline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta) \geq \underline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta)$, $\overline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta) \geq \underline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta)$ must be met for the solution in Eq. (28) to be considered a fuzzy solution. To find out if the solutions satisfy the requirements above, the lengths of $\xi$, $\xi_{\eta}$, $\xi_{\zeta}$, $\xi_{\eta\eta}$, $\xi_{\zeta\zeta}$ are first computed as follows:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta)-\underline{\xi}^{\alpha}(\eta,\zeta) = 6\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}\zeta e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}_{\eta}^{\alpha}(\eta,\zeta)-\underline{\xi}_{\eta}^{\alpha}(\eta,\zeta) = 6\eta(1-\alpha)\left(1-\frac{\zeta^2}{2}\right)+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)2^{n-1}\eta^{\frac{n+2}{2}}\zeta e^{\frac{\eta^2}{2}}[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}] \\ \overline{\xi}_{\zeta}^{\alpha}(\eta,\zeta)-\underline{\xi}_{\zeta}^{\alpha}(\eta,\zeta) = 6\eta(1-\alpha)\zeta+\sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}2^n\eta^{\frac{n}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta)-\underline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta) = \sum_{n=0}^{\infty}\frac{(1-\alpha)}{n!}(n+2)(n+4)2^{n-2}\eta^{\frac{n+4}{2}}e^{\frac{\eta^2}{2}}\left[(-1)^{\frac{n}{2}}+(-1)^{\frac{3n}{2}}\right] \\ \overline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta)-\underline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta) = 6\eta(\alpha-1) \end{cases}$$

We can see from the calculation above that the following conditions are satisfied. Therefore, it can be concluded that Case 4 is a fuzzy solution.
Example 2. Let us consider the following linear second-order non-homogeneous FPDE as follows:

$$\xi_{\eta\eta}^{\alpha}(\eta,\zeta) = \xi_{\zeta\zeta}^{\alpha}(\eta,\zeta)+\eta \tag{29}$$

With boundary and initial conditions:

$$\begin{cases} \xi^{\alpha}(\eta,0) = \eta[\alpha-1,1-\alpha] \\ \xi^{\alpha}(0,\zeta) = \zeta[\alpha-1,1-\alpha] \\ \xi_{\eta}^{\alpha}(\eta,0) = \xi_{\zeta}^{\alpha}(0,\zeta) = 0 \end{cases}$$

Here $\xi: [0,\infty[\times[0,\infty[\to \Re_g$ is a fuzzy-valued function, $\eta,\zeta \geq 0$ and $\alpha \in [0,1]$.
Applying the FST to both sides of the equation. (17), we get:

$S_t[\xi_{\eta\eta}^{\alpha}(\eta,\zeta)] = S_t[\xi_{\zeta\zeta}^{\alpha}(\eta,\zeta)]+S_t[\eta]$
Four scenarios can be distinguished from the solutions of equation (17).
Case 1: Assuming that the function $\xi$ is (i)-differentiable with respect to both $\eta$ and $\zeta$, we yield

$$\begin{cases} S_t\left[\underline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta)\right] = S_t\left[\underline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta)\right]+S_t[\eta] \\ S_t\left[\overline{\xi}_{\eta\eta}^{\alpha}(\eta,\zeta)\right] = S_t\left[\overline{\xi}_{\zeta\zeta}^{\alpha}(\eta,\zeta)\right]+S_t[\eta] \end{cases} \tag{30}$$

Applying theorems, $S_t[\eta] = \eta$ to equation (30) , we yield the following equation:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right]-\frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right]+\eta \\ \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right]-\frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right]+\eta \end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = -\frac{1}{q^2}\eta(\alpha-1)+\eta \\ \frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]-\frac{1}{q^2}S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = -\frac{1}{q^2}\eta(1-\alpha)+\eta \end{cases} \tag{31}$$

Solving eq. (19), we yield $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$ and $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$, satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1,1-\alpha]$, where

$S[\xi^{\alpha}(0,\zeta)] = q[\alpha-1,1-\alpha]$

We then get the solutions of eq. (32) as follows:

$$\begin{cases} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{q}{2}(\alpha-1)\left[e^{\frac{\eta}{q}} + e^{\frac{-\eta}{q}}\right] - \eta q^2 + \eta(\alpha-1) \\ S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{q}{2}(1-\alpha)\left[e^{\frac{\eta}{q}} + e^{\frac{-\eta}{q}}\right] - \eta q^2 + \eta(1-\alpha) \end{cases} \quad (32)$$

By applying the inverse of FST in eq. (32) , we yield:

$$\begin{cases} \underline{\xi}^{\alpha}(\eta,\zeta) = \zeta(\alpha-1)\mathrm{Cosh}(\eta\zeta) - 2\eta\zeta^2 + \eta(\alpha-1) \\ \overline{\xi}^{\alpha}(\eta,\zeta) = \zeta(1-\alpha)\mathrm{Cosh}(\eta\zeta) - 2\eta\zeta^2 + \eta(1-\alpha) \end{cases} \quad (33)$$

The requirements $\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$ and $\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$ must be met in order for the solution in Eq. (33) to be considered fuzzy. To find out if the solutions satisfy the aforementioned requirements, we first compute the lengths of $\xi, \xi_{\eta}, \xi_{\zeta}, \xi_{\eta\eta}$ and $\xi_{\zeta\zeta}$ as follows

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) - \underline{\xi}^{\alpha}(\eta,\zeta) = 2\zeta(1-\alpha)\mathrm{Cosh}(\eta\zeta) + 2\eta(1-\alpha) \\ \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\xi}(\eta,\zeta) = 2\zeta^2(1-\alpha)\mathrm{Sinh}(\eta\zeta) + 2(1-\alpha) \\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) = 2(1-\alpha)\zeta^3\mathrm{Cosh}(\eta\zeta) \\ \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) = 0 \\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) = 0 \end{cases}$$

We can see from the calculation above that $\alpha \leq 1$ for all $\eta, \zeta \geq 0$. So, it is a fuzzy solution having a triangular fuzzy number. The findings for this instance are shown in Fig. 1 and Fig. 2. In Fig. 1, we allow the value of $\eta$ to fluctuate between (a) 0 and 0.1222 and (b) 0 and 0.3333, whereas the values of $\zeta$ are fixed at (a) 0 and (b) 0.8. In Fig. 2, the same procedure is used, with $\eta$ fixed at (a) 0.4 and (b) 0.8 and $\zeta$ allowed to vary between (a) 1.6 and 4 and (b) 2.7 and 4

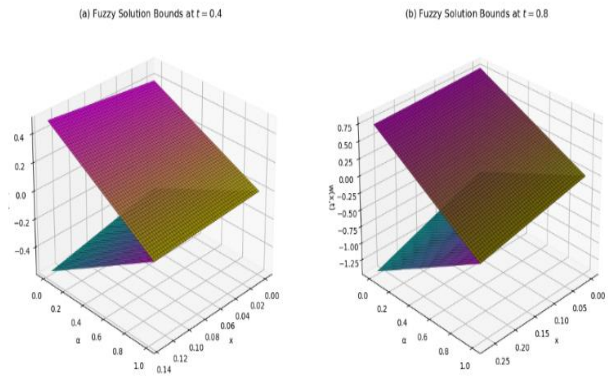


**Fig. 1:** The Solutions of Eq.(30) for Case 1 when (A) Z = 0.4 and (B) Z = 0.8.

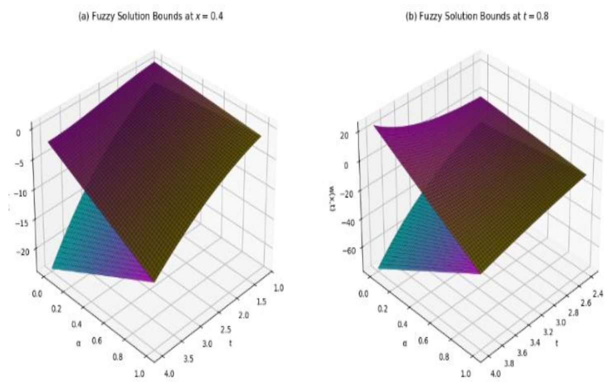


**Fig. 2:** The Solutions of Eq.(30) for Case 1 When (A) H = 0.4 and (B) H = 0.8.

Case 2: Assuming that the function $\xi$ is (i)-differentiable with respect to $\eta$ and (ii)-differentiable w.r.t $\zeta$We yield:

$$\begin{cases} S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \\ S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \end{cases} \quad (34)$$

Applying Theorems 2.1–2.4, $S_t[\eta] = \eta$ to equation (34), we yield the following equation:

$$\begin{cases}\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]=\frac{1}{q^2}S_t\left[\overline{\xi}^\alpha(\eta,\zeta)\right]-\frac{1}{q^2}\left[\overline{\xi}^\alpha(\eta,0)\right]-\frac{1}{q}\left[\overline{\xi}'^\alpha(\eta,0)\right]+\eta\\ \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^\alpha(\eta,\zeta)\right]=\frac{1}{q^2}S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]-\frac{1}{q^2}\left[\underline{\xi}^\alpha(\eta,0)\right]-\frac{1}{q}\left[\underline{\xi}'^\alpha(\eta,0)\right]+\eta\end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases}\frac{\partial^2}{\partial\eta^2}S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]-\frac{1}{q^2}S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]=\frac{1}{q^2}\eta(\alpha-1)+\eta\\ \frac{\partial^2}{\partial\eta^2}S_t\left[\overline{\xi}^\alpha(\eta,\zeta)\right]-\frac{1}{q^2}S_t\left[\overline{\xi}^\alpha(\eta,\zeta)\right]=\frac{1}{q^2}\eta(1-\alpha)+\eta\end{cases} \tag{35}$$

Solving eq. (35), we yield $S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]$ and $S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]$, satisfying the initial condition $\xi^\alpha(0,\zeta)=\zeta[\alpha-1,1-\alpha]$, where

$S[\xi^\alpha(0,\zeta)]=q[\alpha-1,1-\alpha]$

We then get the solutions to the equation. (35) as follows:

$$\begin{cases}S_t\left[\underline{\xi}^\alpha(\eta,\zeta)\right]=q(\alpha-1)\mathrm{Cos}\left(\frac{\eta}{q}\right)+\eta q^2+\eta(\alpha-1)\\ S_t\left[\overline{\xi}^\alpha(\eta,\zeta)\right]=q(1-\alpha)\mathrm{Cos}\left(\frac{\eta}{q}\right)+\eta q^2+\eta(1-\alpha)\end{cases} \tag{36}$$

By applying the inverse of FST, we yield:

$$\begin{cases}\underline{\xi}^\alpha(\eta,\zeta)=\zeta(\alpha-1)J_0(\eta\zeta)+2\eta\zeta^2+\eta(\alpha-1)\\ \overline{\xi}^\alpha(\eta,\zeta)=\zeta(1-\alpha)J_0(\eta\zeta)+2\eta\zeta^2+\eta(1-\alpha)\end{cases} \tag{37}$$

Where $J_0$ is the Bessel function of the first kind of order 0.
The requirements $\overline{\xi}^\alpha(\eta,\zeta)\geq\underline{\xi}^\alpha(\eta,\zeta)$; $\underline{\xi}^\alpha_\eta(\eta,\zeta)\geq\overline{\xi}^\alpha_\eta(\eta,\zeta)$; $\underline{\xi}^\alpha_\zeta(\eta,\zeta)\geq\overline{\xi}^\alpha_\zeta(\eta,\zeta)$; $\underline{\xi}^\alpha_{\eta\eta}(\eta,\zeta)\geq\overline{\xi}^\alpha_{\eta\eta}(\eta,\zeta)$ and $\underline{\xi}^\alpha_{\zeta\zeta}(\eta,\zeta)\geq\overline{\xi}^\alpha_{\zeta\zeta}(\eta,\zeta)$ must be met for the solution in Eq. (34) to be considered a fuzzy solution. To find out if the solutions satisfy the requirements as mentioned earlier, we first compute the lengths of $\xi$, $\xi_\eta$, $\xi_\zeta$, $\xi_{\eta\eta}$ and $\xi_{\zeta\zeta}$ as follows

$$\begin{cases}\overline{\xi}^\alpha(\eta,\zeta)-\underline{\xi}^\alpha(\eta,\zeta)=2\zeta(1-\alpha)J_0(\eta\zeta)+2\eta(1-\alpha)\\ \overline{\xi}^\alpha_\eta(\eta,\zeta)-\underline{\xi}^\alpha_\xi(\eta,\zeta)=-2\zeta^2(1-\alpha)J_1(\eta\zeta)+2(1-\alpha)\\ \underline{\xi}^\alpha_\zeta(\eta,\zeta)-\underline{\xi}^\alpha_\zeta(\eta,\zeta)=-(1-\alpha)\zeta^3[J_0(\eta\zeta)-J_1(\eta\zeta)]\\ \overline{\xi}^\alpha_{\eta\eta}(\eta,\zeta)-\underline{\xi}^\alpha_{\eta\eta}(\eta,\zeta)=2(1-\alpha)[J_0(\eta\zeta)-\eta\zeta J_1(\eta\zeta)]\\ \underline{\xi}^\alpha_{\zeta\zeta}(\eta,\zeta)-\underline{\xi}^\alpha_{\zeta\zeta}(\eta,\zeta)=2(\alpha-1)[-2\eta J_1(\eta\zeta)-\frac{\eta^2\zeta}{2}J_0(\eta\zeta)-J_2(\eta\zeta)]\end{cases}$$

Since the solutions are not determined for $\alpha\in[0,1[$ It may be said that they are not (ii)-differentiable. Since it is only defined when $\alpha=1$The obtained solutions are reduced to crisp solutions. Therefore, from Fig. 3 and Fig. 4, we can conclude that there are no fuzzy solutions as the graph is not a triangular fuzzy number.

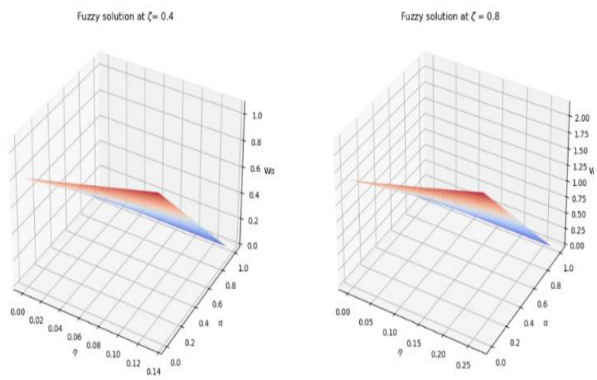


**Fig. 3:** The Solutions of EQ.(34) for Case 2 When (A) Z = 0.4 and (B) Z = 0.8.

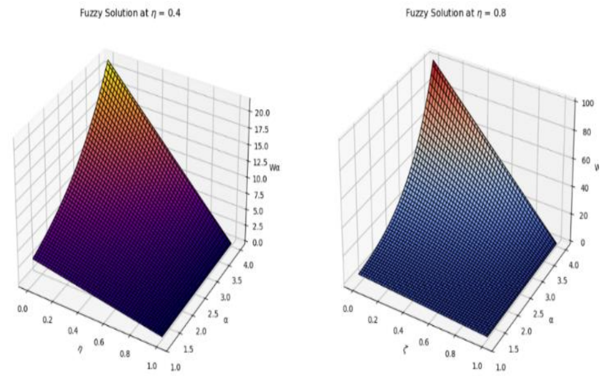


**Fig. 4:** The Solutions of Eq.(34) for Case 2 when (A) H = 0.4 and (B) H = 0.8

Case 3: Assuming that the function ξ is (ii)-differentiable with respect to η and (i)-differentiable w.r.t ζWe yield:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \\ S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \end{cases} \tag{38}$$

Applying the Theorems, $S_t[\eta] = \eta$ to equation (38) , we yield the following equation:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \eta \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \eta \end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}\eta(\alpha-1) + \eta \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2}\eta(1-\alpha) + \eta \end{cases} \tag{39}$$

Solving eq. (39), we yield $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$ and $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$, satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1, 1-\alpha]$, where

$S[\xi^{\alpha}(0,\zeta)] = q[\alpha-1, 1-\alpha]$

We then get the solutions of the equation. (39) as follows:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = q(1-\alpha)\mathrm{Cos}\left(\frac{\eta}{q}\right) + \eta q^2 + \eta(1-\alpha) \\ S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = q(\alpha-1)\mathrm{Cos}\left(\frac{\eta}{q}\right) + \eta q^2 + \eta(\alpha-1) \end{cases} \tag{40}$$

By applying the inverse of FST, we yield:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) = \zeta(1-\alpha)J_0(\eta\zeta) + 2\eta\zeta^2 + \eta(1-\alpha) \\ \underline{\xi}^{\alpha}(\eta,\zeta) = \zeta(\alpha-1)J_0(\eta\zeta) + 2\eta\zeta^2 + \eta(\alpha-1) \end{cases} \tag{41}$$

Where $J_0$ is the Bessel function of the first kind of order 0.

The requirements $\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$ and $\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$ must be met for the solution in Eq. (38) to be considered fuzzy. To find out if the solutions satisfy the requirements as mentioned earlier, we first compute the lengths of $\xi$, $\xi_{\eta}$, $\xi_{\zeta}$, $\xi_{\eta\eta}$ and $\xi_{\zeta\zeta}$ as follows

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) - \underline{\xi}^{\alpha}(\eta,\zeta) = 2\zeta(1-\alpha)J_0(\eta\zeta) + 2\eta(1-\alpha) \\ \underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta) = -2\zeta^2(1-\alpha)J_1(\eta\zeta) + 2(1-\alpha) \\ \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) = -(1-\alpha)\zeta^3[J_0(\eta\zeta) - J_1(\eta\zeta) \\ \underline{w}^{\alpha}_{\eta\eta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) = 2(1-\alpha)[J_0(\eta\zeta) - \eta\zeta J_1(\eta\zeta)] \\ \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) - \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) = 2(\alpha-1)[-2\eta J_1(\eta\zeta) - \frac{\eta^2\zeta}{2}J_0(\eta\zeta) - J_2(\eta\zeta)] \end{cases}$$

We can see from the calculation above that $\alpha \leq 1$ for all $\eta, \zeta \geq 0$. So, it is a fuzzy solution having a triangular fuzzy number. The findings for this instance are shown in Fig. 5 and Fig. 6. In Fig. 5, we allow the value of η to fluctuate between (a) 0 and 0.1222 and (b) 0 and

0.3333, whereas the values of ζ are fixed at (a) 0.4 and (b) 0.8. In Fig. 6, the same procedure is used, with η fixed at (a) 0.4 and (b) 0.8 and ζ allowed to vary between (a) 1.6 and 4 and (b) 2.7 and 4.

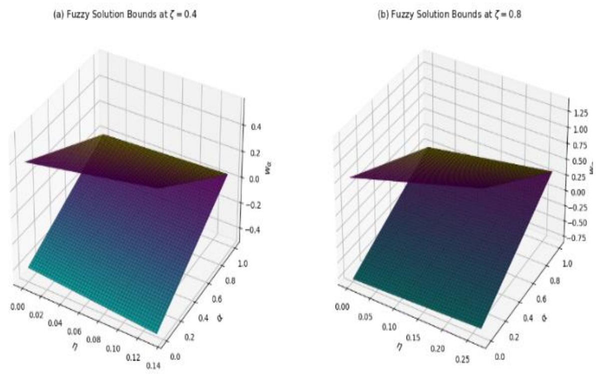


**Fig. 5:** The Solutions of Eq.(38) for Case 3 when (A) H = 0.4 and (B) H = 0.8.

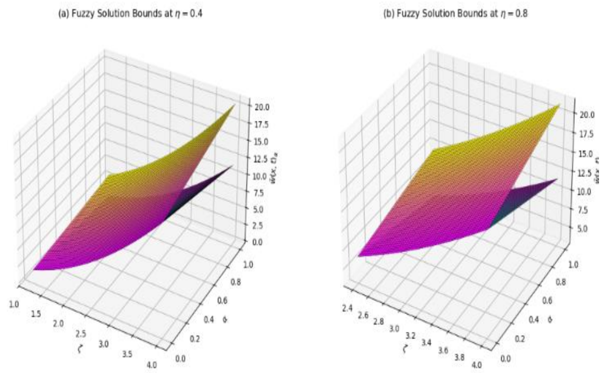


**Fig. 6:** The Solutions of Eq.(38) for Case 3 when (A) Z = 0.4 and (B) Z = 0.8.

Case 4: Assuming that the function ξ is (ii)-differentiable with respect to both η and ζ, we yield

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \\ S_t\left[\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)\right] = S_t\left[\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)\right] + S_t[\eta] \end{cases} \tag{42}$$

Applying the Theorems, $S_t[\eta] = \eta$ to equation (42), we yield the following equations:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\overline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\overline{\xi}'^{\alpha}(\eta,0)\right] + \eta \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2}\left[\underline{\xi}^{\alpha}(\eta,0)\right] - \frac{1}{q}\left[\underline{\xi}'^{\alpha}(\eta,0)\right] + \eta \end{cases}$$

By arranging and positioning the boundary conditions in the above equations, we get:

$$\begin{cases} \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = -\frac{1}{q^2}\eta(1-\alpha) + \eta \\ \frac{\partial^2}{\partial\eta^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] - \frac{1}{q^2} S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = -\frac{1}{q^2}\eta(\alpha-1) + \eta \end{cases} \tag{43}$$

Solving eq. (43), we yield $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$ and $S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right]$, satisfying the initial condition $\xi^{\alpha}(0,\zeta) = \zeta[\alpha-1, 1-\alpha]$, where

$S[\xi^{\alpha}(0,\zeta)] = q[\alpha-1, 1-\alpha]$

We then get the solutions of the equation. (43) as follows:

$$\begin{cases} S_t\left[\overline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{q}{2}(1-\alpha)\left[e^{\frac{\eta}{q}} + e^{\frac{-\eta}{q}}\right] - \eta q^2 + \eta(1-\alpha) \\ S_t\left[\underline{\xi}^{\alpha}(\eta,\zeta)\right] = \frac{q}{2}(\alpha-1)\left[e^{\frac{\eta}{q}} + e^{\frac{-\eta}{q}}\right] - \eta q^2 + \eta(\alpha-1) \end{cases} \tag{44}$$

By applying the inverse of FST, we yield:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) = \zeta(1-\alpha)\mathrm{Cosh}(\eta\zeta) - 2\eta\zeta^2 + \eta(1-\alpha) \\ \underline{\xi}^{\alpha}(\eta,\zeta) = \zeta(\alpha-1)\mathrm{Cosh}(\eta\zeta) - 2\eta\zeta^2 + \eta(\alpha-1) \end{cases} \tag{45}$$

The requirements $\overline{\xi}^{\alpha}(\eta,\zeta) \geq \underline{\xi}^{\alpha}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta)$; $\underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta)$ and $\underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) \geq \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta)$ must be met for the solution in Eq. (42) to be considered fuzzy. To find out if the solutions satisfy the requirements above, we first compute the lengths of $\xi, \xi_{\eta}, \xi_{\zeta}, \xi_{\eta\eta}$ and $\xi_{\zeta\zeta}$ as follows:

$$\begin{cases} \overline{\xi}^{\alpha}(\eta,\zeta) - \underline{\xi}^{\alpha}(\eta,\zeta) = 2\zeta(1-\alpha)\mathrm{Cosh}(\eta\zeta) + 2\eta(1-\alpha) \\ \underline{\xi}^{\alpha}_{\eta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\xi}(\eta,\zeta) = 2\zeta^{2}(\alpha-1)\mathrm{Sinh}(\eta\zeta) + 2(1-\alpha) \\ \underline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\zeta}(\eta,\zeta) = 2(\alpha-1)\zeta^{3}\mathrm{Cosh}(\eta\zeta) \\ \underline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\eta\eta}(\eta,\zeta) = 0 \\ \underline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) - \overline{\xi}^{\alpha}_{\zeta\zeta}(\eta,\zeta) = 0 \end{cases}$$

Since the solutions are not determined for $\alpha \in [0,1[$It may be said that they are not (ii)-differentiable. Since it is only defined when $\alpha = 1$The obtained solutions are reduced to crisp solutions. Therefore, from Fig. 7 and Fig. 8, we can conclude that there is a fuzzy solution as the graph is not a triangular fuzzy number.

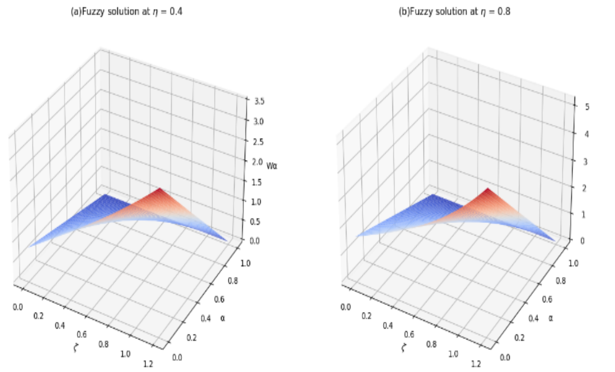


**Fig. 7:** The Solutions of Eq. (42) for Case 4 when (A) H = 0.4 and (B) H = 0.8.

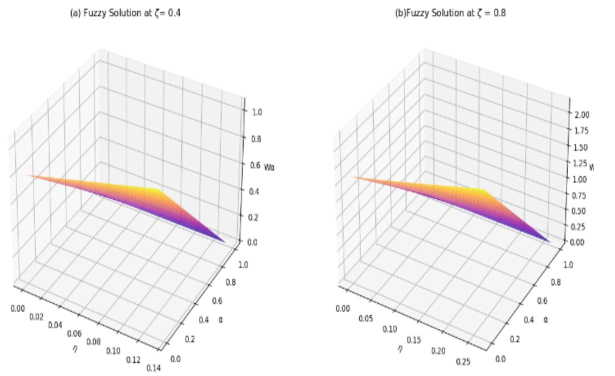


**Fig. 8:** The Solutions of Eq. (42) for Case 4 when (A) Z = 0.4 and (B) Z = 0.8.

## 5. Discussion

The FST possesses several notable advantages over classical fuzzy Laplace and Fourier transforms when applied to fuzzy differential equations. Unlike the Laplace transform, which often requires reformulating initial conditions into derivatives, the Sumudu transform preserves the initial conditions in their original form, simplifying the solution process [7], [14]. The FST maps the time domain directly to the transform domain without scaling, allowing easier inversion and interpretation of the solution [23], [19]. FST allows symbolic manipulation of fuzzy valued terms and supports linearity and convolution-like operations for gH-differentiable functions without altering the fuzziness [22], [25]. In contrast to Laplace or Fourier transforms, the Sumudu transform leads to simpler algebraic expressions and avoids unnecessary domain shifting, thus making it suitable for analytical software implementation.

Fuzzy differential models with Sumudu transform techniques find natural applications in uncertain dynamic systems across various domains. In modelling the flow of fluids through porous or elastic media where material properties (e.g., permeability or viscosity) are imprecisely known, FST provides closed-form fuzzy solutions to PDEs with uncertain coefficients [26], [24]. Many real-time systems are affected by measurement errors and linguistic uncertainties. FST-based FPDE models offer analytical insight into system behaviour under imprecise feedback or fuzzy gain margins [7]. In ecology, where data uncertainties stem from incomplete observations, fuzzy modelling using FST aids in analyzing growth, diffusion, and control under uncertain parameters [30], [28]. FPDEs modelling heat flow or diffusion under vague initial profiles can be efficiently tackled using FST, particularly when the source or boundary conditions are fuzzy-valued [23], [18].

Despite its strengths, the current FST-based approach to FPDEs has several limitations. The method has been primarily developed for linear fuzzy PDEs. Extending it to nonlinear or strongly coupled systems remains a significant challenge. While the transform is extendable, the handling of mixed partial derivatives in fuzzy systems (especially under nontrivial boundary geometries) becomes algebraically complex. Current formulations of FST focus on symbolic or exact solutions. Numerical inversion or approximation strategies need further exploration. The (i)- and (ii)-differentiability influences the solution structure significantly, requiring careful validation of the fuzzy conditions to ensure meaningful interpretation [17], [12].

## 6. Conclusion

In this paper, FST is applied to derive analytical solutions of second-order linear fuzzy partial differential equations (FPDEs) under gH-differentiability. The approach was tested for multiple differentiability cases ((i)–(iv)) with appropriate symbolic validation to ensure the fuzzy solution satisfies the original equation structure. The FST technique preserves initial conditions, supports symbolic computation, and simplifies inversion, offering significant advantages over classical fuzzy Laplace approaches. The results demonstrate that FST provides an efficient and elegant method for tackling linear FPDEs with fuzzy initial and boundary conditions. It has wide-ranging applications in domains involving uncertainty and imprecision, including engineering, fluid dynamics, and biological modelling. Potential extensions of this research include that the proposed method can be used to solve higher-order, fractional, or delay FPDEs.

## Acknowledgement

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