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Research paper

A study on Three-Stage Tandem Queueing Systems with Poisson Input and Load-Dependent Service Mechanisms

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Abstract

Queueing theory plays a crucial role in analyzing congestion and optimizing resource utilization in complex systems. Traditional models often assume stationary arrival and service processes, typically modeled using homogeneous Poisson processes. However, in many practical scenarios, such as hospital operations, manufacturing systems, cloud computing, and airport security, service rates are time-dependent and are more accurately captured by Non-Homogeneous Poisson Process (NHPP). This study presents a three-node tandem queueing model where each node features a time-dependent service mechanism governed by a NHPP. We derive key performance metrics, including the typical number of users in line, the duration users spend before receiving service at each stage and across the entire system, the overall throughput, and the variation in the number of users present. A thorough sensitivity analysis is conducted to explore how different service rate parameters impact these Performance measures. The results highlight the significant effect of time-dependent service dynamics on system behavior, demonstrating that the proposed model offers a more accurate and flexible framework for studying systems with time-varying service processes. Additionally, this three-node model generalizes and extends earlier two-node configurations, providing deeper insights into multi-stage service environments.

Keywords: *Comparative Study; Non-Homogeneous Poisson Process; Performance Measures; Sensitivity Analysis; Tandem Queueing Model.*

1. Introduction

Queueing models serve as essential tools for managing congestion and enhancing service efficiency across multiple areas of application. These models are widely used in design and information transmission networks, transportation logistics, machine maintenance, production lines, and neurophysiological systems, among others. They have been extended to analyze multi-stage systems, particularly valuable in modern communication infrastructures. These earlier works often assumed homogeneous and time-independent arrival patterns.

The study of time-varying arrival and service rates gained momentum following Newell's (1968) introduction of queues with time-dependent arrivals. Massey (1981) and Massey & Whitt (1993, 1994) further explored non-stationary queues and their approximations. Subsequent investigations into traffic characteristics—such as those by Leland et al. (1994), Feldmann (2000), and Crovella & Bestavros (1997)—revealed that network traffic, especially in Ethernet and TCP/IP systems, tends to be bursty and self-similar, defying traditional Poisson-based assumptions. As a result, more realistic models, such as the G/M/1 queue with Weibull interarrival times (Fisher et al., 2001), were developed to capture such behavior. Dinda (2006) and others have shown that traffic in networks such as LANs, MANs, and WANs exhibits long-range dependence and temporal variability.

The primary motivation for this research arises from the scarcity of queueing models that incorporate time-varying arrival and service processes, particularly in systems where service rates are dependent on the load of the system. While some studies have addressed the dynamics of time-dependent service rates, there remains a substantial void in the literature related to the analysis of multi-node queueing systems with load-dependent, Homogeneous Poisson arrival (HPA) and Non-homogeneous service processes (NHSP).

Queueing models with time-dependent service rates are vital to assess a variety of practical systems. For instance, in telecommunications, data traffic varies with time, and network resources (such as bandwidth) are often allocated dynamically based on traffic load. Similarly, in transportation systems, congestion levels change contingent on the time of day, affecting the speed and efficiency of service. Manufacturing systems also experience variability in processing times depending on machine load and other factors.

To address this gap, this thesis develops and analyzes queueing models that incorporate Homogeneous Poisson arrival (HPA) processes and load-dependent, time-varying service rates. These models are extended to a three-node tandem system, where users pass through three sequential service stations. The research aims to fill this gap in the queueing theory literature and offer practical insights for improving the performance of real-world systems. By incorporating load-dependent service rates, we can model more accurately the behavior of complex systems, such as networked computing systems, transportation infrastructures, and assembly lines.

The framework established by Rao and Aparajitha (2018) in their study on the two-node Tandem queueing model (TQM) service rates that vary over time. The primary objective is to extend their analysis to a three-node system, where the service processes is modeled as NHPP. The thesis develops several variations of the three-node TQM, each with different assumptions about user behavior, arrival processes, and service mechanisms.
The thesis extends this framework to a three-node tandem queueing system. In this system, users reach the initial service station and then pass through the 2nd and 3rd service stations in sequence. The arrival process remains an NHPP with a constant arrival rate λ. Both service stations exhibit time-varying and load-dependent service rates. The service rates at each node are modeled as NHPP with rates dependent on the number of users in the system at each node.
The paper titled "Parallel and Series Queueing Model with State and Time Dependent Service" by Dr. J. Durga Aparajitha and Dr. K. Srinivasa Rao (2023) presents a comprehensive analysis of a tandem queueing system where service rates are influenced by both the system state and time. By employing a non-homogeneous Poisson process to model non-stationary service processes, the study derives the joint probability generating function for the queue size distribution across three interconnected queues.
The paper titled "A Hybrid Parallel-Sequential Service Model for Tandem Communication Networks with Load-Dependent and Time-Variant Behaviour" by Dr. J. Durga Aparajitha, Dr. Chakrala Sreelatha, and Dr. K. Srinivasa Rao (2025) presents an advanced queuing model that integrates parallel and sequential service mechanisms to analyze tandem communication networks. This model addresses the complexities introduced by load-dependent and time-varying service rates, providing a more accurate representation of real-world network behaviors. The study offers valuable insights into optimizing network performance and resource allocation in dynamic environments
The mathematical framework for analyzing these queueing models involves the use of difference-differential equations, which are essential for capturing the transient behavior of the system over time. The primary performance metrics of interest include the probability of the system being empty, the average number of users within the system, the utilization of the service stations, and the average user waiting time. Furthermore, the system's throughput, the variance in the number of users in the queue, and the CV of the queue length are also computed.
There is limited literature addressing queueing systems with time-dependent service rates, particularly in the context of tandem configurations involving multiple stages. Non-homogeneous service behavior in such models can be effectively represented using NHPP, which relaxes the restrictive assumption of time-invariant rates inherent in traditional Poisson processes (Parzen, 1965). This paper focuses on a three-node tandem queueing system where each service node operates under a time-dependent service mechanism. Specifically, the service rate at each stage is modeled as a linear function of time, capturing dynamic variations in system performance more accurately. The structure of the paper is as follows: Segment 2 outlines the fundamental assumptions, formulates the system's differential-difference equations, and provides the transient-state analysis. Segment 3 derives key performance indicators, including average queue lengths, system throughput, user waiting times, and the CV for the queue size. Segment 4 offers numerical examples to illustrate the solution methodology. Segment 5 presents a sensitivity analysis that explores how variations in input parameters influence performance outcomes. A comparison between the proposed model and its homogeneous counterpart is discussed in Segment 6. Finally, Segment 7 concludes the paper with key observations and potential directions for future research.

2. Queueing model

This Segment outlines the formulation of the proposed queueing model. We consider a tandem queueing system comprising three service nodes arranged sequentially. In this setup, the output of each queue acts as the input to the next, forming a linear flow of users or jobs through the system. The aim is to model systems where the service dynamics are time-dependent and better represented by NHPP.
To develop the model, the following assumptions are made:

1) Service Mechanism: Each of the three service stations has a time-dependent service rate, modeled as a linear function of ‘t’, i.e., $\mu_i(t) = \alpha_i + \beta_i(t)$, for for i=1,2,3, where α_i, $\beta_i \geq 0$.
2) Queue Discipline: The system follows the FCFS discipline at each node.
3) Interdependence: The user must complete service at one node before proceeding to the next; transitions between nodes are instantaneous and lossless.
4) System Capacity: Each queue has infinite capacity, ensuring no user is lost due to space limitations.
5) Initial Conditions: At t=, the system is assumed to be empty.

Based on these assumptions, a set of differential-difference equations is constructed to describe the time evolution of the system.
The schematic diagram illustrating the structure of the three-node tandem queueing system is presented in Fig. 1. It visually represents the sequential flow of users through the three service nodes, where each stage receives input from the previous one and passes its output to the next, forming a linear service pipeline.

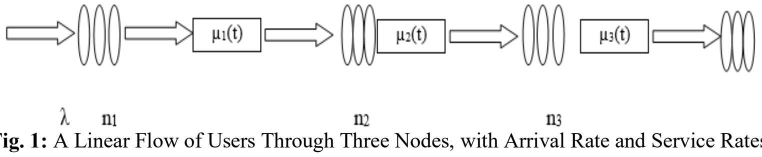


Fig. 1: A Linear Flow of Users Through Three Nodes, with Arrival Rate and Service Rates.

Let $Pn_1 n_2 n_3$ (t) denote the probability that at time t, there are:

- n_1 users in the 1st queue,
- n_2 users in the 2nd queue , and
- n_3 users in the 3rd queue.

The system dynamics are governed by a set of difference-differential equations that describe the evolution of these state probabilities over time, considering the arrival rate λ and the time-dependent service rates $\mu_1(t)$,$\mu_2(t)$, and $\mu_3(t)$for the three queues, respectively.
These equations are derived based on the probability flow into and out of each state (n_1, n_2, n_3), considering the possible transitions due to arrivals and service completions at each node.

$$\frac{\partial Pn_1 n_2 n_3 (t)}{\partial t} = -\left(\lambda + n_1\mu_1(t) + n_2\mu_2(t) + n_3\mu_3(t)\right) Pn_{1,}n_{2,}n_{3,}(t) + \lambda\, Pn_{1-1}, n_{2,}n_{3,}(t) + (n_1+1)\mu_1(t)Pn_{1+1,}n_{2-1,}n_3(t) + (n_2+1)\mu_2(t)Pn_{1,}n_{2+1,}n_{3-1}(t) + (n_3+1)\mu_3(t)Pn_{1,}n_{2,}n_{3+1}(t);\ n_1, n_2, n_3 > 0 \quad (1)$$

$\frac{\partial P_{0 n_2 n_3}(t)}{\partial t} = -(\lambda + n_2\mu_2(t) + n_3\mu_3(t)) P_{0,n_2,n_3}(t) + \mu_1(t)P_{1,n_2-1,n_3}(t) + (n_2+1)\mu_2(t)P_{0,n_2+1,n_3-1}(t) + (n_3 + 1)\mu_3(t)P_{0,n_2,n_3+1}(t);\ n_1 = 0, n_2, n_3 > 0$

$\frac{\partial P_{n_1 0 n_3}(t)}{\partial t} = -(\lambda + n_1\mu_1(t) + n_3\mu_3(t)) P_{n_1,0,n_3}(t) + \lambda P_{n_1-1,0,n_3}(t) + \mu_2(t)P_{n_1,1,n_3-1}(t) + (n_3+1)\mu_3(t)P_{n_1,0,n_3+1}(t);\ n_2 = 0, n_1, n_3 > 0$

$\frac{\partial P_{n_1 n_2 0}(t)}{\partial t} = -(\lambda + n_1\mu_1(t) + n_2\mu_2(t)) P_{n_1,n_2,0}(t) + \lambda P_{n_1-1,n_2,0}(t) + (n_1+1)\mu_1(t)P_{n_1+1,n_2-1,0}(t) + \mu_3(t)P_{n_1,n_2,1}(t);\ n_3 = 0, n_1, n_2 > 0$

$\frac{\partial P_{0 0 n_3}(t)}{\partial t} = -(\lambda + n_3\mu_3(t)) P_{0,0,n_3}(t) + \mu_2(t)P_{0,1,n_3-1}(t) + (n_3+1)\mu_3(t)P_{0,0,n_3+1}(t);\ n_3 > 0, n_1, n_2 = 0$

$\frac{\partial P_{n_1 0 0}(t)}{\partial t} = -(\lambda + n_1\mu_1(t)) P_{n_1,0,0}(t) + \lambda P_{n_1-1,0,0}(t) + \mu_3(t)P_{n_1,0,1}(t);\ n_1 > 0, n_2, n_3 = 0$

$\frac{\partial P_{0 n_2 0}(t)}{\partial t} = -(\lambda + n_2\mu_2(t)) P_{0,n_2,0}(t) + \mu_1(t)P_{1,n_2-1,0}(t) + \mu_3(t)P_{0,n_2,1}(t);\ n_2 > 0, n_1, n_3 = 0$

$\frac{\partial P_{000}(t)}{\partial t} = -(\lambda) P_{0,0,0}(t) + \mu_3(t)P_{0,0,1}(t);\ n_1, n_2, n_3 = 0$

The Probability Generating Function of $P_{n1,n2,n3}(t)$ is

$$P(S1, S2, S3, t) = \sum_{n1=0}^{\inf}\sum_{n2=0}^{\inf}\sum_{n3=0}^{\inf} P_{n1,n2,n3}(t)\, S_1^{n1}S_2^{n2}S_3^{n3} \quad (2)$$

Multiplying the Eq. (1) with $S_1^{n1}S_2^{n2}S_3^{n3}$ and sum overall n_1, n_2, n_3 yields

$$\frac{\partial P_{n_1,n_2,n_3}(t)}{\partial t} S_1^{n1}S_2^{n2}S_3^{n3} = -\Big(\sum_{n1=0}^{inf}\sum_{n2=0}^{inf}\sum_{n3=0}^{inf}(\lambda + n_1\mu_1(t) + n_2\mu_2(t) + n_3\mu_3(t))\, P_{n_1,n_2,n_3}(t)S_1^{n1}S_2^{n2}S_3^{n3} + \sum_{n1=0}^{inf}\sum_{n2=0}^{inf}\sum_{n3=0}^{inf}\lambda P_{n_1-1,n_2,n_3}(t)S_1^{n1}S_2^{n2}S_3^{n3} + \sum_{n1=0}^{inf}\sum_{n2=0}^{inf}\sum_{n3=0}^{inf}(n_1+1)\mu_1(t)P_{n_1+1,n_2-1,n_3}(t)\, S_1^{n1}S_2^{n2}S_3^{n3} + \sum_{n1=0}^{inf}\sum_{n2=0}^{inf}\sum_{n3=0}^{inf}(n_2+1)\mu_2(t)P_{n_1,n_2+1,n_3-1}(t)S_1^{n1}S_2^{n2}S_3^{n3} + \sum_{n1=0}^{inf}\sum_{n2=0}^{inf}\sum_{n3=0}^{inf}(n_3+1)\mu_3(t)P_{n_1,n_2,n_3+1}(t)\, S_1^{n1}S_2^{n2}S_3^{n3}\Big) \quad (3)$$

Upon simplification, we obtain

$$\frac{\partial P(S_1,S_2,S_3,(t)}{\partial t} = \mu_1(t)(S_2 - S_1)\frac{\partial P(S_1,S_2,S_3,(t)}{\partial S1} + \mu_2(t)(S_3 - S_2)\frac{\partial P(S_1,S_2,S_3,(t)}{\partial S2} + \mu_3(t)\frac{\partial P(S_1,S_2,S_3,(t)}{\partial S2}(1 - S_3) - \lambda(1 - S_1)P(S_1,S_2,S_3,t) \quad (4)$$

Analyzing the equation above equation by using the method of Lagrange multipliers, the corresponding auxiliary equations are derived

$$\frac{\partial t}{1} = \frac{\partial S_1}{-\mu_1(t)(S_2 - S_1)} = \frac{\partial S_2}{-\mu_2(t)(S_3 - S_2)} = \frac{\partial S_3}{-\mu_3(t)(1 - S_3)} = \frac{\partial P}{-\lambda(1 - S_1)P(S_1,S_2,S_3,t)} \quad (5)$$

Let the service rates are time-dependent and linear, taking the form of

$\mu_1(t)=\alpha_1+\beta_1 t;$

$\mu_2(t)=\alpha_2+\beta_2 t;$

$\mu_3(t)=\alpha_3+\beta_3 t;$

Analyzing the 1st and 4th components from equation (5), this leads to the following outcomes

$$a = (s_3 - 1)e^{\int \mu_3(t)}dt \quad (6)$$

Analyzing the 1st and 3rd components from equation (5), this leads to the following outcomes

$$b = s_2 e^{-\int \mu_2(t)dt} + (s_3 - 1)e^{-\int \mu_3(t)}dt \int \mu_2(t)\, e^{\int(\mu_3(t) - \mu_2(t))dt}dt + \int \mu_2(t)e^{-\int \mu_2(t)dt}\, dt \quad (7)$$

Analyzing the 1st and 2nd components from equation (5), this leads to the following outcomes

$$c = s_1 e^{-\int \mu_1(t)dt} + \big(s_2 e^{-\int \mu_2(t)dt} + (s_3 - 1)e^{-\int \mu_3(t)}dt \int \mu_2(t)\, e^{\int(\mu_3(t) - \mu_2(t))dt}dt + \int \mu_2(t)e^{-\int \mu_2(t)dt}\, dt\big)\big(\int \mu_1(t)\, e^{\int(\mu_2(t) - \mu_1(t))dt}dt\big) - \Big((s_3 - 1)e^{-\int \mu_3(t)}\big(\int \mu_1(t)\big(\int \mu_2(t)\, e^{\int(\mu_3(t) - \mu_2(t))dt}dt\, e^{\int \mu_2(t)}\big)e^{-\int \mu_1(t)dt}dt\big)\Big) - \big(\int \mu_1(t)\big(\int \mu_2(t)\, e^{-\int \mu_2(t)dt}e^{\int \mu_2(t)dt}dt\big)e^{-\int \mu_1(t)dt}dt\big) \quad (8)$$

Analyzing the 1st and 5th components from equation (5), this leads to the following outcomes

$$d = P(s_1, s_2, s_3, t)exp\Big[s_1 e^{-\int \mu_1(t)dt} + \big(s_2 e^{-\int \mu_2(t)dt} + (s_3 - 1)e^{-\int \mu_3(t)}dt \int \mu_2(t)\, e^{\int(\mu_3(t) - \mu_2(t))dt}dt + \int \mu_2(t)e^{-\int \mu_2(t)dt}\, dt\big)\big(\int \mu_1(t)\, e^{\int(\mu_2(t) - \mu_1(t))dt}dt\big) - \Big((s_3 - 1)e^{-\int \mu_3(t)}\big(\int \mu_1(t)\big(\int \mu_2(t)\, e^{\int(\mu_3(t) - \mu_2(t))dt}dt\, e^{\int \mu_2(t)}\big)e^{-\int \mu_1(t)dt}dt\big)\Big) - \big(\int \mu_1(t)\big(\int \mu_2(t)\, e^{-\int \mu_2(t)dt}e^{\int \mu_2(t)dt}dt\big)e^{-\int \mu_1(t)dt}dt\big)\Big] * \big(\lambda e^{-\int \mu_1(t)dt}dt\big) + \big(s_2 e^{-\int \mu_2(t)dt} + (s_3 -$$

$$1)e^{-\int\mu_3(t)}dt\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt+\int\mu_2(t)e^{-\int\mu_2(t)dt}\,dt\big)*\big(\lambda e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\,e^{\int(\mu_2(t)-\mu_1(t))dt}dt\big)dt\big)-$$
$$(s_3-1)e^{-\int\mu_3(t)dt}\big(\int\lambda e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\big(\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt\;e^{\int\mu_2(t)}\big)e^{-\int\mu_1(t)dt}dt\big)dt\big)-$$
$$\big(\int\lambda e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\,(\int\mu_2(t)\,e^{-\int\mu_2(t)dt}e^{\int\mu_2(t)dt}dt\big)e^{-\int\mu_1(t)dt}dt\big)dt\big)+\int\lambda\,dt \quad (9)$$

Here, the parameters a, b, c and d are arbitrary constants that define the time-dependent service rates of the three service nodes, typically in the form $\mu_i(t)=\alpha_i+\beta_i(t)$, for i=1,2,3. These parameters allow flexibility in modeling the increasing of variable capacity of servers over time.
The initial conditions for the system are specified as: $P_{0,0,0}(0)=1$, and $P_{n1,n2,n3}(0)=0$ for all $(n_1,n_2,n_3)\neq(0,0,0)$, indicating that the system starts empty at time t=0.
To analyze the system, we define the PGF of the number of users in each queue at time t as:

$$P(S_1,S_2,S_3,t)=exp\Big(-\frac{\lambda s_1e^{-\int\mu_1(t)dt}}{\alpha 1}\Big)-\frac{\lambda\big(s_2e^{-\int\mu_2(t)dt}+(s_3-1)e^{-\int\mu_3(t)}dt\int\mu_2(t)e^{\int(\mu_3(t)-\mu_2(t))dt}dt+\int\mu_2(t)e^{-\int\mu_2(t)dt}dt\big)\big(\int\mu\;\;(t)e^{\int(\mu_2(t)-\mu_1(t))dt}dt\big)}{\alpha 1}+$$
$$\frac{\lambda\Big((s_3-1)e^{-\int\mu_3(t)}\big(\int\mu_1(t)\big(\int\mu_2(t)e^{\int(\mu_3(t)-\mu_2(t))dt}dt\;e^{\int\mu_2(t)}\big)e^{-\int\mu_1(t)dt}dt\big)\Big)}{\alpha 1}+\frac{\lambda\big(\int\mu_1(t)\big(\int\mu_2(t)e^{-\int\mu_2(t)dt}e^{\int\mu_2(t)dt}dt\big)e^{-\int\mu_1(t)dt}dt\big)}{\alpha 1}+\frac{\lambda\big(s_2e^{-\int\mu_2(t)dt}\big)}{\alpha 2-\alpha 1}+$$
$$\frac{\lambda\Big((s_3-1)e^{-\int\mu_3(t)}dt\int\mu_2(t)e^{\int(\mu_3(t)-\mu_2(t))dt}dt\Big)}{\alpha 2-\alpha 1}+\frac{\lambda\big(\int\mu_2(t)e^{-\int\mu_2(t)dt}dt\big)}{\alpha 2-\alpha 1}-\frac{\lambda(s_3-1)e^{-\int\mu_3(t)}dt*\alpha 2}{(\alpha 3-\alpha 2)(\alpha 3-\alpha 1)}-\lambda\Big(\frac{s_2e^{-\int\mu_2(t)dt}}{\alpha 2}\Big)-$$
$$\frac{\lambda\Big((s3-1)e^{-\int\mu_3(t)}dt\int\mu_2(t)e^{\int(\mu_3(t)-\mu_2(t))dt}dt\Big)}{\alpha 2}-\frac{\lambda\big(\int\mu_2(t)e^{-\int\mu_2(t)dt}dt\big)}{\alpha 2}-\frac{\lambda\Big((s_3-1)e^{-\int\mu_3(t)}dt\Big)}{\alpha 3}+\frac{\lambda\Big((s_3-1)e^{-\int\mu_3(t)}dt\Big)}{\alpha 3-\alpha 2}+$$
$$\begin{bmatrix} s_1e^{-\int\mu_1(t)dt}+\big(s_2e^{-\int\mu_2(t)dt}+\;(s_3-1)e^{-\int\mu_3(t)}dt\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt+\int\mu_2(t)e^{-\int\mu_2(t)dt}\,dt\big)\\ \big(\int\mu_1(t)\,e^{\int(\mu_2(t)-\mu_1(t))dt}dt\big)-\\ \Big((s_3-1)e^{-\int\mu_3(t)}\big(\int\mu_1(t)\big(\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt\;e^{\int\mu_2(t)}\big)e^{-\int\mu_1(t)dt}dt\big)\Big)-\\ \big(\int\mu_1(t)\big(\int\mu_2(t)\,e^{-\int\mu_2(t)dt}e^{\int\mu_2(t)dt}dt\big)e^{-\int\mu_1(t)dt}dt\big)\end{bmatrix}\big(\int\lambda\,e^{\int\mu_1(t)dt}dt\big)-$$

$$\big(s_2e^{-\int\mu_2(t)dt}+\;(s_3-1)e^{-\int\mu_3(t)}dt\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt+$$
$$\int\mu_2(t)e^{-\int\mu_2(t)dt}\,dt\big)\big(\int\lambda\,e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\,e^{\int(\mu_2(t)-\mu_1(t))dt}dt\big)\,dt\big)+$$

$$\Big((s_3-1)e^{-\int\mu_3(t)}\big(\int\lambda\,e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\big(\int\mu_2(t)\,e^{\int(\mu_3(t)-\mu_2(t))dt}dt\;e^{\int\mu_2(t)}\big)e^{-\int\mu_1(t)dt}dt\big)dt\Big)+$$
$$\big(\int\lambda\,e^{\int\mu_1(t)dt}\big(\int\mu_1(t)\big(\int\mu_2(t)\,e^{-\int\mu_2(t)dt}e^{\int\mu_2(t)dt}dt\big)e^{-\int\mu_1(t)dt}dt\big)dt\big)-\int\lambda\,dt \quad (10)$$

3. Attributes of the queuing model

By carrying out the expansion of the equation P (s_1, s_2, s_3, t) as given in Eq. (10) and isolating the constant components , we derive the expression for the probability that there are no users in the queue.

$$P_{000}(t)=exp*(-\lambda)\Bigg\{\Bigg(e^{-\left(\alpha_1t+\beta_1\frac{t^2}{2}\right)}\Big(\int_0^t e^{\alpha_1v+\beta_1\frac{v^2}{2}}dv-\frac{1}{\alpha_1}\Big)\Bigg)+\Bigg(e^{-\left(\alpha_2t+\beta_2\frac{t^2}{2}\right)}\Bigg(\frac{1}{(\alpha_2-\alpha_1)}-\int_0^t\frac{(\alpha_1+\beta_1v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\Bigg)+$$
$$e^{-\left(\alpha_2t+\beta_2\frac{t^2}{2}\right)}\Big(\int_0^t e^{\alpha_1v+\beta_1\frac{v^2}{2}}dv\int_0^t(\alpha_1+\beta_1v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\;-\Big(\int_0^t e^{\alpha_1v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\Big)dv\Big)-\frac{1}{\alpha_2}\Bigg)+$$
$$\Bigg[e^{-\left(\alpha_3t+\beta_3\frac{t^2}{2}\right)}\Big(\int_0^t(\alpha_1+\beta_1v)\;(\int_0^t(\alpha_2+\beta_2v)\;e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\;e^{\alpha_2v+\beta_2\frac{v^2}{2}})\;e^{-(\alpha_2v+\beta_2\frac{v^2}{2})}dv-\Big(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\Big)\Big)+$$
$$e^{-\left(\alpha_3t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+\beta_2v)\;e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\Bigg(\Big(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\Big)-\int_0^t\frac{(\alpha_1+\beta_1v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\Bigg)+\;e^{-\left(\alpha_3t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+$$
$$\beta_2v)\;e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv(\,(\int_0^t(\alpha_1+\beta_1v)\;e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\int_0^t\;e^{\left(\alpha_1v+\beta_1\frac{v^2}{2}\right)}dv)-\int_0^t\;e^{\left(\alpha_1v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+$$
$$\beta_1v)\;e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv)dv)+\;e^{-\left(\alpha_3t+\beta_3\frac{t^2}{2}\right)}(\int_0^t\;e^{\left(\alpha_1v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+\beta_1v)\;(\int_0^t(\alpha_2+$$
$$\beta_2v)\;e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\;e^{\alpha_2v+\beta_2\frac{v^2}{2}})e^{-(\alpha_2v+\beta_2\frac{v^2}{2})}dv)dv)-(\int_0^t(\alpha_1+\beta_1v)\Big(\int_0^t(\alpha_2+$$
$$\beta_2v)\;e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\;e^{\alpha_2v+\beta_2\frac{v^2}{2}}\Big)e^{-(\alpha_1v+\beta_1\frac{v^2}{2})}dv\;)\;(\int_0^t\;e^{\left(\alpha_1v+\beta_1\frac{v^2}{2}\right)}dv)\Bigg]\Bigg\}$$

Taking $s_2=1$, $s_3=1$ in P (s_1,s_2, s_3, t) , we obtain the PGF of the 1st queue size as

$$P(s_{1,}t)=exp\Bigg(\lambda\,(s_1-1)e^{-\left(\alpha_1t+\beta_1\frac{t^2}{2}\right)}\Big(\int_0^t\;e^{\alpha_1v+\beta_1\frac{v^2}{2}}dv-\frac{1}{\alpha_1}\Big)\Bigg)\;;\;\lambda<\alpha_1,\beta_1 \quad (12)$$

By developing P (s_1, t) and gathering the constant components, we obtain the probability that the 1st queue is empty as

$$P_{0..}(t) = exp\left(-\lambda\, e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)\right) \quad (13)$$

The average number of users in the 1st queue is

$$L_1(t) = \lambda e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right) \quad (14)$$

The utilization of the 1st service station is

$$U_1(t) = 1 - exp\left(-\lambda\, e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)\right) \quad (15)$$

The throughput of the 1st service station is

$$ThP_1(t) = (\alpha_1+\beta_1 t)\left[1 - exp\left(-\lambda\, e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)\right)\right] \quad (16)$$

The average waiting time of a user in the 1st queue is

$$W_1(t) = \frac{\lambda e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)}{(\alpha_1+\beta_1 t)\left[1-exp\left(-\lambda e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)\right)\right]} \quad (17)$$

The variance of the number of users in the 1st queue is

$$V_1(t) = \lambda e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right) \quad (18)$$

The coefficient of variation (CV) of the number of users in the 1st system is

$$CV_1(t) = \left(\lambda e^{-\left(\alpha_1 t+\beta_1\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv - \frac{1}{\alpha_1}\right)\right)^{-1/2} * 100 \quad (19)$$

Taking s_1=1, s_3=1 in P (s_1, s_2, s_3, t) , we obtain the PGF of the 2nd queue size as

$$P(s_2,t) = exp.\lambda\left((s_2-1)e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + (s_2-1)e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1 + \beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right) - \frac{1}{\alpha_2}\right);\lambda < \alpha_1, \beta_1 \quad (20)$$

By developing P (s_2, t) and gathering the constant components, we obtain the probability that the 2nd queue is empty as

$$P_{.0.}(t) = -exp.\lambda\left(e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1 + \beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right) - \frac{1}{\alpha_2}\right);\lambda < \alpha_1, \beta_1 \quad (21)$$

The average number of users in the 2nd queue is

$$L_2(t) = \lambda\left(e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1 + \beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right) - \frac{1}{\alpha_2}\right);\lambda < \alpha_1, \beta_1 \quad (22)$$

The utilization of the 2nd service station is

$$U_2(t) = 1 - exp.\lambda\left(-e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right)-\frac{1}{\alpha_2}\right); \lambda<\alpha_1,\beta_1 \tag{23}$$

The throughput of the 2nd service station is

$$ThP_2(t) = (\alpha_1+\beta_1 t)\left[1 - exp.\lambda\left(e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right)-\frac{1}{\alpha_2}\right)\right] \tag{24}$$

The average waiting time of a user in the 2nd queue is

$$W_2(t) = \frac{L_2(t)}{ThP_2(t)} \tag{25}$$

The variance of the number of users in the 2nd queue is

$$V_2(t) = \lambda\left(e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right)-\frac{1}{\alpha_2}\right); \lambda<\alpha_1,\beta_1 \tag{26}$$

The coefficient of variation (CV) of the number of users in the 2nd system is

$$CV_2(t) = \left(\lambda\left(e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\frac{1}{(\alpha_2-\alpha_1)} - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_2 t+\beta_2\frac{t^2}{2}\right)}\left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}dv \int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv - \left(\int_0^t e^{\alpha_1 v+\beta_1\frac{v^2}{2}}\int_0^t(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\right)dv\right)-\frac{1}{\alpha_2}\right)\right)^{-1/2} * 100 \tag{27}$$

Taking $s_1=1$, $s_2=1$ in P (s_1, s_2, s_3, t), we obtain the pgf of the 3rd queue size as

$$P(s_3,t) = exp.\lambda\left[(s_3-1)\, e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\left(\int_0^t(\alpha_1+\beta_1 v)\,(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})\, e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + (s_3-1)\, e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t\frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + (s_3-1)\, e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv((\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv)dv) + + (s_3-1)\, e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+\beta_1 v)\,(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv)dv) - (\int_0^t(\alpha_1+\beta_1 v)\left(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right)e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})}dv\,)\,(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv)\right] \tag{28}$$

By developing P ($s_{3,}$, t) and gathering the constant components, we obtain the probability that the 3rd queue is empty as

$$P_{..0}(t) = exp.(-\lambda)\left[e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\left(\int_0^t(\alpha_1+\beta_1 v)\,(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})\, e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t\frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv((\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv)dv) + + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}(\int_0^t(\alpha_1+\beta_1 v)\,(\int_0^t(\alpha_2+$$

$$\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv)dv) - (\int_0^t (\alpha_1+\beta_1 v)\left(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})} dv\,)\,(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv)\Bigg] \quad (29)$$

The average number of users in the 3rd queue is

$$L_3(t) = \lambda\Bigg[e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\left(\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})\, e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv((\int_0^t (\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1 + \beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv)dv) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} (\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv)dv) - (\int_0^t (\alpha_1+\beta_1 v)\left(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})} dv\,)\,(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv)\Bigg] \quad (30)$$

The utilization of the 3rd service station is

$$U_3(t) = 1 - exp.(-\lambda)\Bigg[e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\left(\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})\, e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv((\int_0^t (\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1 + \beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv)dv) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} (\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv)dv) - (\int_0^t (\alpha_1+\beta_1 v)\left(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})} dv\,)\,(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv)\Bigg] \quad (31)$$

The throughput of the 3rd service station is

$$ThP_3(t) = (\alpha_3 + \beta_3 t)\Bigg\{1 - exp.(-\lambda)\Bigg[e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\left(\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}})\, e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv((\int_0^t (\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1 + \beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}} dv)dv) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} (\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} (\int_0^t (\alpha_1+\beta_1 v)\,(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})} dv)dv) - (\int_0^t (\alpha_1+\beta_1 v)\left(\int_0^t (\alpha_2 + \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}} dv\, e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})} dv\,)\,(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)} dv)\Bigg]\Bigg\} \quad (32)$$

The average waiting time of a user in the 3rd queue is

$$W_3(t) = \frac{L_3(t)}{ThP_3(t)} \quad (33)$$

The variance of the number of users in the 3rd queue is

$$\begin{aligned} V_3(t) = \lambda \Bigg[& e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \left(\int_0^t(\alpha_1+\beta_1 v)\left(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + \\ & e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t(\alpha_2 + \\ & \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\Big(\Big(\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv\Big) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}\Big(\int_0^t(\alpha_1 + \\ & \beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\Big)dv\Big) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\Big(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}\Big(\int_0^t(\alpha_1+\beta_1 v)\Big(\int_0^t(\alpha_2 + \\ & \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\Big)e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv\Big)dv\Big) - \Big(\int_0^t(\alpha_1+\beta_1 v)\left(\int_0^t(\alpha_2 + \right. \\ & \left.\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right)e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})}dv\Big)\Big(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv\Big)\Bigg] \end{aligned} \tag{34}$$

The coefficient of variation (CV) of the number of users in the 3rd system is

$$\begin{aligned} CV_3(t) = \Bigg(\lambda \Bigg[& e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \left(\int_0^t(\alpha_1+\beta_1 v)\left(\int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right) e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv - \left(\frac{\alpha_1\alpha_2}{\alpha_3(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right)\right) + \\ & e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t(\alpha_2+\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}\right) - \int_0^t \frac{(\alpha_1+\beta_1 v)e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv}{\alpha_1}\right) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)} \int_0^t(\alpha_2 + \\ & \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\Big(\Big(\int_0^t(\alpha_1+\beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv\Big) - \int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}\Big(\int_0^t(\alpha_1 + \\ & \beta_1 v)\, e^{(\alpha_2-\alpha_1)v+(\beta_2-\beta_1)\frac{v^2}{2}}dv\Big)dv\Big) + e^{-\left(\alpha_3 t+\beta_3\frac{t^2}{2}\right)}\Big(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}\Big(\int_0^t(\alpha_1+\beta_1 v)\Big(\int_0^t(\alpha_2 + \\ & \beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\Big)e^{-(\alpha_2 v+\beta_2\frac{v^2}{2})}dv\Big)dv\Big) - \Big(\int_0^t(\alpha_1+\beta_1 v)\left(\int_0^t(\alpha_2 + \right. \\ & \left.\beta_2 v)\, e^{(\alpha_3-\alpha_2)v+(\beta_3-\beta_2)\frac{v^2}{2}}dv\; e^{\alpha_2 v+\beta_2\frac{v^2}{2}}\right)e^{-(\alpha_1 v+\beta_1\frac{v^2}{2})}dv\Big)\Big(\int_0^t e^{\left(\alpha_1 v+\beta_1\frac{v^2}{2}\right)}dv\Big)\Bigg]\Bigg)^{-1/2} * 100 \end{aligned} \tag{35}$$

The average number of users present in the entire queueing system at any time t is denoted by L(t), and is given by the sum of the expected number of users in each of the three queues:

$$L(t) = L_1(t) + L_2(t) + L_3(t) \tag{36}$$

Where:

- $L_1(t)$ is the average number of users in the 1st queue at time t,
- $L_2(t)$ is the average number of users in the 2nd queue, and
- $L_3(t)$ is the average number of users in the 3rd queue.

4. Numerical illustration and sensitivity analysis

This Segment presents a numerical study to analyze the performance of the proposed three-node tandem queueing system. In this model, users arrive at the 1st queue and receive service at the 1st service station. Upon completion, they proceed to the 2nd queue, and then subsequently to the 3rd queue, each connected sequentially. The arrival process of users is assumed to follow a Poisson distribution, while the service processes at all three service stations follow NHPP with time-dependent service rates defined as $\mu_1(t) = \alpha_1 + \beta_1(t)$, $\mu_2(t) = \alpha_2 + \beta_2(t)$ and $\mu_3(t) = \alpha_3 + \beta_3(t)$ respectively. Given that the system's dynamics are highly sensitive to time variations, the transient behavior of the model is examined by evaluating performance metrics using selected parameter values for the model. t=0.100, 0.101, 0.103, 0.104, 0.105; λ=0.05, 0.10, 0.15, 0.20, 0.30, 0.40, 1.00; α_1=20 to 20.9, 23, 24, 25, 26, 27, β_1=10, 12, 14, 15, 16, 17, 18; α_2=22, 23, 24, 25, 26, 27, 28, 29, 30; β_2=18, 19, 20, 21, 22, 23, 24, 35, 45; α_3=21, 24, 25, 26, 27, 28 and β_3=14, 15, 16, 17, 18, 19, 20, 22, 23, 35, 70.

For various values of the parameters t, λ, α_1, α_2, α_3, β_1, β_2, β_3, several performance metrics are computed. These include the probability that a queue is empty, the average number of users in each queue, the utilization rates of all three service stations, the throughput at each service node, the variance in the number of users per queue, and the CV of queue lengths. The computed results for these performance indicators, corresponding to different parameter values, are summarized in Table 1. The relationships between the parameters and the observed performance metrics are visually depicted in Figure 2.

From Table 1, as time (t) varies from 0.100 to 0.105, the probability of emptiness of the system increases from 2.9289 to 2.96024. The probability of emptiness of the 1st queue decreases from 0.99824 to 0.99819, the probability that the emptiness of the 2nd queue increases from 0.99991 to 0.99995, and the probability that the emptiness of the 3rd queue increases from 0.93075 to 0.96210. The average number of users in the 1st queue increases from 0.00176 to 0.00181, in the 2nd queue it decreases from 0.00009 to 0.00005, and in the 3rd queue it decreases from 0.07177 to 0.03864 when all other variables are held constant. These results highlight that the system's overall emptiness probability is highly sensitive to changes in time.

As the arrival rate (λ) changes from 0.05 to 1.00, the probability of emptiness of the system, 1st queue decreases from 0.99824 to 0.96545. The probability of emptiness of the system, 2nd queue decreases from 0.99991 to 0.99821, and the probability of emptiness of the system, 3rd queue, decreases from 0.93075 to 0.23803. The average number of users in the system, 1st, 2nd, and 3rd queues increase from 0.00176 to 0.03516, 0.00009 to 0.00180, and 0.07177 to 1.43536, respectively, when all other variables are held constant.

As the service rate parameter (α_1) changes from 20 to 20.8, the probability of emptiness of the system, 1st queue, increases from 0.99824 to 0.99826. The probability of emptiness of the system, 2nd queue decreases from 0.99991 to 0.99974, and the probability of emptiness of the system, 3rd queue decreases from 0.93051 to 0.41974. The average number of users in system, 1st queue decreases from 0.00176 to 0.00174, the average number of users in system 2nd queue increases from 0.00009 to 0.00026, and the average number of users in system 3rd queue increases from 0.07203 to 0.86812, when all other variables are held constant.

As the service rate parameter (β_1) changes from 17 to 10, the probability of emptiness of the system, 1st queue, decreases from 0.99824 to 0.99821. The probability of emptiness of the system, 2nd queue has no change, i.e., 0.99991, and the probability of emptiness of the system, 3rd queue decreases from 0.93051 to 0.92887. The average number of users in system, 1st, queue decreases from 0.00176 to 0.00179. The average number of users in system 2nd queue remains unchanged, i.e., 0.00009, and the average number of users in system 3rd queue increases from 0.07203 to 0.07379, when all other variables are held constant.

As the service rate parameter (α_2) changes from 22 to 25, the probability of emptiness of the system, 1st queue remains unchanged, i.e., 0.99821. The probability of emptiness of the system, 2nd queue increases from 0.99828 to 0.99991, and the probability of emptiness of the system, 3rd queue increases from 0.61343 to 0.92887.The average number of users in system, 1st, queue is the same, i.e., 0.00179. The average number of users in system 2nd queue decreases from 0.00172 to 0.00009, and the average number of users in system 3rd queue decreases from 0.48869 to 0.07379, when all other variables are held constant.

As the service rate parameter (β_2) changes from 19 to 45, the probability of emptiness of the system, 1st queue remains unchanged, i.e., 0.99821. The probability of emptiness of the system, 2nd queue increases from 0.99991 to 0.99994, and the probability of emptiness of the system, 3rd queue decreases from 0.92871 to 0.92457. The average number of users in system, 1st, queue has no change, i.e., 0.00179, The average number of users in system, 2nd queue decreases from 0.00009 to 0.00006 and The average number of users in the system, 3rd queue increases from 0.07396 to 0.07842, when all other variables are held constant.

As the service rate parameter (α_3) changes from 24 to 24.9, the probability of emptiness of the system, 1st queue remains unchanged, i.e., 0.99832. The probability of emptiness of the system, 2nd queue has no change, i.e., 0.99941, and the probability of emptiness of the system, 3rd queue increases from 0.83571 to 0.87758. The average number of users in system, 1st, queue is has no change i.e., 0.00169, The average number of users in system, 2nd queue has no change i.e., 0.00059 and The average number of users in system, 3rd queue decrease from 0.17948 to 0.13059, when all other variables are held constant. S

As the service rate parameter (β_3) changes from 14 to 23, the probability of emptiness of the system, 1st queue remains unchanged, i.e., 0.99832. The probability of emptiness of the system, 2nd queue has no change, i.e., 0.99941, and the probability of emptiness of the system, 3rd queue increases from 0.86269 to 0.87353. The average number of users in system, 1st, queue is has no change i.e., 0.00169, The average number of users in system, 2nd queue has no change i.e., 0.00059 and The average number of users in system, 3rd queue decrease from 0.14769 to 0.13521, when all other variables are held constant.

Table 1: Values of P000 (t), P0..(t), P.0.(t), P..0(t), L_1(t), L_2(t), L_3(t) and L(t) for Various Values of Parameters

t	λ	α_1	β_1	α_2	β_2	α_3	β_3	P000(t)	P0..(t)	P.0.(t)	P..0(t)	L1(t)	L2(t)	L3(t)	L(t)
0.100	0.05	20	18	25	18	21	19	2.9289	0.99824	0.99991	0.93075	0.00176	0.00009	0.07177	0.07362
0.101	0.05	20	18	25	18	21	19	2.93455	0.99823	0.99992	0.93640	0.00177	0.00008	0.06572	0.06757
0.102	0.05	20	18	25	18	21	19	2.94049	0.99822	0.99993	0.94234	0.00178	0.00007	0.05938	0.06123
0.103	0.05	20	18	25	18	21	19	2.94675	0.99821	0.99994	0.94860	0.00179	0.00006	0.05277	0.05462
0.104	0.05	20	18	25	18	21	19	2.95332	0.99820	0.99994	0.95518	0.00180	0.00006	0.04585	0.04771
0.105	0.05	20	18	25	18	21	19	2.96024	0.99819	0.99995	0.96210	0.00181	0.00005	0.03864	0.04050
0.100	0.10	20	18	25	18	21	19	2.8626	0.99649	0.99982	0.86629	0.00352	0.00018	0.14354	0.14724
0.100	0.15	20	18	25	18	21	19	2.80077	0.99474	0.99973	0.80630	0.00527	0.00027	0.21530	0.22084
0.100	0.20	20	18	25	18	21	19	2.74309	0.99299	0.99964	0.75046	0.00703	0.00036	0.28707	0.29446
0.100	0.30	20	18	25	18	21	19	2.63908	0.98951	0.99946	0.65011	0.01055	0.00054	0.43061	0.44170
0.100	0.40	20	18	25	18	21	19	2.5485	0.98603	0.99928	0.56319	0.01406	0.00072	0.57414	0.58892
0.100	1.00	20	18	25	18	21	19	2.20169	0.96545	0.99821	0.23803	0.03516	0.00180	1.43536	1.47232
0.100	0.05	20	17	25	18	21	19	2.92866	0.99824	0.99991	0.93051	0.00176	0.00009	0.07203	0.07388
0.100	0.05	20.1	17	25	18	21	19	2.90874	0.99824	0.99989	0.91061	0.00176	0.00011	0.09365	0.09552
0.100	0.05	20.2	17	25	18	21	19	2.88430	0.99824	0.99987	0.88619	0.00176	0.00013	0.12083	0.12272
0.100	0.05	20.3	17	25	18	21	19	2.85370	0.99825	0.99985	0.85560	0.00175	0.00015	0.15595	0.15785
0.100	0.05	20.4	17	25	18	21	19	2.81435	0.99825	0.99983	0.81627	0.00175	0.00017	0.20301	0.20493
0.100	0.05	20.5	17	25	18	21	19	2.76210	0.99825	0.99981	0.76404	0.00175	0.00019	0.26913	0.27107
0.100	0.05	20.8	17	25	18	21	19	2.41774	0.99826	0.99974	0.41974	0.00174	0.00026	0.86812	0.87012
0.100	0.05	20	16	25	18	21	19	2.92842	0.99824	0.99991	0.93027	0.00177	0.00009	0.07229	0.07415
0.100	0.05	20	15	25	18	21	19	2.92817	0.99823	0.99991	0.93003	0.00177	0.00009	0.07254	0.07440
0.100	0.05	20	14	25	18	21	19	2.92793	0.99823	0.99991	0.92979	0.00177	0.00009	0.07279	0.07465
0.100	0.05	20	12	25	18	21	19	2.92746	0.99822	0.99991	0.92933	0.00178	0.00009	0.07330	0.07517
0.100	0.05	20	10	25	18	21	19	2.92699	0.99821	0.99991	0.92887	0.00179	0.00009	0.07379	0.07567
0.100	0.05	20	10	24	18	21	19	2.88247	0.99821	0.99967	0.88459	0.00179	0.00033	0.12263	0.12475
0.100	0.05	20	10	23	18	21	19	2.80302	0.99821	0.99923	0.80558	0.00179	0.00077	0.21620	0.21876
0.100	0.05	20	10	22	18	21	19	2.60992	0.99821	0.99828	0.61343	0.00179	0.00172	0.48869	0.49220
0.100	0.05	20	10	25	19	21	19	2.92683	0.99821	0.99991	0.92871	0.00179	0.00009	0.07396	0.07584
0.100	0.05	20	10	25	20	21	19	2.92667	0.99821	0.99991	0.92855	0.00179	0.00009	0.07413	0.07601
0.100	0.05	20	10	25	21	21	19	2.92651	0.99821	0.99991	0.92839	0.00179	0.00009	0.07430	0.07618
0.100	0.05	20	10	25	22	21	19	2.92636	0.99821	0.99991	0.92824	0.00179	0.00009	0.07447	0.07635
0.100	0.05	20	10	25	23	21	19	2.9262	0.99821	0.99991	0.92808	0.00179	0.00009	0.07464	0.07652
0.100	0.05	20	10	25	24	21	19	2.92605	0.99821	0.99992	0.92792	0.00179	0.00008	0.07481	0.07668
0.100	0.05	20	10	25	35	21	19	2.92431	0.99821	0.99993	0.92617	0.00179	0.00007	0.07670	0.07856
0.100	0.05	20	10	25	45	21	19	2.92272	0.99821	0.99994	0.92457	0.00179	0.00006	0.07842	0.08027
0.100	0.05	23	15	26	18	24	19	2.83344	0.99832	0.99941	0.83571	0.00169	0.00059	0.17948	0.18176
0.100	0.05	23	15	26	18	24.2	19	2.86186	0.99832	0.99941	0.86413	0.00169	0.00059	0.14604	0.14832
0.100	0.05	23	15	26	18	24.4	19	2.87804	0.99832	0.99941	0.88031	0.00169	0.00059	0.12748	0.12976
0.100	0.05	23	15	26	18	24.6	19	2.88431	0.99832	0.99941	0.88658	0.00169	0.00059	0.12038	0.12266
0.100	0.05	23	15	26	18	24.9	19	2.87531	0.99832	0.99941	0.87758	0.00169	0.00059	0.13059	0.13287
0.100	0.05	23	16	26	18	25	14	2.86042	0.99832	0.99941	0.86269	0.00169	0.00059	0.14769	0.14997

0.100	0.05	23	16	26	18	25	15	2.86164	0.99832	0.99941	0.86391	0.00169	0.00059	0.14629	0.14857
0.100	0.05	23	16	26	18	25	16	2.86285	0.99832	0.99941	0.86512	0.00169	0.00059	0.14489	0.14717
0.100	0.05	23	16	26	18	25	17	2.86405	0.99832	0.99941	0.86632	0.00169	0.00059	0.14350	0.14578
0.100	0.05	23	16	26	18	25	18	2.86526	0.99832	0.99941	0.86753	0.00169	0.00059	0.14211	0.14439
0.100	0.05	23	16	26	18	25	20	2.86767	0.99832	0.99941	0.86994	0.00169	0.00059	0.13933	0.14161
0.100	0.05	23	16	26	18	25	22	2.87007	0.99832	0.99941	0.87234	0.00169	0.00059	0.13658	0.13886
0.100	0.05	23	16	26	18	25	23	2.87126	0.99832	0.99941	0.87353	0.00169	0.00059	0.13521	0.13749
0.100	0.05	24	16	27	19	25	35	2.87265	0.99835	0.99947	0.87483	0.00165	0.00053	0.13373	0.13591
0.100	0.05	25	17	28	20	26	35	2.89758	0.99839	0.99952	0.89967	0.00161	0.00048	0.10572	0.10781
0.100	0.05	26	18	29	21	27	35	2.92878	0.99842	0.99956	0.93080	0.00158	0.00044	0.07171	0.07373
0.100	0.05	27	19	30	22	28	35	2.96737	0.99846	0.99960	0.96931	0.00154	0.00040	0.03117	0.03311
0.100	0.05	24	16	27	19	25	70	2.9172	0.99835	0.99947	0.91938	0.00165	0.00053	0.08406	0.08624
0.100	0.05	25	16	28	19	26	70	2.94499	0.99838	0.99951	0.94710	0.00162	0.00049	0.05436	0.05647
0.100	0.05	26	16	29	19	27	70	2.97919	0.99842	0.99956	0.98121	0.00159	0.00044	0.01897	0.02100
0.100	0.05	27	16	29	19	28	70	2.76913	0.99845	0.99912	0.77156	0.00155	0.00088	0.25934	0.26177
0.100	0.10	24	16	27	19	25	35	2.76095	0.99670	0.99893	0.76532	0.00330	0.00107	0.26746	0.27183
0.101	0.15	25	17	28	20	26	35	2.75532	0.99515	0.99860	0.76157	0.00487	0.00140	0.27238	0.27865
0.102	0.20	26	18	29	21	27	35	2.8578	0.99366	0.99838	0.86576	0.00636	0.00162	0.14414	0.15212

According to the data presented in Table 2,

The utilization rates, throughput levels, and user waiting times at the service stations exhibit significant time-dependent variations. As the time variable (t) increases from 0.100 to 0.105:

- The utilization of the 1st service station shows a slight increase from 0.00176 to 0.00181. In contrast, the 2nd and 3rd stations experience a decrease, dropping from 0.00009 to 0.00005 and from 0.06925 to 0.0379, respectively.
- The throughput rises modestly at the 1st station, increasing from 0.03829 to 0.03965. However, it declines at both the 2nd and 3rd stations—from 0.00241 to 0.00131 and from 1.58589 to 0.87157, respectively.
- The average waiting time for users in all three queues shows a slight downward trend: from 0.04591 to 0.04572 in the 1st queue, 0.03732 to 0.03719 in the 2nd, and 0.04525 to 0.04433 in the 3rd.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the arrival rate (λ). As λ increases from 0.05 to 1.00:

- The utilization of the 1st service station increases slightly from 0.00181 to 0.03455. Average while, the 2nd and 3rd stations experience rising values, with utilization growing from 0.00005 to 0.00179 and from 0.0379 to 0.76197, respectively.
- The throughput at the 1st station increases significantly from 0.03965 to 0.75321. A similar upward trend is observed in the 2nd and 3rd stations, where throughput grows from 0.00131 to 0.04807 and from 0.87157 to 17.4491, respectively.
- The average waiting time for users also increases slightly across all queues: from 0.04572 to 0.04668 in the 1st queue, from 0.03719 to 0.03735 in the 2nd, and from 0.04433 to 0.08226 in the 3rd.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (α_1). As the service rate parameter (α_1) changes from 20 to 20.8:

- The utilization of the 1st service station shows a slight decrease from 0.00176 to 0.00174. In contrast, the 2nd and 3rd stations experience increases in utilization, rising from 0.00009 to 0.00026 and from 0.06949 to 0.58026, respectively.
- The throughput increases across all three stations. At the 1st station, it grows modestly from 0.03819 to 0.03916. The 2nd and 3rd stations show more significant increases, from 0.00241 to 0.00685 and from 1.59143 to 13.28799, respectively.
- The average waiting time exhibits mixed behavior. It slightly decreases in the 1st queue, from 0.04612 to 0.04448, remains constant in the 2nd queue at 0.03732, and increases in the 3rd queue from 0.04526 to 0.06533.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (β_1). As the service rate parameter (β_1) decreases from 17 to 10,

- The utilization of the 1st service station shows a slight increase from 0.00176 to 0.00179. The 2nd station's utilization remains constant at 0.00009, while the 3rd station sees a small increase from 0.06949 to 0.07113.
- The throughput at the 1st station experiences a slight decline from 0.03819 to 0.03749. In contrast, the 2nd and 3rd stations record modest increases, with throughput rising from 0.00241 to 0.00248 and from 1.59143 to 1.62892, respectively.
- The average waiting time for users increases slightly in all three queues: from 0.04612 to 0.04766 in the 1st queue, remains unchanged at 0.03732 in the 2nd, and increases marginally from 0.04526 to 0.0453 in the 3rd.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (α_2). As the service rate parameter (α_2) changes from 22 to 25,

- The utilization of the 1st service station remains unchanged at 0.00179. However, the 2nd and 3rd stations exhibit increases in utilization, rising from 0.00009 to 0.00172 and from 0.07113 to 0.38657, respectively.
- The throughput at the 1st station also remains constant at 0.03749. In contrast, the 2nd and 3rd stations show significant increases in throughput, from 0.00248 to 0.04086 and from 1.62892 to 8.85251, respectively.
- The average waiting time stays the same in the 1st queue at 0.04766, but rises slightly in the 2nd and 3rd queues—from 0.03732 to 0.04205 and from 0.0453 to 0.0552, respectively

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (β_2).As the service rate parameter (β_2) changes from 19 to 45,

- The utilization of the 1st service station remains unchanged at 0.00179. For the 2nd station, utilization slightly decreases from 0.00009 to 0.00006, while the 3rd station experiences a small increase from 0.07129 to 0.07543.
- The throughput at the 1st station remains constant at 0.03749. However, the 2nd station sees a slight decline in throughput from 0.00245 to 0.00168, whereas the 3rd station shows a modest increase from 1.63254 to 1.72724.

- The average waiting time remains the same in the 1st queue at 0.04766. In the 2nd queue, it decreases slightly from 0.03718 to 0.0339, while the 3rd queue experiences a minimal increase from 0.0453 to 0.0454.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (α_3). As the service rate parameter (α_3) changes from 24 to 24.9,

- The utilization of the 1st and 2nd service stations remains unchanged at 0.00168 and 0.00059, respectively. However, the 3rd station shows a slight decrease in utilization from 0.16429 to 0.12242.
- The throughput remains constant at the 1st and 2nd stations, at 0.04126 and 0.01637, respectively. Conversely, the 3rd station experiences a reduction in throughput from 4.25522 to 3.28084.
- The average waiting time shows no change in the 1st and 2nd queues, staying at 0.04085 and 0.03598, respectively. In the 3rd queue, it decreases slightly from 0.04218 to 0.0398.

These trends are observed while keeping all other variables constant.

The utilization rates, throughput levels, and average user waiting times at the service stations exhibit significant variations in response to changes in the service rate parameter (β_3). As the service rate parameter (β_3) changes from 14 to 23,

- The utilization of both the 1st and 2nd service stations remains constant at 0.00168 and 0.00059, respectively. The 3rd station shows a slight decrease in utilization from 0.13731 to 0.12647.
- The throughput values also remain unchanged at the 1st and 2nd stations, holding steady at 0.04126 and 0.01637, respectively. However, a slight decrease is noted at the 3rd station, where throughput drops from 3.62485 to 3.45250.
- The average waiting time stays the same in the 1st and 2nd queues, at 0.04085 and 0.03598, respectively, while it decreases marginally in the 3rd queue from 0.04074 to 0.03916.

Table 2: Values of $U_1(t)$, $U_2(t)$, $U_3(t)$, $Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$, $W_1(t)$, $W_2(t)$ and $W_3(t)$ for Different Values of Parameters

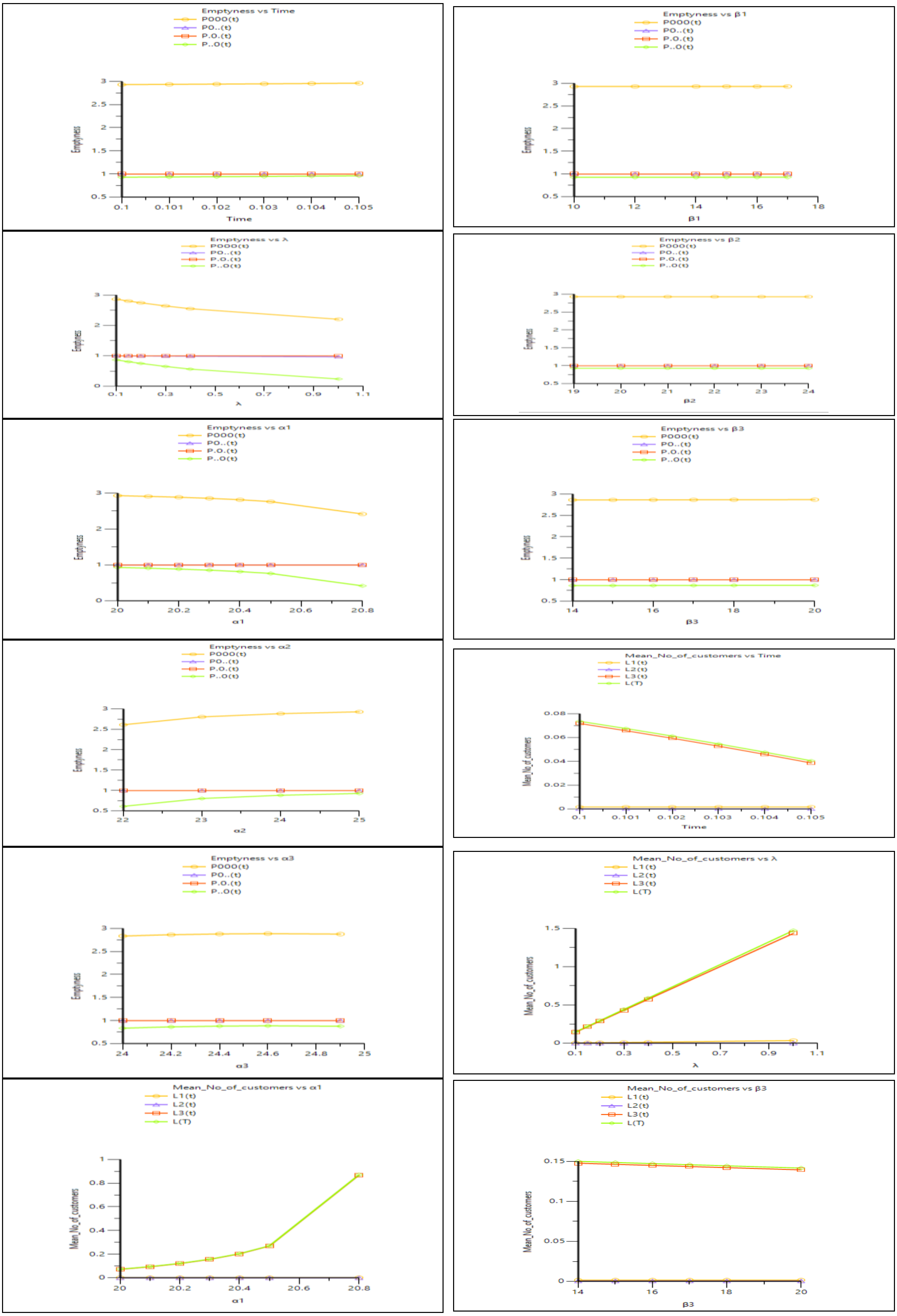
t	λ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Thp_1(t)$	$Thp_2(t)$	$Thp_3(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.100	0.05	20	18	25	18	21	19	0.00176	0.00009	0.06925	0.03829	0.00241	1.58589	0.04591	0.03732	0.04525
0.101	0.05	20	18	25	18	21	19	0.00177	0.00008	0.0636	0.03858	0.00217	1.45771	0.04587	0.03729	0.04508
0.102	0.05	20	18	25	18	21	19	0.00178	0.00007	0.05766	0.03885	0.00194	1.3225	0.04584	0.03726	0.0449
0.103	0.05	20	18	25	18	21	19	0.00179	0.00006	0.0514	0.03912	0.00172	1.17997	0.0458	0.03724	0.04472
0.104	0.05	20	18	25	18	21	19	0.0018	0.00006	0.04482	0.03939	0.00151	1.02977	0.04576	0.03721	0.04453
0.105	0.05	20	18	25	18	21	19	0.00181	0.00005	0.0379	0.03965	0.00131	0.87157	0.04572	0.03719	0.04433
0.100	0.10	20	18	25	18	21	19	0.00351	0.00018	0.13371	0.07652	0.00481	3.06196	0.04595	0.03732	0.04688
0.100	0.15	20	18	25	18	21	19	0.00526	0.00027	0.1937	0.11468	0.00722	4.4358	0.04599	0.03732	0.04854
0.100	0.20	20	18	25	18	21	19	0.00701	0.00036	0.24954	0.15277	0.00962	5.71451	0.04603	0.03732	0.05024
0.100	0.30	20	18	25	18	21	19	0.01049	0.00054	0.34989	0.22875	0.01443	8.01238	0.04611	0.03732	0.05374
0.100	0.40	20	18	25	18	21	19	0.01397	0.00072	0.43681	0.30447	0.01924	10.003	0.04619	0.03733	0.0574
0.100	1.00	20	18	25	18	21	19	0.03455	0.00179	0.76197	0.75321	0.04807	17.4491	0.04668	0.03735	0.08226
0.100	0.05	20	17	25	18	21	19	0.00176	0.00009	0.06949	0.03819	0.00241	1.59143	0.04612	0.03732	0.04526
0.100	0.05	20.1	17	25	18	21	19	0.00176	0.00011	0.08939	0.03832	0.0029	2.04714	0.04591	0.03732	0.04574
0.100	0.05	20.2	17	25	18	21	19	0.00176	0.00013	0.11381	0.03844	0.00341	2.6063	0.0457	0.03732	0.04636
0.100	0.05	20.3	17	25	18	21	19	0.00175	0.00015	0.1444	0.03857	0.00393	3.30681	0.04549	0.03732	0.04716
0.100	0.05	20.4	17	25	18	21	19	0.00175	0.00017	0.18373	0.03869	0.00447	4.20732	0.04529	0.03732	0.04825
0.100	0.05	20.5	17	25	18	21	19	0.00175	0.00019	0.23596	0.03881	0.00503	5.40341	0.04508	0.03732	0.04981
0.100	0.05	20.8	17	25	18	21	19	0.00174	0.00026	0.58026	0.03916	0.00685	13.28799	0.04448	0.03732	0.06533
0.100	0.05	20	16	25	18	21	19	0.00176	0.00009	0.06973	0.03809	0.00242	1.59692	0.04634	0.03732	0.04527
0.100	0.05	20	15	25	18	21	19	0.00177	0.00009	0.06997	0.03799	0.00243	1.60236	0.04655	0.03732	0.04527
0.100	0.05	20	14	25	18	21	19	0.00177	0.00009	0.07021	0.03789	0.00244	1.60776	0.04677	0.03732	0.04528
0.100	0.05	20	12	25	18	21	19	0.00178	0.00009	0.07067	0.03769	0.00246	1.61843	0.04721	0.03732	0.04529
0.100	0.05	20	10	25	18	21	19	0.00179	0.00009	0.07113	0.03749	0.00248	1.62892	0.04766	0.03732	0.0453
0.100	0.05	20	10	24	18	21	19	0.00179	0.00033	0.11541	0.03749	0.00858	2.64288	0.04766	0.03877	0.0464
0.100	0.05	20	10	23	18	21	19	0.00179	0.00077	0.19442	0.03749	0.01913	4.45231	0.04766	0.04034	0.04856
0.100	0.05	20	10	22	18	21	19	0.00179	0.00172	0.38657	0.03749	0.04086	8.85251	0.04766	0.04205	0.0552
0.100	0.05	20	10	25	19	21	19	0.00179	0.00009	0.07129	0.03749	0.00245	1.63254	0.04766	0.03718	0.0453
0.100	0.05	20	10	25	20	21	19	0.00179	0.00009	0.07145	0.03749	0.00242	1.63617	0.04766	0.03704	0.04531
0.100	0.05	20	10	25	21	21	19	0.00179	0.00009	0.07161	0.03749	0.00239	1.63979	0.04766	0.0369	0.04531
0.100	0.05	20	10	25	22	21	19	0.00179	0.00009	0.07176	0.03749	0.00237	1.64342	0.04766	0.03677	0.04531
0.100	0.05	20	10	25	23	21	19	0.00179	0.00009	0.07192	0.03749	0.00234	1.64704	0.04766	0.03663	0.04532
0.100	0.05	20	10	25	24	21	19	0.00179	0.00008	0.07208	0.03749	0.00231	1.65067	0.04766	0.0365	0.04532
0.100	0.05	20	10	25	35	21	19	0.00179	0.00007	0.07383	0.03749	0.00198	1.69068	0.04766	0.03509	0.04536
0.100	0.05	20	10	25	45	21	19	0.00179	0.00006	0.07543	0.03749	0.00168	1.72724	0.04766	0.0339	0.0454
0.100	0.05	23	15	26	18	24	19	0.00168	0.00059	0.16429	0.04126	0.01637	4.25522	0.04085	0.03598	0.04218
0.100	0.05	23	15	26	18	24.2	19	0.00168	0.00059	0.13587	0.04126	0.01637	3.54632	0.04085	0.03598	0.04118
0.100	0.05	23	15	26	18	24.4	19	0.00168	0.00059	0.11969	0.04126	0.01637	3.14776	0.04085	0.03598	0.0405
0.100	0.05	23	15	26	18	24.6	19	0.00168	0.00059	0.11342	0.04126	0.01637	3.00564	0.04085	0.03598	0.04005
0.100	0.05	23	15	26	18	24.9	19	0.00168	0.00059	0.12242	0.04126	0.01637	3.28084	0.04085	0.03598	0.0398
0.100	0.05	23	16	26	18	25	14	0.00168	0.00059	0.13731	0.04126	0.01637	3.62485	0.04085	0.03598	0.04074
0.100	0.05	23	16	26	18	25	15	0.00168	0.00059	0.13609	0.04126	0.01637	3.60648	0.04085	0.03598	0.04056
0.100	0.05	23	16	26	18	25	16	0.00168	0.00059	0.13488	0.04126	0.01637	3.58792	0.04085	0.03598	0.04038
0.100	0.05	23	16	26	18	25	17	0.00168	0.00059	0.13368	0.04126	0.01637	3.56916	0.04085	0.03598	0.0402
0.100	0.05	23	16	26	18	25	18	0.00168	0.00059	0.13247	0.04126	0.01637	3.55021	0.04085	0.03598	0.04003
0.100	0.05	23	16	26	18	25	20	0.00168	0.00059	0.13006	0.04126	0.01637	3.51171	0.04085	0.03598	0.03968
0.100	0.05	23	16	26	18	25	22	0.00168	0.00059	0.12766	0.04126	0.01637	3.47243	0.04085	0.03598	0.03933
0.100	0.05	23	16	26	18	25	23	0.00168	0.00059	0.12647	0.04126	0.01637	3.4525	0.04085	0.03598	0.03916
0.100	0.05	24	16	27	19	25	35	0.00165	0.00053	0.12517	0.04222	0.01544	3.56746	0.03909	0.03461	0.03749
0.100	0.05	25	17	28	20	26	35	0.00161	0.00048	0.10033	0.04308	0.01454	2.95966	0.03748	0.03334	0.03572
0.100	0.05	26	18	29	21	27	35	0.00158	0.00044	0.0692	0.04386	0.01366	2.11059	0.036	0.03216	0.03398
0.100	0.05	27	19	30	22	28	35	0.00154	0.0004	0.03069	0.04456	0.01281	0.96664	0.03463	0.03106	0.03224
0.100	0.05	24	16	27	19	25	70	0.00165	0.00053	0.08062	0.04222	0.01544	2.5799	0.03909	0.03461	0.03258
0.100	0.05	25	16	28	19	26	70	0.00162	0.00049	0.0529	0.04301	0.01458	1.74586	0.03762	0.03345	0.03113

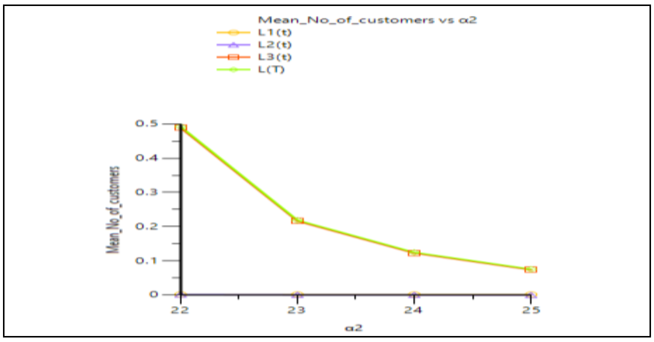
0.100	0.05	26	16	29	19	27	70	0.00158	0.00044	0.01879	0.04372	0.01374	0.63896	0.03626	0.03237	0.02969
0.100	0.05	27	16	29	19	28	70	0.00155	0.00088	0.22844	0.04436	0.02711	7.9955	0.03499	0.03238	0.03244
0.100	0.10	24	16	27	19	25	35	0.0033	0.00107	0.23468	0.08437	0.03088	6.68836	0.03913	0.03462	0.03999
0.101	0.15	25	17	28	20	26	35	0.00485	0.0014	0.23843	0.12966	0.04198	7.04217	0.03752	0.03333	0.03868
0.102	0.20	26	18	29	21	27	35	0.00634	0.00162	0.13424	0.17656	0.05052	4.10365	0.03604	0.03214	0.03513

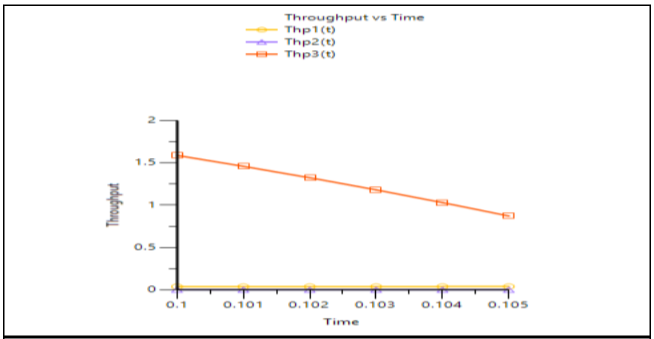
Table3: Values of $V_1(t)$, $V_2(t)$, $V_3(t)$, $CV_1(t)$, $CV_2(t)$ and $CV_3(t)$, for Different Values of Parameters

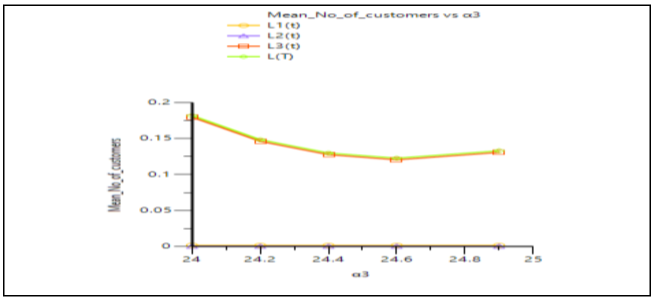
t	λ	α1	β1	α2	β2	α3	β3	V1(t)	V2(t)	V3(t)	CV1(t)	CV2(t)	CV3(t)
0.100	0.05	20	18	25	18	21	19	0.00176	0.00009	0.07177	23.84939	105.5432	3.7328
0.101	0.05	20	18	25	18	21	19	0.00177	0.00008	0.06572	23.77156	111.23807	3.90091
0.102	0.05	20	18	25	18	21	19	0.00178	0.00007	0.05938	23.69635	117.66252	4.10359
0.103	0.05	20	18	25	18	21	19	0.00179	0.00006	0.05277	23.62367	124.99333	4.3533
0.104	0.05	20	18	25	18	21	19	0.0018	0.00006	0.04585	23.5534	133.47487	4.6699
0.105	0.05	20	18	25	18	21	19	0.00181	0.00005	0.03864	23.48547	143.45675	5.08725
0.100	0.10	20	18	25	18	21	19	0.00352	0.00018	0.14354	16.86407	74.63031	2.63949
0.100	0.15	20	18	25	18	21	19	0.00527	0.00027	0.2153	13.76945	60.93539	2.15514
0.100	0.20	20	18	25	18	21	19	0.00703	0.00036	0.28707	11.9247	52.7716	1.8664
0.100	0.30	20	18	25	18	21	19	0.01055	0.00054	0.43061	9.73647	43.08783	1.52391
0.100	0.40	20	18	25	18	21	19	0.01406	0.00072	0.57414	8.43203	37.31515	1.31975
0.100	1.00	20	18	25	18	21	19	0.03516	0.0018	1.43536	5.33289	23.60018	0.83468
0.100	0.05	20	17	25	18	21	19	0.00176	0.00009	0.07203	23.82542	105.34232	3.72607
0.100	0.05	20.1	17	25	18	21	19	0.00176	0.00011	0.09365	23.84125	96.08103	3.26781
0.100	0.05	20.2	17	25	18	21	19	0.00176	0.00013	0.12083	23.85738	88.68346	2.87686
0.100	0.05	20.3	17	25	18	21	19	0.00175	0.00015	0.15595	23.8738	82.58208	2.53222
0.100	0.05	20.4	17	25	18	21	19	0.00175	0.00017	0.20301	23.8905	77.42509	2.21946
0.100	0.05	20.5	17	25	18	21	19	0.00175	0.00019	0.26913	23.90748	72.98125	1.9276
0.100	0.05	20.8	17	25	18	21	19	0.00174	0.00026	0.86812	23.96002	62.54889	1.07327
0.100	0.05	20	16	25	18	21	19	0.00177	0.00009	0.07229	23.80145	105.14027	3.71942
0.100	0.05	20	15	25	18	21	19	0.00177	0.00009	0.07254	23.77749	104.93707	3.71286
0.100	0.05	20	14	25	18	21	19	0.00177	0.00009	0.07279	23.75353	104.73274	3.70639
0.100	0.05	20	12	25	18	21	19	0.00178	0.00009	0.0733	23.70565	104.32071	3.6937
0.100	0.05	20	10	25	18	21	19	0.00179	0.00009	0.07379	23.6578	103.9043	3.68134
0.100	0.05	20	10	24	18	21	19	0.00179	0.00033	0.12263	23.6578	54.84299	2.85562
0.100	0.05	20	10	23	18	21	19	0.00179	0.00077	0.2162	23.6578	36.00016	2.15067
0.100	0.05	20	10	22	18	21	19	0.00179	0.00172	0.48869	23.6578	24.1243	1.43048
0.100	0.05	20	10	25	19	21	19	0.00179	0.00009	0.07396	23.6578	104.7167	3.6771
0.100	0.05	20	10	25	20	21	19	0.00179	0.00009	0.07413	23.6578	105.54321	3.67287
0.100	0.05	20	10	25	21	21	19	0.00179	0.00009	0.0743	23.6578	106.38425	3.66865
0.100	0.05	20	10	25	22	21	19	0.00179	0.00009	0.07447	23.6578	107.24028	3.66445
0.100	0.05	20	10	25	23	21	19	0.00179	0.00009	0.07464	23.6578	108.11175	3.66026
0.100	0.05	20	10	25	24	21	19	0.00179	0.00008	0.07481	23.6578	108.99916	3.65608
0.100	0.05	20	10	25	35	21	19	0.00179	0.00007	0.0767	23.6578	119.97743	3.61088
0.100	0.05	20	10	25	45	21	19	0.00179	0.00006	0.07842	23.6578	132.49006	3.57094
0.100	0.05	23	15	26	18	24	19	0.00169	0.00059	0.17948	24.35762	41.19728	2.36044
0.100	0.05	23	15	26	18	24.2	19	0.00169	0.00059	0.14604	24.35762	41.19728	2.61679
0.100	0.05	23	15	26	18	24.4	19	0.00169	0.00059	0.12748	24.35762	41.19728	2.80081
0.100	0.05	23	15	26	18	24.6	19	0.00169	0.00059	0.12038	24.35762	41.19728	2.88214
0.100	0.05	23	15	26	18	24.9	19	0.00169	0.00059	0.13059	24.35762	41.19728	2.76726
0.100	0.05	23	16	26	18	25	14	0.00169	0.00059	0.14769	24.35762	41.19728	2.60207
0.100	0.05	23	16	26	18	25	15	0.00169	0.00059	0.14629	24.35762	41.19728	2.61451
0.100	0.05	23	16	26	18	25	16	0.00169	0.00059	0.14489	24.35762	41.19728	2.62711
0.100	0.05	23	16	26	18	25	17	0.00169	0.00059	0.1435	24.35762	41.19728	2.63985
0.100	0.05	23	16	26	18	25	18	0.00169	0.00059	0.14211	24.35762	41.19728	2.65274
0.100	0.05	23	16	26	18	25	20	0.00169	0.00059	0.13933	24.35762	41.19728	2.67898
0.100	0.05	23	16	26	18	25	22	0.00169	0.00059	0.13658	24.35762	41.19728	2.70587
0.100	0.05	23	16	26	18	25	23	0.00169	0.00059	0.13521	24.35762	41.19728	2.71957
0.100	0.05	24	16	27	19	25	35	0.00165	0.00053	0.13373	24.61403	43.25192	2.73455
0.100	0.05	25	17	28	20	26	35	0.00161	0.00048	0.10572	24.88443	45.42254	3.07547
0.100	0.05	26	18	29	21	27	35	0.00158	0.00044	0.07171	25.16641	47.7147	3.7343
0.100	0.05	27	19	30	22	28	35	0.00154	0.0004	0.03117	25.45802	50.13434	5.66431
0.100	0.05	24	16	27	19	25	70	0.00165	0.00053	0.08406	24.61403	43.25192	3.44914
0.100	0.05	25	16	28	19	26	70	0.00162	0.00049	0.05436	24.85933	45.27743	4.28921
0.100	0.05	26	16	29	19	27	70	0.00159	0.00044	0.01897	25.11594	47.41332	7.26013
0.100	0.05	27	16	29	19	28	70	0.00155	0.00088	0.25934	25.38199	33.75445	1.96364
0.100	0.10	24	16	27	19	25	35	0.0033	0.00107	0.26746	17.40475	30.58373	1.93362
0.101	0.15	25	17	28	20	26	35	0.00487	0.0014	0.27238	14.33698	26.73267	1.91608
0.102	0.20	26	18	29	21	27	35	0.00636	0.00162	0.14414	12.53636	24.81822	2.63391

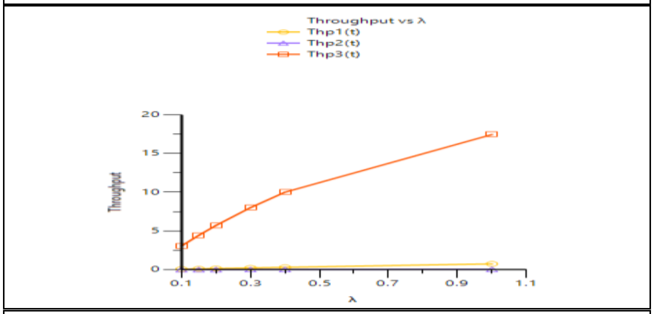
For various values of the parameters t, λ, α_1, β_1, α_2, β_2, α_3 and β_3 the calculated results for key performance metrics—such as the probability that the queue is empty (P0..(t), P.0.(t), P..0(t)), the average number of customers in each queue ($L_1(t)$, $L_2(t)$, $L_3(t)$ and L(t)), the utilization rates of the service stations ($U_1(t)$, $U_2(t)$, $U_3(t)$), the throughput for each service station ($Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$), and the waiting time for each service station ($W_1(t)$, $W_2(t)$ and $W_3(t)$) —are summarized in Table 1,2 & 3. The relationships between these parameters and the corresponding performance indicators are shown in Fig. 2.

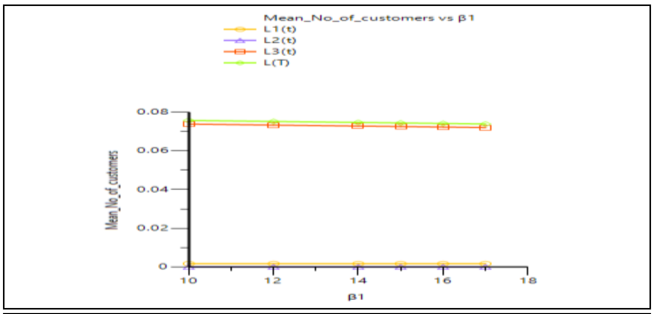

Emptyness vs Time
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
Time
Emptyness vs β1
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
β1
Emptyness vs λ
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
λ
Emptyness vs β2
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
β2
Emptyness vs α1
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
α1
Emptyness vs β3
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
β3
Emptyness vs α2
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
α2
Mean_No_of_customers vs Time
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
Time
Emptyness vs α3
P000(t)
P0..(t)
P.0.(t)
P..0(t)
Emptyness
α3
Mean_No_of_customers vs λ
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
λ
Mean_No_of_customers vs α1
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
α1
Mean_No_of_customers vs β3
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
β3

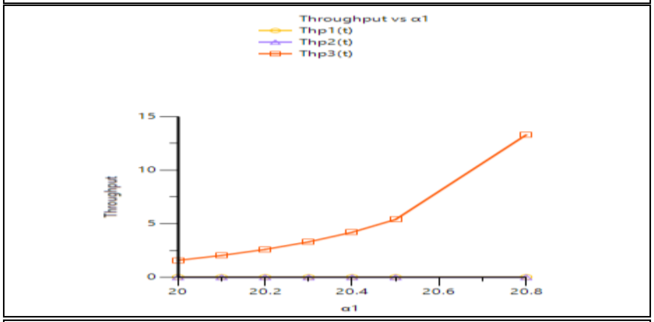

Mean_No_of_customers vs α2
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
α2

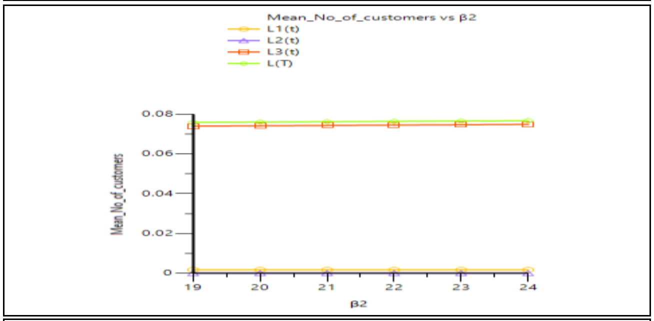

Throughput vs Time
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
Time

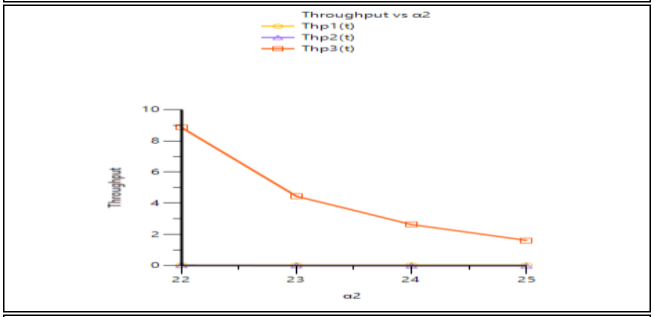

Mean_No_of_customers vs α3
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
α3

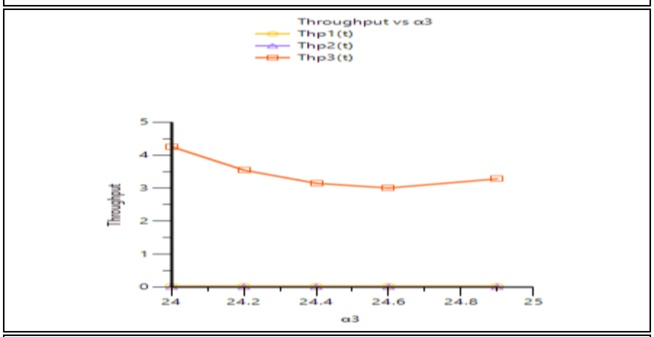

Throughput vs λ
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
λ

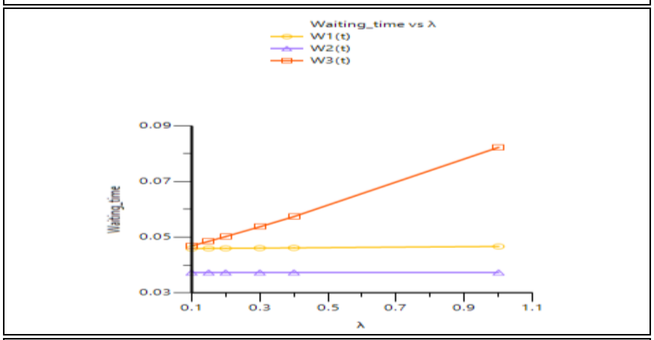

Mean_No_of_customers vs β1
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
β1

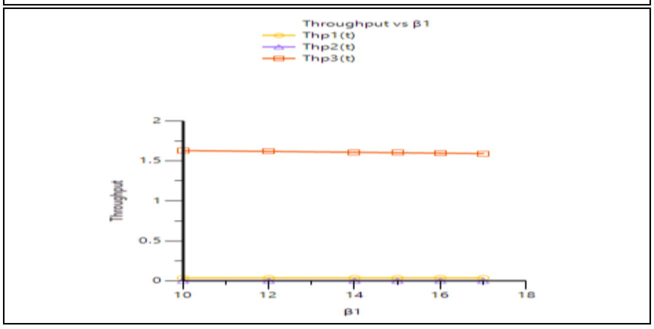

Throughput vs α1
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
α1

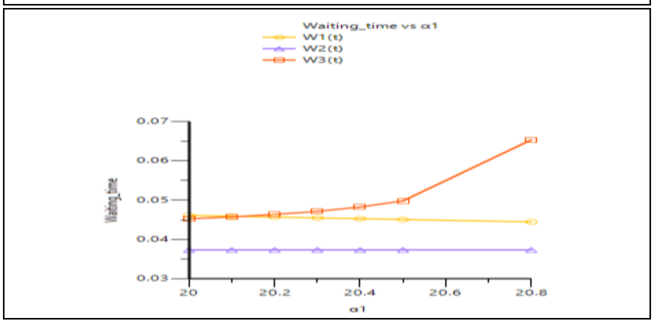

Mean_No_of_customers vs β2
L1(t)
L2(t)
L3(t)
L(T)
Mean_No_of_customers
β2

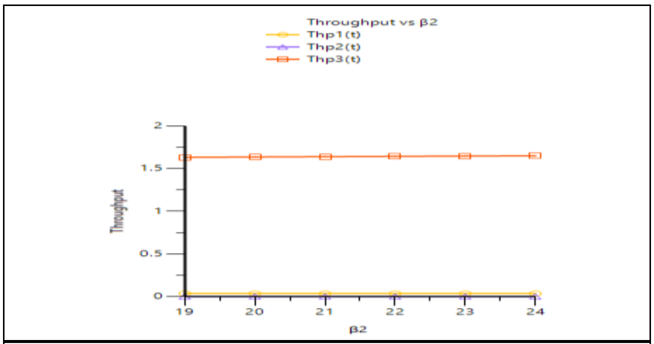

Throughput vs α2
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
α2

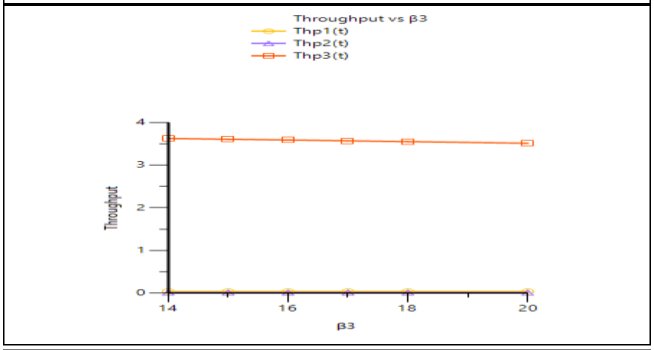

Throughput vs α3
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
α3

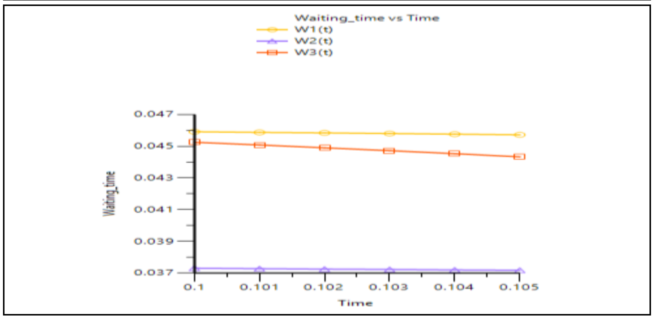

Waiting_time vs λ
W1(t)
W2(t)
W3(t)
Waiting_time
λ

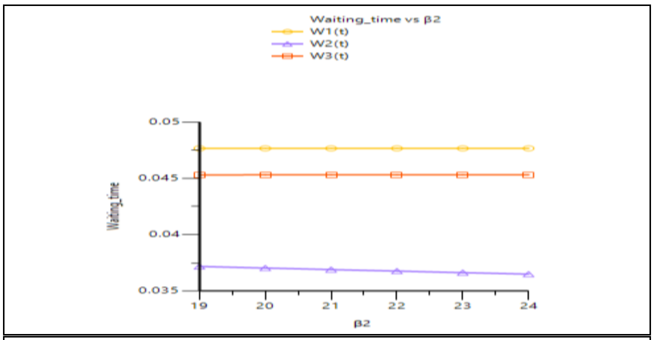

Throughput vs β1
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
β1

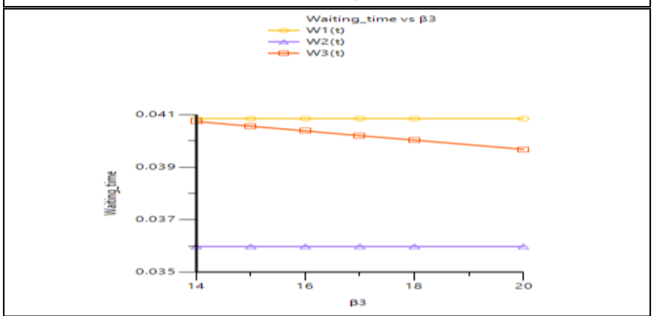

Waiting_time vs α1
W1(t)
W2(t)
W3(t)
Waiting_time
α1

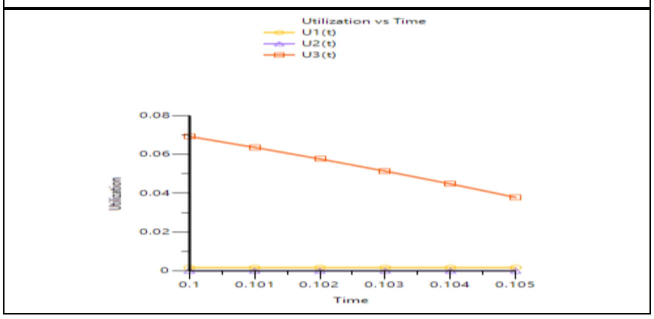

Throughput vs β2
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
β2

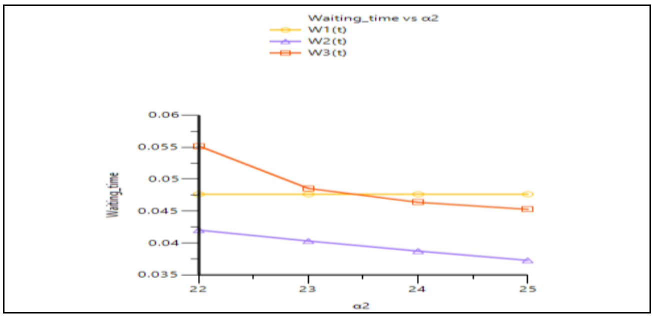

Throughput vs β3
Thp1(t)
Thp2(t)
Thp3(t)
Throughput
β3

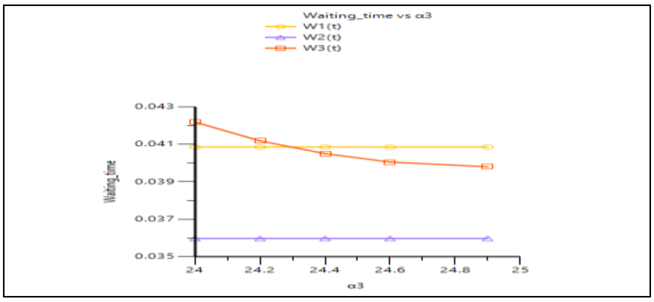

Waiting_time vs Time
W1(t)
W2(t)
W3(t)
Waiting_time
Time

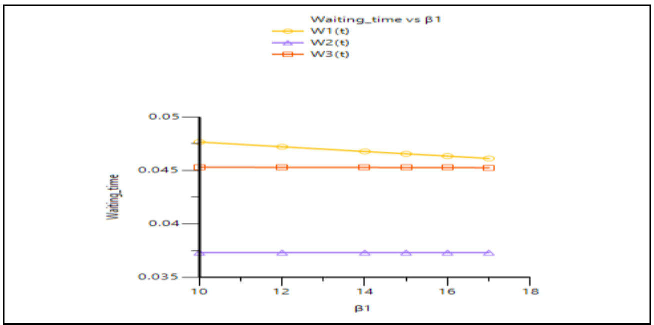

Waiting_time vs β2
W1(t)
W2(t)
W3(t)
Waiting_time
β2

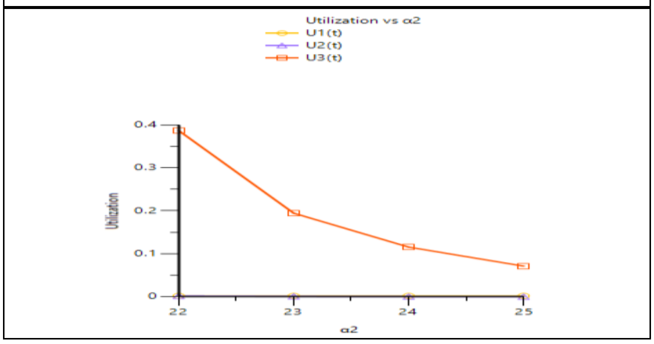

Waiting_time vs β3
W1(t)
W2(t)
W3(t)
Waiting_time
β3

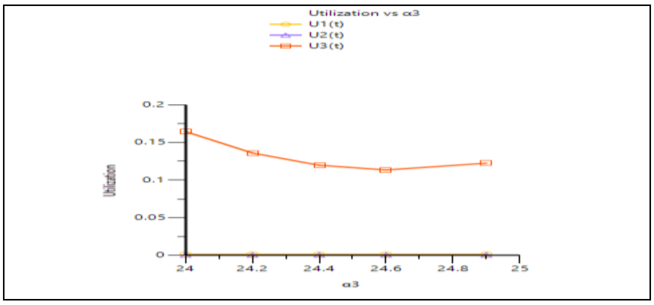

Utilization vs Time
U1(t)
U2(t)
U3(t)
Utilization
Time

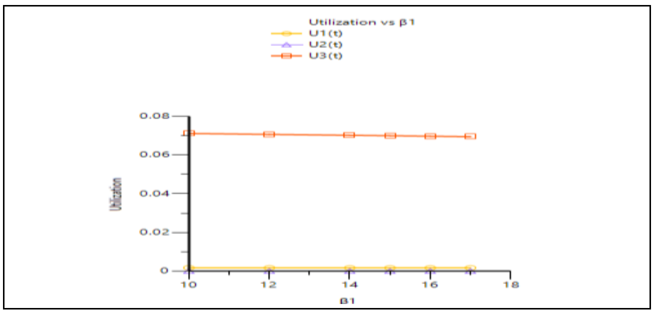

Waiting_time vs α2
W1(t)
W2(t)
W3(t)
Waiting_time
α2


Waiting_time vs α3
W1(t)
W2(t)
W3(t)
Waiting_time
α3


Waiting_time vs β1
W1(t)
W2(t)
W3(t)
Waiting_time
β1


Utilization vs α2
U1(t)
U2(t)
U3(t)
Utilization
α2


Utilization vs α3
U1(t)
U2(t)
U3(t)
Utilization
α3


Utilization vs β1
U1(t)
U2(t)
U3(t)
Utilization
β1

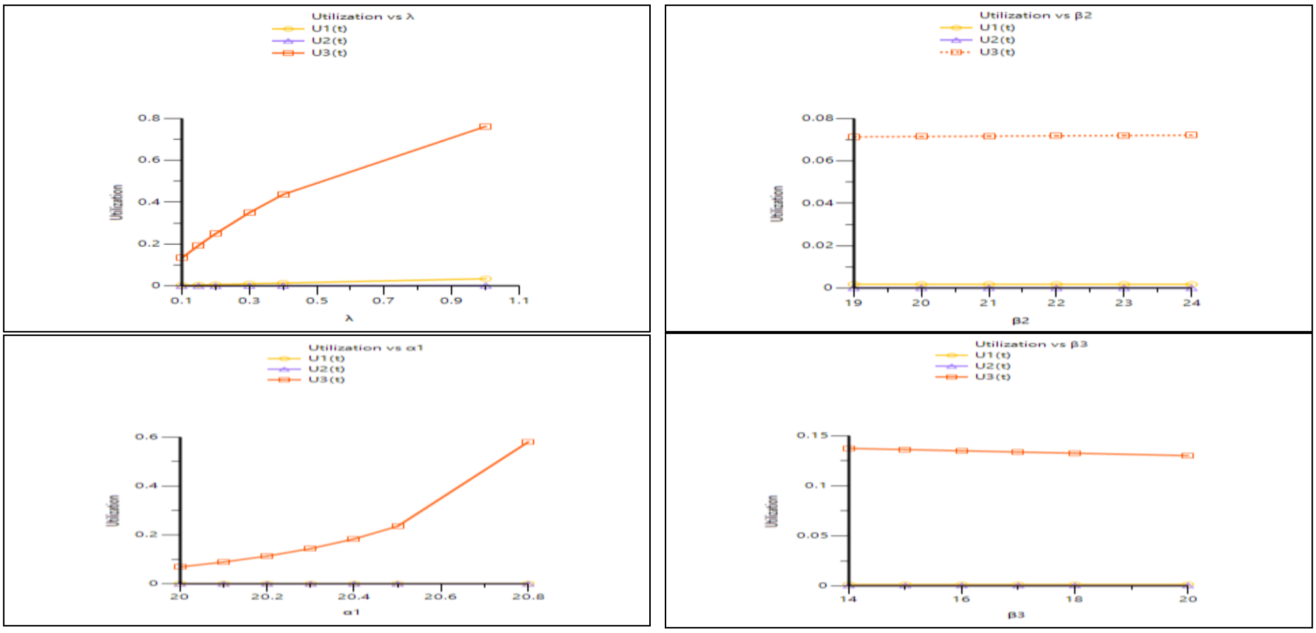


Fig. 2: The Relationship between the Parameters and Corresponding Performance Measures.

5. Sensitivity analysis

Sensitivity analysis of the three-station TQM is conducted with respect to variations in time (t), arrival rate (λ), and the time-dependent service rates of the 1st , 2nd and 3rd servers, denoted as μ1(t), μ2(t) and μ3(t). The analysis examines the effects of these parameters—both individually and collectively—on key performance metrics, including the average number of users in the 1st and 2nd queues, service station utilization, average delay in the 1st and 2nd queues, and the throughput at each service station.
To analyze the sensitivity of the three-station TQM, the average number of users in the 1st , 2nd , and 3rd queues, the utilization of all three service stations, the average delay in each of the three queues, and the throughput at each service station are computed. This analysis is conducted by varying the parameters t, λ, α_1, β_1, α_2, and β_2 by ±10%, ±5%, and 0% from their baseline values. The corresponding results are summarized in Table 4. The baseline parameter values considered for the sensitivity analysis are as follows: t=0.103, λ=0.05, α_1=22, β_1=10, α_2 =24.5, β_2=20, α3=24, and β_3=35.
This sensitivity analysis investigates how variations in different parameters impact the average number of users in the queues, utilization of the service stations, throughput, and average waiting times in a three-station TQM and is presented in Table 4.

5.1. Effect of time (t) (as time increases from -10% to +10%)

The average number of users in the 1st queue increases, while it decreases in the 2nd and 3rd queues.

- The utilization of the 1st service station increases, whereas it decreases for the 2nd and 3rd stations.
- Throughput increases at the 1st station but declines at the 2nd and 3rd.
- Waiting times decrease across all three queues, indicating improved service efficiency over time.

5.2. Effect of arrival rate (λ) (as the arrival rate increases)

- The average number of users increases across all three queues.
- Utilization rises for all three service stations, reflecting a higher workload.
- Throughput also increases across the board.
- Waiting times remain unchanged at the 1st two stations but increase at the 3rd, indicating a potential queue buildup at the final stage.

5.3. Effect of service rate parameter α1 (with increases in α1)

- The average number of users decreases in the 1st queue but increases in the 2nd and 3rd.
- The utilization of the 1st station decreases, while it increases at the 2nd and 3rd.
- Throughput increases at all stations.
- Waiting time decreases in the 1st queue and increases in the 2nd and 3rd.

5.4. Effect of service rate parameter β1 (with increases in β1)

- A decrease in the average number of users in the 1st and 3rd queues; no change in the 2nd.
- Utilization decreases at the 3rd station but remains stable at the 1st and 2nd.
- Throughput decreases for all three stations.
- Waiting times drop in the 1st and 3rd queues, with no change in the 2nd.

5.6. Effect of service rate parameter α2 (with increases in α2)

- No change in the average number of users in the 1st queue, but a decrease in the 2nd and 3rd.
- Utilization and throughput decrease at the 2nd and 3rd stations.
- Waiting times remain unchanged in the 1st queue but decrease in the 2nd and 3rd.

5.7. Effect of service rate parameter β2 (with increases in β2)

- The average number of users in the 1st queue remains the same, decreases in the 2nd, and increases in the 3rd.
- Utilization and throughput decrease at the 2nd station and increase at the 3rd.
- Waiting time in the 2nd queue decreases, while it increases in the 3rd.

5.8. Effect of service rate parameter α3 (with increases in α3)

- The average number of users, utilization, throughput, and waiting time all increase in the 3rd station, with no impact on the 1st and 2nd.

5.9. Effect of service rate parameter β3 (with increases in β3)

- The average number of users, utilization, and waiting time decrease in the 3rd station, while throughput increases.
- No significant effect is seen in the 1st and 2nd stations.

Table 4: The Values of $L_1(t)$, $L_2(t)$, $L_3(t)$, $U_1(t)$, $U_2(t)$, $U_3(t)$, $Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$, $W_1(t)$, $W_2(t)$ and $W_3(t)$ for Different Values of t, λ α_1, β_1, α_2, β_2, α_3, and β_3

Parameter	Selected Value	% change in Selected value	Performance measures L1(t)	L2(t)	L3(t)	U1(t)	U2(t)	U3(t)	Thp1(t)	Thp2(t)	Thp3(t)	W1(t)	W2(t)	W3(t)
		-10%	0.00165	0.00119	0.48024	0.00165	0.00119	0.38137	0.03785	0.03129	10.39010	0.04365	0.03797	0.04622
		-5%	0.00171	0.00101	0.43740	0.00171	0.00101	0.35429	0.03928	0.02664	9.71625	0.04356	0.03782	0.04502
t	0.103	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00181	0.00072	0.31795	0.00181	0.00072	0.27236	0.04169	0.01919	7.56752	0.04336	0.03752	0.04201
		10%	0.00185	0.00061	0.23876	0.00185	0.00061	0.21240	0.04271	0.01624	5.93978	0.04327	0.03737	0.04020
		-10%	0.00159	0.00077	0.34530	0.00159	0.00077	0.29199	0.03650	0.02038	8.06048	0.04346	0.03767	0.04284
		-5%	0.00167	0.00081	0.36449	0.00167	0.00081	0.30545	0.03853	0.02151	8.43183	0.04346	0.03767	0.04323
λ	0.05	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00185	0.00090	0.40285	0.00185	0.00089	0.33159	0.04258	0.02377	9.15352	0.04346	0.03767	0.04401
		10%	0.00194	0.00094	0.42204	0.00194	0.00094	0.34429	0.04461	0.02490	9.50410	0.04346	0.03767	0.04441
		-10%	0.00183	0.00012	0.08899	0.00183	0.00012	0.08514	0.03805	0.00316	2.35043	0.04805	0.03765	0.03786
		-5%	0.00180	0.00038	0.18094	0.00180	0.00038	0.16552	0.03937	0.01019	4.56905	0.04564	0.03766	0.03960
α1	22	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00173	0.00201	1.10500	0.00172	0.00201	0.66879	0.04161	0.05338	18.46192	0.04148	0.03769	0.05985
		10%	0.00170	0.00546	11.71325	0.00170	0.00545	0.99999	0.04230	0.14471	27.60477	0.04015	0.03775	0.42432
		-10%	0.00177	0.00085	0.38398	0.00176	0.00085	0.31886	0.04046	0.02264	8.80202	0.04366	0.03767	0.04362
		-5%	0.00176	0.00085	0.38383	0.00176	0.00085	0.31875	0.04051	0.02264	8.79909	0.04356	0.03767	0.04362
β1	10	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00176	0.00085	0.38351	0.00176	0.00085	0.31854	0.04060	0.02264	8.79316	0.04336	0.03767	0.04361
		10%	0.00176	0.00085	0.38335	0.00176	0.00085	0.31843	0.04065	0.02263	8.79016	0.04327	0.03767	0.04361
		-10%	0.00176	0.00118	2.53412	0.00176	0.00118	0.92067	0.04056	0.03091	25.41506	0.04346	0.03825	0.09971
		-5%	0.00176	0.00092	0.51874	0.00176	0.00092	0.40473	0.04056	0.02444	11.17252	0.04346	0.03781	0.04643
α2	24.5	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00176	0.00057	0.11095	0.00176	0.00057	0.10502	0.04056	0.01549	2.89902	0.04346	0.03697	0.03827
		10%	0.00176	0.00034	0.00074	0.00176	0.00034	0.00074	0.04056	0.00933	0.02031	0.04346	0.03611	0.03624
		-10%	0.00176	0.00086	0.38202	0.00176	0.00086	0.31752	0.04056	0.02275	8.76511	0.04346	0.03796	0.04358
		-5%	0.00176	0.00086	0.38284	0.00176	0.00086	0.31808	0.04056	0.02269	8.78062	0.04346	0.03781	0.04360
β2	20	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00176	0.00085	0.38449	0.00176	0.00085	0.31920	0.04056	0.02258	8.81165	0.04346	0.03752	0.04363
		10%	0.00176	0.00084	0.38532	0.00176	0.00084	0.31977	0.04056	0.02253	8.82716	0.04346	0.03738	0.04365
		-10%	0.00176	0.00085	2.63046	0.00176	0.00085	0.92795	0.04056	0.02264	23.85307	0.04346	0.03767	0.11028
		-5%	0.00176	0.00085	0.29471	0.00176	0.00085	0.25525	0.04056	0.02264	6.73991	0.04346	0.03767	0.04373
α3	24	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00176	0.00085	1.00458	0.00176	0.00085	0.63380	0.04056	0.02264	17.68620	0.04346	0.03767	0.05680
		10%	0.00176	0.00085	21.29595	0.00176	0.00085	1.00000	0.04056	0.02264	28.09500	0.04346	0.03767	0.75800
		-10%	0.00176	0.00085	0.39172	0.00176	0.00085	0.32411	0.04056	0.02264	8.83021	0.04346	0.03767	0.04436
		-5%	0.00176	0.00085	0.38768	0.00176	0.00085	0.32137	0.04056	0.02264	8.81359	0.04346	0.03767	0.04399
β3	35	0%	0.00176	0.00085	0.38367	0.00176	0.00085	0.31864	0.04056	0.02264	8.79614	0.04346	0.03767	0.04362
		5%	0.00176	0.00085	0.37968	0.00176	0.00085	0.31592	0.04056	0.02264	8.77784	0.04346	0.03767	0.04325
		10%	0.00176	0.00085	0.37571	0.00176	0.00085	0.31320	0.04056	0.02264	8.75870	0.04346	0.03767	0.04290

6. Comparative study

In this Segment, a comparative analysis is carried out between the proposed three-station TQM and a model assuming homogeneous Poisson service rates. The comparison is based on key Performance measures such as average number of users, utilization, throughput, and average waiting time at each service station.

To ensure a consistent evaluation, both models are assessed under varying values of the time parameter (t), specifically for t=0.100, 0.101, 0.102, 0.103, 0.104. The resulting performance metrics from both models are summarized and compared in Table 5.

This comparative study aims to highlight the impact of introducing time-dependent and non-homogeneous service rates in the proposed model, demonstrating its potential for more accurate and dynamic performance evaluation in real-world queueing systems.

At t = 0.100, the average number of users in all three queues is consistently lower in the proposed model compared to the traditional Poisson-based model. Specifically, Queue 1 shows a reduction of approximately 2.31%, Queue 2 by 14.06%, and Queue 3 by 18.01%. Similar trends are observed in utilization rates, particularly in the 3rd service station, where a drop of over 15% is evident. Throughput values in the 1st and 2nd stations also improved slightly, while a modest increase of approximately 0.71% is seen in the 3rd station.

As time progresses, these improvements become more pronounced. For example, at t = 0.102, the 3rd queue shows a congestion reduction of over 20.96%, and the utilization in the 3rd station decreases by around 17.73% when compared with the Poisson model. Waiting times across the three queues also consistently favor the proposed model, showing reductions of 4–18%, further underlining its efficiency.

At t = 0.103, the benefits of the non-homogeneous model become especially prominent. The throughput of the 3rd service station increases significantly by approximately 23.83%, indicating the model's adaptability to fluctuating service dynamics. Furthermore, the average waiting time in the 3rd queue is reduced by about 9.31%, enhancing overall system responsiveness.

By t = 0.104, the performance gap between the two models remains evident. The 3rd queue's congestion is reduced by nearly 24.7%, while throughput and utilization metrics continue to reflect the superiority of the proposed system.

These findings confirm that the non-homogeneous service process model provides a more realistic and efficient representation, particularly for short-term system dynamics. The conventional Poisson model, while it approaches equilibrium over time, struggles to capture transient performance fluctuations. In contrast, the proposed time-dependent framework remains sensitive to variations in system parameters, leading to more accurate predictions of Performance measures like queue length, utilization, and delay.

Table 5: Comparative Study of Models with Non-Homogeneous and Homogeneous Poisson Service Rates

$\lambda=0.05$, $\alpha_1=22$, $\alpha_2=25$, $\alpha_3=24.5$, $\beta_1=10$, $\beta_2=20$, $\beta_3=35$

t	Performance measures	Models with Non-Homogeneous arrival and service processes	Models with Homogeneous arrival and service processes	Difference	Percentage of Variation
0.100	L1(t)	0.00173	0.00177	0.00004	2.31214
	L2(t)	0.00064	0.00073	0.00009	14.06250
	L3(t)	0.29273	0.34545	0.05272	18.00977
	U1(t)	0.00173	0.00177	0.00004	2.31214
	U2(t)	0.00064	0.00073	0.00009	14.06250
	U3(t)	0.25378	0.29209	0.03831	15.09575
	Thp1(t)	0.03983	0.03889	0.00094	2.36003
	Thp2(t)	0.01721	0.01813	0.00092	5.34573
	Thp3(t)	7.10574	7.15633	0.05059	0.71196
	W1(t)	0.04352	0.04549	0.00197	4.52665
	W2(t)	0.03705	0.04001	0.00296	7.98920
	W3(t)	0.04120	0.04827	0.00707	17.16019
0.101	L1(t)	0.00174	0.00178	0.00004	2.29885
	L2(t)	0.00062	0.00070	0.00008	12.90323
	L3(t)	0.28237	0.33716	0.05479	19.40362
	U1(t)	0.00174	0.00178	0.00004	2.29885
	U2(t)	0.00062	0.00070	0.00008	12.90323
	U3(t)	0.24601	0.28620	0.04019	16.33673
	Thp1(t)	0.04008	0.03913	0.00095	2.37026
	Thp2(t)	0.01662	0.01754	0.00092	5.53550
	Thp3(t)	6.89677	7.01197	0.1152	1.67035
	W1(t)	0.04350	0.04550	0.002	4.59770
	W2(t)	0.03702	0.04001	0.00299	8.07672
	W3(t)	0.04094	0.04808	0.00714	17.44016
0.102	L1(t)	0.00175	0.00179	0.00004	2.28571
	L2(t)	0.00059	0.00068	0.00009	15.25424
	L3(t)	0.27158	0.32851	0.05693	20.96252
	U1(t)	0.00175	0.00179	0.00004	2.28571
	U2(t)	0.00059	0.00068	0.00009	15.25424
	U3(t)	0.23783	0.28000	0.04217	17.73115
	Thp1(t)	0.04032	0.03936	0.00096	2.38095
	Thp2(t)	0.01605	0.01698	0.00093	5.79439
	Thp3(t)	6.67580	6.86008	0.18428	2.76042
	W1(t)	0.04348	0.04550	0.00202	4.64581
	W2(t)	0.03699	0.04001	0.00302	8.16437
	W3(t)	0.04068	0.04789	0.00721	17.72370
0.103	L1(t)	0.00176	0.00180	0.00004	2.27273
	L2(t)	0.00085	0.00066	0.00019	22.35294
	L3(t)	0.38367	0.31949	0.06418	16.72792
	U1(t)	0.00176	0.00180	0.00004	2.27273
	U2(t)	0.00085	0.00066	0.00019	22.35294
	U3(t)	0.31864	0.27348	0.04516	14.17273
	Thp1(t)	0.04056	0.03959	0.00097	2.39152
	Thp2(t)	0.02264	0.01643	0.00621	27.42933
	Thp3(t)	8.79614	6.70035	2.09579	23.82625
	W1(t)	0.04346	0.04550	0.00204	4.69397

	W2(t)	0.03767	0.04001	0.00234	6.21184
	W3(t)	0.04362	0.04768	0.00406	9.30766
0.104	L1(t)	0.00177	0.00181	0.00004	2.25989
	L2(t)	0.00055	0.00064	0.00009	16.36364
	L3(t)	0.24868	0.31011	0.06143	24.70243
	U1(t)	0.00177	0.00181	0.00004	2.25989
	U2(t)	0.00055	0.00064	0.00009	16.36364
	U3(t)	0.22017	0.26663	0.04646	21.10188
	Thp1(t)	0.04079	0.03982	0.00097	2.37803
	Thp2(t)	0.01496	0.01590	0.00094	6.28342
	Thp3(t)	6.19550	6.53246	0.33696	5.43879
	W1(t)	0.04344	0.04550	0.00206	4.74217
	W2(t)	0.03694	0.04001	0.00307	8.31077
	W3(t)	0.04014	0.04747	0.00733	18.26109

7. Conclusion

This study presents the design and analysis of a three-node TQM incorporating time- and state-dependent service rates. The model assumes that both arrival and service processes follow non-homogeneous Poisson distributions, with service rates influenced by time and the number of users in each queue. This approach captures the dynamic nature of real-world systems, such as hospital operations, manufacturing systems, cloud computing, and airport security, where service capabilities fluctuate over time and depend on system load.

Explicit expressions for key Performance measures—including the average number of users in each queue, average waiting times, service station utilizations, and throughput—have been derived. Sensitivity analyses reveal that variations in parameters like time (t) and arrival rate (λ) significantly impact system performance. Notably, increases in time lead to a higher average number of users in the 1st queue and reductions in the 2nd and 3rd queues, along with decreased utilization and throughput in the latter stages. Conversely, an increase in arrival rate results in a higher average number of users, increased delays, and elevated utilization and throughput across all service stations.

Comparative evaluations indicate that the proposed model outperforms traditional models with homogeneous Poisson service processes, especially for smaller values of t. The non-homogeneous approach more accurately reflects the transient behaviors of systems, providing better predictions of Performance measures and enabling more effective congestion management.

In summary, the three-node TQM with time- and state-dependent service rates offers a robust framework for analyzing and optimizing complex systems where service dynamics are influenced by temporal factors and system state. This model serves as a valuable tool for designing efficient service systems and enhancing overall performance.

Limitations

- Computational complexity when scaling to more nodes or incorporating additional stochastic behaviors.
- The assumption of infinite queue capacity and its implications.
- Challenges in real-world NHPP parameter estimation, including data availability and model calibration.

Future research endeavors will focus on extending the analysis to more complex systems. Specifically, we aim to develop and analyze:

Three-Node Tandem Queueing Model with Phase-Type State and Time-Dependent Service Rates: This model will incorporate phase-type service processes where service rates are influenced by both the system's state and time-varying factors. Such an approach allows for a more nuanced representation of service dynamics in communication networks.

Three-Node Tandem Queueing Model with Direct Arrivals and Phase-Type State and Time-Dependent Service Rates: Extending the previous model, this framework will consider direct arrivals to the system, where incoming packets bypass intermediate nodes, and service rates remain state and time-dependent. This scenario is particularly relevant for analyzing networks with dynamic routing and varying traffic patterns.

These models aim to provide deeper insights into the performance and optimization of communication networks, especially in environments characterized by fluctuating traffic loads and service capabilities.

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Abbreviations & notations

Notation	Description
FCFS	First-Come, First-Served
HSR	Homogeneous Service Rate
LANs	Local Arrival Networks
MANs	Metropolitan Arrival Networks
WANs	Wide Arrival Networks
NHPP	Non-Homogeneous Poisson Process
NHSP	Non-homogeneous service processes
HPA	Homogeneous Poisson arrival
TQM	Tandem queueing model
NHSR	Non-Homogeneous Service Rate
PGF	Probability Generating Function
t	Time
n_1	Customers in the first queue
n_2	Customers in the second queue

n_3	Customers in the third queue
λ	Arrival rate parameter
α_1	Service Rate Parameter
α_2	Service Rate Parameter
α_3	Service Rate Parameter
β_1	Time dependent service rate parameter
β_2	Time dependent service rate parameter
β_3	Time dependent service rate parameter
$\mu_1(t)$	The service rates at the first service station
$\mu_2(t)$	The service rates at the second service station
$\mu_3(t)$	The service rates at the third service station
$P_{000}(t)$	probability of the queue being empty
$P_{0..}(t)$	initial queue is empty of elements
$P_{.0.}(t)$	Second queue is empty of elements
$P_{..0}(t)$	Third queue is empty of elements
$L_1(t)$	The average number of customers in the initial queue
$L_2(t)$	The average number of customers in the second queue
$L_3(t)$	The average number of customers in the third queue
$U_1(t)$	The Utilization in the initial service station
$U_2(t)$	The Utilization in the second service station
$U_3(t)$	The Utilization in the third service station
$Thp_1(t)$	The throughput in the initial service station
$Thp_2(t)$	The throughput in the second service station
$Thp_3(t)$	The throughput in the third service station
$W_1(t)$	The average waiting time of customers in the initial queue
$W_2(t)$	The average waiting time of customers in the second queue
$W_3(t)$	The average waiting time of customers in the third queue
$V_1(t)$	The variance of the number of customers in the initial queue
$V_2(t)$	The variance of the number of customers in the second queue
$V_3(t)$	The variance of the number of customers in the third queue
Var.	Variance
CV	Coefficient Of Variation
$CV_1(t)$	The Coefficient Of Variation of the number of customers in the initial queue
$CV_2(t)$	The Coefficient Of Variation of the number of customers in the second queue
$CV_3(t)$	The Coefficient Of Variation of the number of customers in the third queue

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