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Research paper

Asymptotic Behavior of The Third-Order Damped Nonlinear Differential Equations with Distributed Deviating Arguments

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Abstract

In this article, we investigate the oscillation of a class of third-order damped nonlinear differential equations with distribution deviating arguments. Using the integral average and generalized Riccati technique, new sufficient criteria for oscillation of the equations' solutions are established. The main results are illustrated by some examples.

Keywords: *Oscillation; Nonlinear, Differential Equations; Third-Order; Distribution Deviating Arguments; Damping.*

1. Introduction

Third-order differential equations are useful in a variety of fields, including physics, engineering, and biology, where they are used to simulate processes affected by third-order derivatives. Third-order differential equations are more complicated than lower-order equations, needing advanced solution techniques and, in many cases, numerical methods for precision computation. Solutions to these equations can take the form of specific functions or series, with the uniqueness or multiplicity of solutions determined by the equation's properties and initial conditions. Third-order differential equations, for example, can help us comprehend vibrations in mechanical systems because they describe the link between applied forces and motion or structural deformations. Third-order delay differential equations, a notable subclass of these equations, have temporal delays and solutions based on prior variable values. Such equations are especially useful when time lags are included in cause-and-effect relationships, such as those seen in physical systems, biological processes, or economic models. Over the last three decades, substantial scholarly attention has been directed to understanding the oscillatory behavior of third-order differential equations with variable coefficients. These efforts have produced a large body of theoretical conclusions, many of which have been published in current academic journals.

The study of oscillation theory within the area of differential equations is a complicated and sophisticated field aiming at understanding the properties and trends of solutions that exhibit oscillatory behaviour. This subject of mathematics includes ordinary and partial differential equations, including fractional forms, which serve as fundamental mathematical models for numerous dynamical systems encountered in physics, engineering, biology, and other fields. Oscillation theory seeks to answer questions on the presence, singularity, amplitude, frequency, and stability of oscillations within such systems. By diving into the dynamics of systems regulated by these equations, oscillation theory provides fundamental insights into their temporal behavior, allowing for the prediction, analysis, and regulation of oscillatory phenomena across a wide range of scientific and technical disciplines. Although differential equations are frequently used for modeling in various domains, it can be challenging to find general solutions for most of them. Consequently, scholars have concentrated on these equations' qualitative components. Asymptotic behavior and oscillation are two important components of this qualitative study that are still of interest. When solving problems, it is frequently clear that the system's current and past states have an impact on its future state.

As a result, when addressing the problem, include a time delay in the equation. Works by O. Arino et al. [12], G. S. Ladde et al. [13], and J.K. Hale [14] provide useful information for anyone interested in the theoretical and practical components of qualitative research on neutral equations. Real-world applications make use of a wide range of models that represent oscillatory events. Some mathematical biology models employ cross-diffusion terms to represent oscillation and/or delay behaviours. For further details on this subject, read publications [10], [11]. Extensive research into oscillation in fourth- or higher-order differential equations has resulted in the development of numerous approaches for defining oscillation criteria. Important results about the oscillatory characteristics of differential equations and damped delay differential equation solutions, with or without distributed deviating arguments, are presented in several works, cited as [15–21].

The oscillatory solutions of third-order damped nonlinear differential equations with distributed deviating arguments of the following type are the main subject of this study:

$$(\kappa_1(u)([\kappa_2(u)z'(u)]')^\nu)' + p(u)([\kappa_2(u)z'(u)]')^\nu + \int_a^b q(u,\xi)f\big(z(\varkappa(u,\xi))\big)\,d\xi = 0, \tag{1.1}$$

For $u \geq u_0$, where $\kappa_1, \kappa_2, p, q \in C(I, \mathbb{R}_+), f \in C(\mathbb{R}, \mathbb{R})$ satisfying $zf(z) > 0, \frac{f(z)}{z^\nu} \geq L > 0$ for $z \neq 0, \varkappa \in C(\mathbb{R}, \mathbb{R})$ satisfying $\varkappa(u) \leq u, \varkappa'(u) \geq 0$ and $\lim_{u\to\infty}\varkappa(u) = \infty, \nu \geq 1$ is a quotient of two odd positive integers.
A solution of Eq.(1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Eq.(1.1) is said to be oscillatory in case all its solutions are oscillatory. Using a generalized Riccati transformation and integral average technique, we will construct some novel oscillatory and asymptotic conditions for Eq. (1.1) in Section 3.1. In Section 4.1, we will demonstrate various applications based on our findings. The following conclusion is ensured by the conditions we provide in the sequel.
(H) Then every solution of Eq. (1.1) is oscillatory or tends to zero.

2. Main results

In this section, we define some of the auxiliary functions and lemmas that we call while proving our main results. For convenience, we denote:

$$\mathcal{A}(u) = \kappa_1(u)([\kappa_2(u)z'(u)]')^\nu,$$

$$\mathcal{B}(u,u_0) = \exp\Big(-\int_{u_0}^{u}\frac{p(s)}{\kappa_1(s)}ds\Big),$$

$$\psi_1(u,\kappa_1) = \int_{\kappa_1}^{u}\frac{[\mathcal{B}(s,u_0)]^{\frac{1}{\nu}}}{\kappa_1^{\frac{1}{\nu}}(s)}ds,$$

$$\psi_2(u,\kappa_1) = \int_{\kappa_1}^{u}\frac{\psi_1(s,\kappa_1)}{\kappa_2(s)}ds, \Omega(u) = \int_a^b q(u,\xi)d\xi$$

$$\mathcal{C}(u) = L\frac{\Omega(u)\zeta_1(u)}{\mathcal{B}(u,u_0)} - \zeta_1(u)[\kappa_1(u)\zeta_2(u)]'$$

$$\mathcal{D}(u) = \frac{\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),T)[\kappa_1(u)\zeta_2(u)]^{1+\frac{1}{\nu}}}{\kappa_2(\varkappa(u))},$$

$$\mathcal{D}^*(u) = \frac{\nu\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),T)\psi_2^{\nu-1}(\varkappa(u,T))\kappa_1^2(u)\zeta_2^2(u)}{\kappa_2(\varkappa(u))}$$

$$\mathcal{E}(u) = \frac{\kappa_2(\varkappa(u))\zeta_1'(u)+(\nu+1)\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),T)[\kappa_1(u)\zeta_2(u)]^{\frac{1}{\nu}}}{(\nu+1)\kappa_2^{\frac{1}{\nu+1}}(\varkappa(u))\zeta_1^{\frac{\nu}{\nu+1}}(u)\Big(\varkappa'(u)\frac{\nu}{\nu+1}\psi_1^{\frac{\nu}{\nu+1}}(\varkappa(u),T)}$$

$$\mathcal{E}^*(u) = \frac{[\kappa_2(\varkappa(u))\zeta_1'(u)+2\nu\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),T)\psi_2^{\nu-1}(\varkappa(u,T))\kappa_1(u)\zeta_2(u)]^2}{4\nu\kappa_2(\varkappa(u))\zeta_1(u)\cdot\varkappa'(u)\psi_1(\varkappa(u),T)\psi_2^{\nu-1}(\varkappa(u,T))}.$$

Lemma 2.1: *Assume that*

$$\int_{u_0}^{\infty}\Big(\frac{\mathcal{B}(s,u_0)}{\kappa_1(s)}\Big)^{\frac{1}{\nu}}ds = \infty, \tag{2.1}$$

$$\int_{u_0}^{\infty}\frac{ds}{\kappa_2(s)} = \infty, \tag{2.2}$$

And z is eventually a positive solution (EPS) of Eq. (1.1). Then, z(u) satisfies either
(C_I) $z(u) > 0, z'(u) > 0, [\kappa_2(u)z'(u)]' > 0,$ (2.3)

Or

(C_{II}) $z(u) > 0, z'(u) < 0, [\kappa_2(u)z'(u)]' > 0.$ (2.4)

Proof. Let (1.1) has a positive solution $z(u)$ on $[u_0,\infty)$, say $z(u) > 0, z(\varkappa(u)) > 0$.

$$\Big[\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)}\Big]' = \frac{\mathcal{B}(u,u_0)\mathcal{A}'(u) - (\mathcal{B}(u,u_0))'\mathcal{A}(u)}{\mathcal{B}(u,u_0)^2}$$
$$= \frac{\mathcal{A}'(u)+p(u)([\kappa_2(u)z'(u)]')^\nu)}{\mathcal{B}(u,u_0)}.$$

Then $\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)}$ is strictly decreasing on $[u_1,\infty)$, and together with $\kappa_1(u) > 0, \mathcal{B}(u,u_0) > 0$, we deduce that $[\kappa_2(u)z'(u)]'$ is eventually on sign. We claim $[\kappa_2(u)z'(u)]' > 0$ on $[u_2,\infty)$, where $u_2 > u_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $u_3 > u_2$ such that $[\kappa_2(u)z'(u)]' < 0$ on $[u_3,\infty)$. Then

$$\kappa_2(u)z'(u) - \kappa_2(u_3)z'(u) = \int_{u_3}^{u} \frac{[\mathcal{B}(s,u_0)]^{1/\nu}[\kappa_2(s)z'(s)]'}{[\mathcal{B}(s,u_0)\kappa_1(s)]^{1/\nu}} ds.$$

From (2.1), we have $\lim_{u\to\infty}\kappa_2(u)z'(u) = -\infty$, and thus there exists $u_4 \in [u_3,\infty)$ such that $\kappa_2(u)z'(u) < 0$ on $[u_4,\infty)$. By the assumption $[\kappa_2(u)z'(u)]' < 0$ and hence $\kappa_2(u)z'(u)$ is strictly decreasing on $[u_4,\infty)$. Then

$$z(u) - z(u_4) = \int_{u_4}^{u} \frac{\kappa_2(s)z'(s)}{\kappa_2(s)} ds \leq \kappa_2(u_4)z'(u_4)\int_{u_4}^{u} \frac{1}{\kappa_2(s)} ds.$$

Using (2.2), we have $\lim_{u\to\infty} z(u) = -\infty$, which contradicts the fact that $[\kappa_2(u)z'(u)]' < 0$ on $[u_2,\infty)$.
Lemma 2.2. Let z(u) satisfies case (C_{II}). Suppose

$$\limsup_{u\to\infty} \int_{u_0}^{u} \left(\frac{1}{\kappa_2(\xi)}\int_{\xi}^{\infty} \left(\frac{\mathcal{B}(\tau,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)}\right)^{\frac{1}{\nu}} d\tau\right) d\xi = \infty, \tag{2.5}$$

And then $\lim_{u\to\infty} z(u) = 0$.
Proof. Suppose, to contrary, $z'(u) < 0$, together with z(u) is EPS of Eq. (1.1), assume on the contrary that $\lim_{u\to\infty} z(u) = \alpha \geq 0$ and $\lim_{u\to\infty}\kappa_2(u)z'(u) = \varrho \leq 0$. An integration of (2.3) from u to ∞ yields

$$\begin{aligned}
-\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)} &= -\lim_{u\to\infty} \frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)} + \int_u^{\infty} \frac{-\int_a^b q(u,\xi)f(z(\varkappa(u,\xi)))d\xi}{\mathcal{B}(s,u_0)} ds \\
&\leq -\lim_{u\to\infty} \frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)} + \int_u^{\infty} \frac{-L\int_a^b q(u,\xi)z^{\nu}(\varkappa(u,\xi))d\xi}{\mathcal{B}(s,u_0)} ds \\
&\leq -L\int_u^{\infty} \frac{\int_a^b q(u,\xi)z^{\nu}(\varkappa(u,\xi))d\xi}{\mathcal{B}(s,u_0)} ds \\
&\leq -L\alpha^{\nu}\int_u^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds,
\end{aligned}$$

Which is followed by

$$-[\kappa_2(u)z'(u)]' \leq -\left[L\alpha^{\nu}\left(\frac{\mathcal{B}(u,u_0)}{\kappa_1(u)}\int_u^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)\right]^{\frac{1}{\nu}}. \tag{2.6}$$

Integrating the last inequality from u to ∞ yields

$$\begin{aligned}
\kappa_2(u)z'(u) &= \lim_{u\to\infty}\kappa_2(u)z'(u) - \alpha L^{\frac{1}{\nu}}\int_u^{\infty}\left(\frac{\mathcal{B}(s,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)^{\frac{1}{\nu}} d\tau \\
&= \varrho - \alpha L^{\frac{1}{\nu}}\int_u^{\infty}\left(\frac{\mathcal{B}(\tau,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)^{\frac{1}{\nu}} d\tau \\
&\leq -\alpha L^{\frac{1}{\nu}}\int_u^{\infty}\left(\frac{\mathcal{B}(\tau,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)^{\frac{1}{\nu}} d\tau,
\end{aligned}$$

Which implies

$$z'(u) \leq -\alpha L^{\frac{1}{\nu}}\frac{1}{\kappa_2(u)}\int_u^{\infty}\left(\frac{\mathcal{B}(\tau,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)^{\frac{1}{\nu}} d\tau. \tag{2.7}$$

Finally, integrating the inequality (2.7) from u_6 to u yields

$$z(u) - z(u_6) \leq -\alpha L^{\frac{1}{\nu}}\int_{u_6}^{u}\left[\frac{1}{\kappa_2(\xi)}\int_{\xi}^{\infty}\left(\frac{\mathcal{B}(\tau,u_0)}{\kappa_1(\tau)}\int_{\tau}^{\infty} \frac{\Omega(s)}{\mathcal{B}(s,u_0)} ds\right)^{\frac{1}{\nu}} d\tau\right] d\xi, \tag{2.8}$$

Which contradicts (2.5). So we have $\alpha = 0$, and then $\lim_{u\to\infty} z(u) = 0$.

Lemma 2.3: *Let (2.1) and (2.2) hold. If $z(u)$ is a EPS of Eq. (1.1) such that with case (C) for $u \geq u_1 \geq u_0$, where u_1 is sufficiently large, for $u \in [u_1,\infty)$, we have*

$$z'(u) \geq \frac{\psi_1(u,T_3^*)}{\kappa_2(u)}\left(\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\right)$$

And

$$z(u) \geq \psi_2(u,T_3^*)\left(\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\right).$$

Proof. By Lemma 2.1 we have $\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)}$ is strictly decreasing on $[T_3^*,\infty)$. So

$$\begin{aligned}\kappa_2(u)z'(u) &\geq \kappa_2(u)z'(u) - \kappa_2(T_3^*)z'(T_3^*)\\ &= \int_{T_3^*}^{u} \frac{[\mathcal{B}(s,u_0)\kappa_1(s)]^{\frac{1}{\nu}}[\kappa_2(s)z'(s)]'}{[\mathcal{B}(s,u_0)\kappa_1(s)]^{\frac{1}{\nu}}} ds\\ &\geq \frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\int_{T_3^*}^{u} \frac{[\mathcal{B}(s,u_0)]^{\frac{1}{\nu}}}{\kappa_1^{\frac{1}{\nu}}(s)} ds\\ &= \psi_1(u,T_3^*)\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}},\end{aligned}$$

And then

$$z'(u) \geq \frac{\psi_1(u,T_3^*)}{\kappa_2(u)}\left(\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\right).$$

Furthermore,

$$\begin{aligned}z(u) \geq z(u) - z(T_3^*) &= \int_{T_3^*}^{u} z'(s)ds\\ &\geq \int_{T_3^*}^{u} \frac{\psi_1(s,T_3^*)}{\kappa_2(s)}\left(\frac{\mathcal{A}^{\frac{1}{\nu}}(s)}{[\mathcal{B}(s,u_0)]^{\frac{1}{\nu}}}\right) ds\\ &\geq \left(\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\right)\int_{T_3^*}^{u} \frac{\psi_1(s,T_3^*)}{\kappa_2(s)} ds\\ &= \psi_2(u,T_3^*)\left(\frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}}\right).\end{aligned}$$

Which is the desired result.

Lemma 2.4: *If U and V are non-negative real numbers. Then*

$\lambda U V^{\lambda-1} - U^{\lambda} \leq (\lambda - 1)V^{\lambda}$,

For all $\lambda > 1$.

Theorem 2.1: *Let (2.1), (2.2), and (2.5) hold. If there exist two nonnegative functions $\zeta_1(u)$, $\zeta_2(u) \in C^1([u_0,\infty),\mathbb{R})$ with $\zeta_1(u) > 0$, such that*

$$\limsup_{u\to\infty} \int_{T}^{u} \{\mathcal{C}(s) + \mathcal{D}(s) - \mathcal{E}(s)\}ds = \infty \tag{2.9}$$

Then conclusion (H) hold.

Proof. Assume (1.1) has a non-oscillatory solution z on I, such that $z(u) > 0, z(\varkappa(u)) > 0$ on $[u_1,\infty)$, where u_1 is sufficiently large. By Lemma 2.1, conditions 2.2 are satisfied for large u_2 such that $[\kappa_2(u)z'(u)]' > 0$ on $[u_2,\infty)$, and either $z'(u) > 0$ on $[u_2,\infty)$ or $\lim_{u\to\infty} z(u) = 0$.

First, assume $z'(u) > 0$ on $[u_2,\infty)$. Since $\lim_{u\to\infty}\varkappa(u) = \infty$, there exists $u_3 > u_2$ such that $\varkappa(u) > u_2$ on $[u_3,\infty)$. So $z'(\varkappa(u)) > 0$ on $[u_3,\infty)$.

Define a generalized Riccati function:

$$\omega(u) = \zeta_1(u)\kappa_1(u)\left[\frac{([\kappa_2(u)z'(u)]')^{\nu}}{z^{\nu}(\varkappa(u))\mathcal{B}(u,u_0)} + \zeta_2(u)\right].$$

Then for $u \in [u_3,\infty)$, we e obtain

$$\omega'(r) = \frac{\zeta_1(r)}{z^{\nu}(\varkappa(r))}\left[\frac{\mathcal{A}(r)}{\mathcal{B}(r,r_0)}\right]' + \left[\frac{\zeta_1(r)}{z^{\nu}(\varkappa(r))}\right]'\left(\frac{\mathcal{A}(r)}{\mathcal{B}(r,r_0)}\right) + \zeta_1(r)[\kappa_1(r)\zeta_2(r)]' + \zeta_1'(r)\kappa_1(r)\zeta_2(r)$$

$$= \frac{\zeta_1(u)}{z^\nu(\varkappa(u))}\left\{\frac{\mathcal{B}(u,u_0)(\mathcal{A}(u))' - (\mathcal{B}(u,u_0))'\mathcal{A}(u)}{\mathcal{B}(u,u_0)^2}\right\}$$
$$+\left[\frac{z^\nu(\varkappa(u))\zeta_1'(u) - (z^\nu(\varkappa(u)))'\zeta_1(u)}{z^\nu(\varkappa(u))z^\nu(\varkappa(u))}\right]\left(\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)}\right)$$
$$+\zeta_1(u)[\kappa_1(u)\zeta_2(u)]' + \zeta_1'(u)\kappa_1(u)\zeta_2(u)$$
$$= \frac{\zeta_1(u)}{z^\nu(\varkappa(u))}\left[\frac{(\mathcal{A}(u))' + p(u)([\kappa_2(u)z'(u)]')^\nu}{\mathcal{B}(u,u_0)}\right] + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u)$$
$$-\left[\frac{\zeta_1(u)(z^\nu(\varkappa(u)))'}{z^\nu(\varkappa(u))}\right]\frac{\mathcal{A}(u)}{z^\nu(\varkappa(u))\mathcal{B}(u,u_0)} + \zeta_1(u)[\kappa_1(u)\zeta_2(u)]'$$
$$= -\frac{\zeta_1(u)}{z^\nu(\varkappa(u))}\left[\frac{\int_a^b q(u,\xi)f(z(\varkappa(u,\xi)))d\xi}{\mathcal{B}(u,u_0)}\right] + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u)$$
$$-\left[\frac{\zeta_1(u)(z^\nu(\varkappa(u)))'}{z^\nu(\varkappa(u))}\right]\frac{\mathcal{A}(u)}{z^\nu(\varkappa(u))\mathcal{B}(u,u_0)} + \zeta_1(u)[\kappa_1(u)\zeta_2(u)]'$$
$$\le -L\frac{\Omega(u)\zeta_1(u)}{\mathcal{B}(u,u_0)} + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \left[\frac{\zeta_1(u)(z^\nu(\varkappa(u)))'}{z^\nu(\varkappa(u))}\right]\frac{\mathcal{A}(u)}{z^\nu(\varkappa(u))\mathcal{B}(u,u_0)} + \zeta_1(u)[\kappa_1(u)\zeta_2(u)]'.$$

Then

$$\omega'(u) \le -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \zeta_1(u)\left[\frac{\nu z^{\nu-1}(\varkappa(u))z'(\varkappa(u))\varkappa'(u)}{z^\nu(\varkappa(u))}\right]\frac{\mathcal{A}(u)}{z^\nu(\varkappa(u))\mathcal{B}(u,u_0)}. \tag{2.10}$$

By Lemma 2.3 and $z'(u) > 0$, we have

$$\omega'(u) \le -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \left[\frac{\nu\zeta_1(u)\varkappa'(u)}{z(\varkappa(u))}\right]\left\{\frac{\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))}\frac{\kappa_1^{\frac{1}{\nu}}(\varkappa(u))[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{[\mathcal{B}(\varkappa(u),u_0)]^{\frac{1}{\nu}}}\right\}\frac{\mathcal{A}(u)}{z^\nu(\varkappa(u))\mathcal{B}(u,u_0)}.$$

By Lemma 2.1, $\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)}$ is strictly decreasing on $[u_2,\infty)$. So

$$\frac{(\kappa_1(\varkappa(u)))^{\frac{1}{\nu}}[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{[\mathcal{B}(\varkappa(u),u_0)]^{\frac{1}{\nu}}} > \frac{(\kappa_1(u))^{\frac{1}{\nu}}[\kappa_2(u)z'(u)]'}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}},$$

For $u \in [u_3,\infty)$, and we get

$$\omega'(u) \le -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \nu\frac{\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))}\left[\frac{\omega(u)}{\zeta_1(u)} - \kappa_1(u)\zeta_2(u)\right]^{1+\frac{1}{\nu}}. \tag{2.11}$$

By applying the inequality

$$(u-v)^{1+\frac{1}{\nu}} \ge u^{1+\frac{1}{\nu}} + \frac{1}{\nu}v^{1+\frac{1}{\nu}} - \left(1+\frac{1}{\nu}\right)v^{\frac{1}{\nu}}u,$$

We obtain

$$\left[\frac{\omega(u)}{\zeta_1(u)} - \kappa_1(u)\zeta_2(u)\right]^{1+\frac{1}{\nu}} \ge \frac{\omega^{1+\frac{1}{\nu}}(u)}{\zeta_1^{1+\frac{1}{\nu}}(u)} + \frac{1}{\nu}[\kappa_1(u)\zeta_2(u)]^{1+\frac{1}{\nu}}.$$

Using (2.12) in (2.11), which yields,

$$\omega'(u) \le -\mathcal{C}(u) - \frac{\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)[\kappa_1(u)\zeta_2(u)]^{1+\frac{1}{\nu}}}{\kappa_2(\varkappa(u))}$$
$$+\frac{\kappa_2(\varkappa(u))\zeta_1'(u) + (\nu+1)\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)[\kappa_1(u)\zeta_2(u)]^{\frac{1}{\nu}}}{\kappa_2(\varkappa(u))\zeta_1(u)}\omega(u).$$

Setting

$$\lambda = 1+\frac{1}{\nu},\ X^\lambda = \nu\frac{\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))}\frac{\omega^{1+\frac{1}{\nu}}(u)}{\zeta_1^{1+\frac{1}{\nu}}(u)},\ Y^{\lambda-1} = \nu^{\frac{1}{\nu+1}}\mathcal{E}(u)$$

Using Lemma 2.4 in (2.12) we get that

$$\omega'(u) \le -\mathcal{C}(u) - \mathcal{D}(u) + \mathcal{E}(u). \tag{2.14}$$

Integrating the inequality (2.14) from u_3 to u, yields

$$\int_{u_3}^{u}\{\mathcal{C}(s) + \mathcal{D}(u) - \mathcal{E}(u)\}\,ds \le \omega(u_3) - \omega(u) \le \omega(u_3) < \infty,$$

Which contradicts (2.9).

Theorem 2.2: *Let (2.1), (2.2), and (2.5) hold. If there exist two nonnegative functions* $\zeta_1(u)$, $\zeta_2(u) \in C^1([u_0,\infty),\mathbb{R})$ *with* $\zeta_1(u) > 0$, *such that*

$$\limsup_{u\to\infty} \int_T^u \{\mathcal{C}(s) + \mathcal{D}^*(s) - \mathcal{E}^*(s)\}ds = \infty \tag{2.15}$$

Then conclusion (H) hold.
Proof. Assume (1.1) has a non-oscillatory solution z on I, such that $z(u) > 0, z(\varkappa(u)) > 0$ on $[u_1,\infty)$, where u_1 is sufficiently large. By Lemma 2.1, conditions 2.2 are satisfied for large u_2 such that $[\kappa_2(u)z'(u)]' > 0$ on $[u_2,\infty)$, and either $z'(u) > 0$ on $[u_2,\infty)$ or $\lim_{u\to\infty} z(u) = 0$. First, we assume $z'(u) > 0, z'(\varkappa(u)) > 0$ on $[u_3,\infty)$, where $u_3 > u_2$ is sufficiently large.
Let $\omega(u)$ be defined as in Theorem 2.1 by Lemma (2.3), for $u \in [u_3,\infty)$We have the following observation:

$$\begin{aligned}
\frac{z'(\varkappa(u))}{z(\varkappa(u))} &\geq \frac{z'(\varkappa(u))}{z(\varkappa(u))} = \frac{z'(\varkappa(u))}{z^{\nu}(\varkappa(u))} z^{\nu-1}(\varkappa(u)) \\
&\geq \frac{\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))z^{\nu}(\varkappa(u))} \left\{ \frac{\kappa_1^{\frac{1}{\nu}}(\varkappa(u))[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{\left[e_{-\frac{p}{\kappa_1}}(\varkappa(u),u_0)\right]^{\frac{1}{\nu}}} \right\} z^{\nu-1}(\varkappa(u)) \\
&\geq \frac{\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))z^{\nu}(\varkappa(u))} \left\{ \frac{\kappa_1^{\frac{1}{\nu}}(\varkappa(u))[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{\left[e_{-\frac{p}{\kappa_1}}(\varkappa(u),u_0)\right]^{\frac{1}{\nu}}} \right\} \\
&\quad \times \psi_2^{\nu-1}(\varkappa(u),u_3) \left\{ \frac{\kappa_1^{\frac{1}{\nu}}(\varkappa(u))[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{\left[e_{-\frac{p}{\kappa_1}}(\varkappa(u),u_0)\right]^{\frac{1}{\nu}}} \right\}^{\nu-1} \\
&\geq \frac{\psi_1(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))z^{\nu}(\varkappa(u))} \left\{ \frac{\kappa_1^{\frac{1}{\nu}}(\varkappa(u))[\kappa_2(\varkappa(u))z'(\varkappa(u))]'}{\left[e^{\kappa_1}_{-\frac{p}{\kappa_1}}(\varkappa(u),u_0)\right]^{\frac{1}{\nu}}} \right\} \psi_2^{\nu-1}(\varkappa(u),u_3) \left\{ \frac{\mathcal{A}^{\frac{1}{\nu}}(u)}{[\mathcal{B}(u,u_0)]^{\frac{1}{\nu}}} \right\}^{\nu-1}
\end{aligned}$$

Using (2.16) in (2.10) we get that

$$\begin{aligned}
\omega'(u) &\leq -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \zeta_1(u)\left[\frac{\nu z^{\nu-1}(\varkappa(u))z'(\varkappa(u))\varkappa'(u)}{z^{\nu}(\varkappa(u))}\right]\frac{\mathcal{A}(u)}{z^{\nu}(\varkappa(u))\mathcal{B}(u,u_0)} \\
&\leq -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \nu\zeta_1(u)\varkappa'(u)\frac{\psi_1(\varkappa(u),u_3)\psi_2^{\nu-1}(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))}\left\{\frac{\mathcal{A}(u)}{\mathcal{B}(u,u_0)z^{\nu}(\varkappa(u))}\right\}^2 \\
&= -\mathcal{C}(u) + \frac{\zeta_1'(u)}{\zeta_1(u)}\omega(u) - \nu\zeta_1(u)\varkappa'(u)\frac{\psi_1(\varkappa(u),u_3)\psi_2^{\nu-1}(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))}\left[\frac{\omega(u)}{\zeta_1(u)} - \kappa_1(u)\zeta_2(u)\right]^2 \\
&= -\mathcal{C}(u) - \mathcal{D}^*(u) + \left[\frac{\kappa_2(\varkappa(u))\zeta_1'(u) + 2\nu\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)\psi_2^{\nu-1}(\varkappa(u),u_3)\kappa_1(u)\zeta_2(u)}{\kappa_2(\varkappa(u))\zeta_1(u)}\right]\omega(u) \\
&\quad - \frac{\nu\zeta_1(u)\varkappa'(u)\psi_1(\varkappa(u),u_3)\psi_2^{\nu-1}(\varkappa(u),u_3)}{\kappa_2(\varkappa(u))\zeta_1^2(u)}\omega^2(u)
\end{aligned}$$

Integrating the inequality (2.17) from u_3 to u yields

$$\int_{u_3}^u \{\mathcal{C}(s) + \mathcal{D}^*(s) - \mathcal{E}^*(s)\}ds < \omega(u_3) - \omega(u) \leq \omega(u_3) < \infty$$

Which contradicts (2.15).

Theorem 2.3: *Let (2.1), (2.2), and (2.5) hold and define*

$$\mathbb{D} = \{(u,s) \mid u \geq s \geq u_0, u,s \in \mathbb{T}\}$$

A kernel function $\mathcal{H} \in C(\mathbb{D},\mathbb{R})$ such that

$$\mathcal{H}(u,u) = 0, \text{ for } u \geq u_0, \quad \mathcal{H}(u,s) > 0$$

For $u > s \geq u_0$, and $\mathcal{H}$ has a non-positive partial derivative $\mathcal{H}'(u,s)$ concerning the second variable, such that

$$\limsup_{u\to\infty} \frac{1}{\mathcal{H}(u,u_0)} \int_{u_0}^u \mathcal{H}(u,s)\{\mathcal{C}(s) + \mathcal{D}(s) - \mathcal{E}(s)\}ds = \infty \tag{2.18}$$

Then conclusion (H) hold.
Proof. Assume (1.1) has a non-oscillatory solution z on I, such that $z(u) > 0, z(\varkappa(u)) > 0$ on $[u_1,\infty)$, where u_1 is sufficiently large. By Lemma 2.1, conditions 2.2 are satisfied for large u_2 such that $[\kappa_2(u)z'(u)]' > 0$ on $[u_2,\infty)$, and either $z'(u) > 0$ on $[u_2,\infty)$ or $\lim_{u\to\infty} z(u) = 0$. First, we assume $z'(u) > 0, z'(\varkappa(u)) > 0$ on $[u_3,\infty)$, where $u_3 > u_2$ is sufficiently large. Let $\omega(u)$ be defined as in

Theorem 2.1: *by Lemma (2.3), for* $u \in [u_3,\infty)$, *we have*

$$\mathcal{C}(\mathrm{u})+\mathcal{D}(\mathrm{u})-\frac{\kappa_2(\varkappa(\mathrm{u}))\zeta_1'(\mathrm{u})+(\nu+1)\zeta_1(\mathrm{u})\varkappa'(\mathrm{u})\psi_1(\varkappa(\mathrm{u}),\mathrm{u}_3)[\kappa_1(\mathrm{u})\zeta_2(\mathrm{u})]^{\frac{1}{\nu}}}{(\nu+1)\kappa_2^{\frac{1}{\nu+1}}}(\varkappa(\mathrm{u}))\zeta_1^{\frac{\nu}{\nu+1}}(\mathrm{u})$$

Substituting u with s in (2.19), multiplying (2.19) by $\mathcal{H}(\mathrm{u},\mathrm{s})$ and integrating the resulting inequality from u_3 to u, we find that

$$\begin{aligned}\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}-\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\omega'(\mathrm{s})\mathrm{ds}\\ =\mathcal{H}(\mathrm{u},\mathrm{u}_3)\omega(\mathrm{u}_3)+\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}^{\mathrm{ds}_\mathrm{s}}(\mathrm{u},\mathrm{s})\omega(\mathrm{s})\mathrm{ds}\\ \le\mathcal{H}(\mathrm{u},\mathrm{u}_3)\omega(\mathrm{u}_3)\le\mathcal{H}(\mathrm{u},\mathrm{u}_0)\omega(\mathrm{u}_3)\end{aligned}$$

Then

$$\begin{aligned}&\int_{\mathrm{u}_0}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}\\ &=\int_{\mathrm{u}_0}^{\mathrm{u}_3}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}+\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}\\ &\le\mathcal{H}(\mathrm{u},\mathrm{u}_0)\omega(\mathrm{u}_3)+\mathcal{H}(\mathrm{u},\mathrm{u}_0)\int_{\mathrm{u}_0}^{\mathrm{u}_3}|\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})|\mathrm{ds}.\end{aligned}$$

So

$$\begin{aligned}&\limsup_{\mathrm{u}\to\infty}\frac{1}{\mathcal{H}(\mathrm{u},\mathrm{u}_0)}\left\{\int_{\mathrm{u}_0}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}\right\}\\ &\qquad\le\omega(\mathrm{u}_3)+\int_{\mathrm{u}_0}^{\mathrm{u}_3}|\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})|\mathrm{ds}<\infty,\end{aligned}$$

Which contradicts (2.18).

Theorem 2.4: *Let (2.1), (2.2), and (2.5) hold and $\mathcal{H}$ be defined as in Theorem 2.3, such that*

$$\limsup_{\mathrm{u}\to\infty}\frac{1}{\mathcal{H}(\mathrm{u},\mathrm{u}_0)}\int_{\mathrm{u}_0}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}^*(\mathrm{s})-\mathcal{E}^*(\mathrm{s})\}\mathrm{ds}=\infty \tag{2.20}$$

Then conclusion (H) hold.

Proof. Assume (1.1) has a non-oscillatory solution z on I, such that $\mathrm{z}(\mathrm{u})>0, \mathrm{z}(\varkappa(\mathrm{u}))>0$ on $[\mathrm{u}_1,\infty)$, where u_1 is sufficiently large. By Lemma 2.1, conditions 2.2 are satisfied for large u_2 such that $[\kappa_2(\mathrm{u})\mathrm{z}'(\mathrm{u})]'>0$ on $[\mathrm{u}_2,\infty)$, and either $\mathrm{z}'(\mathrm{u})>0$ on $[\mathrm{u}_2,\infty)$ or $\lim_{\mathrm{u}\to\infty}\mathrm{z}(\mathrm{u})=0$. First, we assume $\mathrm{z}'(\mathrm{u})>0, \mathrm{z}'(\varkappa(\mathrm{u}))>0$ on $[\mathrm{u}_3,\infty)$, where $\mathrm{u}_3>\mathrm{u}_2$ is sufficiently large. Let $\omega(\mathrm{u})$ be defined as in Theorem 2.1. Then by (2.17), for $\mathrm{u}\in[\mathrm{u}_3,\infty)$, we have

$$\mathcal{C}(\mathrm{u})+\mathcal{D}^*(\mathrm{u})-\mathcal{E}^*(\mathrm{u})\le-\omega'(\mathrm{u}) \tag{2.21}$$

Substituting u with s in (2.21), multiplying (2.21) by $\mathcal{H}(\mathrm{u},\mathrm{s})$ and integrating the resulting inequality from u_3 to u, we find that

$$\begin{aligned}\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}^*(\mathrm{s})-\mathcal{E}^*(\mathrm{s})\}\mathrm{ds}\le-\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\omega'(\mathrm{s})\mathrm{ds}\\ =\mathcal{H}(\mathrm{u},\mathrm{u}_3)\omega(\mathrm{u}_3)+\int_{\mathrm{u}_3}^{\mathrm{u}}\mathcal{H}'^{\mathrm{s}}(\mathrm{u},\mathrm{s})\omega(\mathrm{s})\mathrm{ds}\\ \le\mathcal{H}(\mathrm{u},\mathrm{u}_3)\omega(\mathrm{u}_3)\le\mathcal{H}(\mathrm{u},\mathrm{u}_0)\omega(\mathrm{u}_3)\end{aligned}$$

Similarly to Theorem 2.3, we get

$$\begin{aligned}&\limsup_{\mathrm{u}\to\infty}\frac{1}{\mathcal{H}(\mathrm{u},\mathrm{u}_0)}\int_{\mathrm{u}_0}^{\mathrm{u}}\mathcal{H}(\mathrm{u},\mathrm{s})\{\mathcal{C}(\mathrm{s})+\mathcal{D}^*(\mathrm{s})-\mathcal{E}^*(\mathrm{s})\}\mathrm{ds}\\ &\le\omega(\mathrm{u}_3)+\int_{\mathrm{u}_0}^{\mathrm{u}_3}|\mathcal{C}(\mathrm{s})+\mathcal{D}^*(\mathrm{s})-\mathcal{E}^*(\mathrm{s})|\mathrm{ds}\\ &<\infty\end{aligned}$$

Which contradicts (2.20).

In Theorems 2.3 and 2.4, we can find various corollaries by using $\mathcal{H}(\mathrm{u},\mathrm{s})$ for some specific functions, such as $(\mathrm{u}-\mathrm{s})^{\mathrm{n}}$. For instance,

$$\mathcal{H}(\mathrm{u},\mathrm{s})=(\mathrm{u}-\mathrm{s})^{\mathrm{n}}, \mathrm{n}\ge1$$

Yields the following corollaries.

Corollary 2.1. Let (2.1), (2.2) and (2.5) holds. If

$$\limsup_{\mathrm{u}\to\infty}\frac{1}{(\mathrm{u}-\mathrm{u}_0)^{\mathrm{n}}}\int_{\mathrm{u}_0}^{\mathrm{u}}(\mathrm{u}-\mathrm{s})^{\mathrm{n}}\{\mathcal{C}(\mathrm{s})+\mathcal{D}(\mathrm{s})-\mathcal{E}(\mathrm{s})\}\mathrm{ds}=\infty.$$

Then conclusion (H) hold.

Corollary 2.2. Let (2.1), (2.2) and (2.5) holds. If

$$\limsup_{\mathrm{u}\to\infty}\frac{1}{(\mathrm{u}-\mathrm{u}_0)^{\mathrm{n}}}\int_{\mathrm{u}_0}^{\mathrm{u}}(\mathrm{u}-\mathrm{s})^{\mathrm{n}}\{\mathcal{C}(\mathrm{s})+\mathcal{D}^*(\mathrm{s})-\mathcal{D}^*(\mathrm{s})\}\mathrm{ds}=\infty.$$

Then conclusion (H) hold.

Example 2.1. Consider the equation

$$[(uz''(u))^{\nu}]' + \frac{1}{u^{\nu+1}}(z''(u))^{\nu} + \int_2^3 \frac{q_0}{u^{4/3}} z^{\nu}\left(\frac{u}{6}+\frac{1}{\xi}\right) d\xi = 0, u \in [2,\infty)$$

Where $\nu = \frac{3}{2} \geq 1$ is a quotient of two odd positive integers. By Corollary 2.1, we deduce that every solution of Eq. (1.1) is oscillatory or tends to zero.

3. Conclusion

In this work, we explored the oscillatory nature of a special type of third-order nonlinear differential equations that include both damping and distributed delays. Our focus was on a simplified, or canonical, form of these equations, which we enhanced by adding damping terms to better reflect real-world behavior. To understand when and why oscillations occur in their solutions, we applied two key mathematical tools: Riccati's transformation and the integral averaging method. These techniques helped us set clear and reliable conditions that ensure the solutions will oscillate. Overall, this study provides deeper insight into the behavior of such complex systems and contributes to the growing understanding of how delayed and damped dynamics interact in mathematical models.

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