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Research paper

Complete Extension and Join of Two Graphs

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Abstract

In 2016 [5], Suresh Singh G and Sunitha Grace Zacharia introduced the concept of graph extension by adding edges to G in a particular way. In this paper, we try to find out some classes of graphs whose joint is completely extendable.

***Keywords**: Graph extension, completely extendable graphs, path Graph, Join of graphs.*

1. Introduction

Field extensions involve the fundamental concept of beginning with a base field and then, through various techniques, expanding it to create a larger field that includes the base field and fulfills additional properties. Various methods of extending fields through graphs have been investigated by different researchers. Wesley Pegden and Alan Frieze [11] introduce the preferential attachment graph of G as a sequence of random graphs $G_1, G_2, \dots, G_n$ where G_{t+1} is obtained from G, by adding a vertex $t+1$ and m random edges incident on a vertex $t+1$. Suresh Singh G and Sunitha Grace Zacharia [5] introduced the concept of graph extension by adding edges to G in a particular way and proved some characterizations

Definition 1.1. [5] Let G be a simple (p,q) graph. Extension on G is defined as follows: in the first extension, add one edge to G, denoted as G_1, $G_1 = G \cup \{e_1\}$. In the second extension, add two edges with G_1 denoted by G_2, $G_2 = G \cup \{e_1, e_2, e_3\}$ and so on until no such extension remains

Definition 1.2. [5] If $G^n \cong K_p$, for some n, then G is said to be a completely extendable graph, and n is known as the order of extension.

In the study of network growth modeling, the evolution of a system can be examined by analyzing how new connections are incrementally established over time. This dynamic process is effectively captured by the concept of graph extension, wherein a network, mathematically represented as a graph-begins from an initial state defined by a specific set of nodes and edges and progressively incorporates new edges. A pertinent example is the structure of the Internet, composed of autonomous systems (AS) that interconnect to form a sparse graph. Over the years, the emergence of new Internet Service Providers (ISPs) has led to the formation of additional interconnection agreements with existing AS, thereby generating new links and expanding the graph. This growth can be modeled by sequentially adding one or more edges during discrete time intervals, reflecting real-world network development. Similarly, mobile communication networks evolve not as pre-completed structures, but through the gradual deployment of base stations and communication links in response to increased demand or technological advancement. Graph extension models are particularly well-suited to this scenario, as they conceptualize each new base station or link as an additional edge in the existing graph. Power grids offer another illustrative example, where transmission lines connecting power plants, substations, and end users are added progressively over time. Through the lens of graph extension, designers and engineers can better understand how incremental construction influences the robustness and efficiency of the overall network. [3, 8, 9, 12].
Infrastructure development in urban planning and public utilities is a classic example where graph extension plays a critical role in modeling and managing growth. Cities often begin with a basic transportation network, such as roads or railways connecting key locations. This network can be represented as a graph, where nodes correspond to intersections or stations, and edges represent roads or rail lines [3]. As the city grows and demand increases, the network expands through the addition of new connections, such as roads, bridges, or rail links. Each new connection corresponds to the addition of an edge in the existing graph.
For instance, a city might start with a simple road network linking the downtown area to nearby residential neighborhoods. As new suburbs develop, planners add roads that connect these areas to the city center and one another. This gradual growth can be modeled by progressively extending the graph initially adding one or two edges, followed by more extensive additions over time. Graph extension models help planners evaluate how each new connection improves overall accessibility, reduces travel time, and eases traffic congestion [1]. A similar

approach applies to electrical grid development. Over time, new transmission lines are built to connect power plants, industrial areas, and residential zones. Each new line strengthens the capacity and reliability of the grid. By using graph extension techniques, engineers can simulate this step-by-step growth, plan the optimal placement and timing of new infrastructure, and assess the system's performance at every stage. This enables more sustainable, resilient, and cost-effective development in urban environments [9]. In [5], the authors characterize the extension as follows.

Theorem 1.3. *[5] Let G be a (p,q) graph, and let $G^k = G \cup \{e_1, e_2, \ldots, e_m\}$. If G^k is the extension of G, then*

$$m = \frac{k(k+1)}{2}$$

Definition 1.4. *[5] Let G be a (p,q) graph that is not completely extendable. Let r be the maximum possible number of extensions in G such that G^r is not complete. Then the deficiency number of G is defined as the number of edges required to make G^r as a complete graph.*

Definition 1.5. *[4, 6] Let $G1$ and $G2$ be two disjoint graphs. Then the join of $G1$ and $G2$ is denoted by $G1 \vee G2$ and has a vertex set $V(G1) \cup V(G2)$ and edge set $E(G1) \cup E(G2) \cup \{xy \mid x \in V(G1), y \in V(G2)\}$.*

The joint of the two graphs is a fundamental operation in graph theory with wide-ranging applications across computer science, social sciences, and systems engineering. This operation constructs a new graph by taking two disjoint graphs and adding edges between every vertex of the first graph and every vertex of the second. Conceptually, the join models complete the interaction between two distinct systems. For instance, in network design, the joining of two separate computer networks yields a unified structure where every node is interconnected, which is essential during corporate acquisitions or integration processes in cloud computing environments [2]. In social network analysis, the join operation facilitates the modeling of full collaboration between two distinct groups [10], such as newly formed research teams or merging communities where everyone from one group interacts with all members of the other. In the context of distributed and parallel computing, the join enables the association of tasks from one graph with processors from another, representing scenarios where any task can be executed on any processor, and it is an important framework for optimal task scheduling [7]. Additionally, in interdisciplinary research mapping, graph joins are used to represent collaborative networks between distinct academic domains, allowing for analysis of the scope and depth of interdisciplinary engagement. These diverse applications underscore the theoretical and practical significance of the graph join operation, particularly in systems where full bipartite interaction is a critical feature. For basic definitions and results, we refer to [4, 6].

2. Main Results

The graph extension is becoming a critical tool in modeling scalable and adaptable systems, offering practical benefits in both theory and application. In this section, we find out some classes of graphs whose join is completely extendable.

Theorem 2.1. The graph $Pn \vee P_2$ is completely extendable. And the order of extension is $n-2$.
Proof. Let the vertex set of Pn is $\{v_1, v_2, \ldots, v_n\}$` and the vertex set of $P2$ is $\{u_1, u_2\}$. Then $Pn \vee P2$ has a vertex set $\{v_1, v_2, \ldots, v_n, u_1, u_2\}$. If
$P_n \vee P_2$ is completely extendable, then $P_n \vee P_2 \cong K_{n+2}$. In $P_n \vee P_2$, the vertex ui has degree $n+1$.
Number of edges needed to $P_n \vee P_2$to for K_{n+2}

$$= \frac{(n+1)(n+2)}{2} - (n + 2n)$$
$$= \frac{n^2 + 3n + 2 - 6n}{2}$$
$$= \frac{n^2 - 3n + 2}{2}$$
$$= \frac{(n-1)(n-2)}{2}$$

Hence $P_n \vee P_2$ is completely extendable, and order of extension is $n - 2$.

Theorem 2.1. can be generalized as follows

Theorem 2.2. *Let G be a completely extendable graph. Then $G \vee P_2$ is completely extendable and the order of extension of $G \vee P_2$ is the order of the extension of G.*
Proof. Let G be a (p,q) graph. Suppose order of extension of G is k. The theorem 1.3, $q = \frac{p(p-1)}{2} - \frac{k(k+1)}{2}$ for some k. We have $G \vee P_2$ has $P+2$ vertices.

The number of edges of $G \vee P_2 = \frac{p(p-1)}{2} - \frac{k(k+1)}{2} + 1 + 2p$

$$= \frac{p^2 - p + 4p + 2}{2} - \frac{k(k+1)}{2}$$
$$= \frac{p^2 + 3p + 2}{2} - \frac{k(k+1)}{2}$$
$$= \frac{(p+1)(p+2)}{2} - \frac{k(k+1)}{2}$$

Hence $G \vee P_2$ is completely extendable and the order of extension is 2.

Theorem 2.3. *Let G be a completely extendable graph and K_n be the complete graph. Then $G \vee K_n$ is completely extendable and the order of extension of $G \vee K_n$ is same as the order of extension of G.*

Proof. Let G be a (p, q) graph. Since G is a completely extendable Graph, so, $q = \frac{p(p-1)}{2} - \frac{k(k+1)}{2}$ for some k.

The number of edges of $G \vee Kn = \frac{p(p-1)}{2} + \frac{n(n-1)}{2} + np - \frac{k(k+1)}{2}$

$$= \frac{p(p-1)}{2} + \frac{n(n-1)}{2} + np - \frac{k(k+1)}{2}$$

$$= \frac{p^2 - p + n^2 - n + 2np}{2} - \frac{k(k+1)}{2}$$

$$= \frac{n^2 + 2np + p^2 - (n+p)}{2} - \frac{k(k+1)}{2}$$

$$= \frac{(n+p)^2 - (n+p)}{2} - \frac{k(k+1)}{2}$$

$$= \frac{(n+p)(n+p-1)}{2} - \frac{k(k+1)}{2}$$

Hence by Theorem 1.3, $G \vee Kn$ is completely extendable and the order of extension of $G \vee Kn$ is *k*.

Consider the following graphs C4 and G

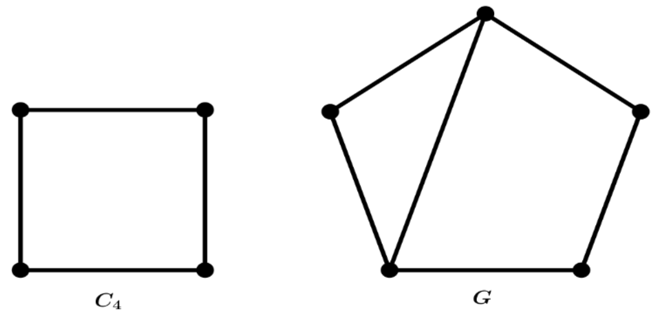

Fig. 1: C_4 and G

Since C_4 contains 4 vertices and 4, which is not completely extendable and the deficiency number is 1. If we add one edge to G we get $G^{(1)}$, if we add two edges to G^1 we get G^2. G^2 is complete if we add one more edge. Hence, G is not completely extendable and the deficiency number is 1. If we consider $C_4 \vee G$. It has 9 vertices and $4+6+4\times5=30$ edges.

Number of edges required for $C4 \vee G$ to form K_9

$$= \frac{9 \times 8}{2} - 30$$

$=36-30$

$=6$

Hence, $C_4 \vee G$ is completely extendable.

In general join of two completely extendable graphs need not be completely extendable. For that consider the completely extendable graph G with p vertices and q edges. Then we have,

$$q = \frac{p(p-1)}{2} - \frac{k(k+1)}{2}$$

for some k.

Then the graph $G \vee G$ has $2p$ vertices and $2q+p^2$ edges.

$2q + p = p(p-1) - k(k+1) + p^2$

$= p(2p-1) - k(k+1)$

$$= \frac{2p(2p-1)}{2} - k(k+1)$$

Hence, G is not completely extendable for $k \neq 2$. If $k=2$, then $G \vee G$ is completely extendable. This can be concluded as follows.

Theorem 2.4. *Let G be a (p, q) graph. If the order of extension of G is k, then $G \vee G$ is completely extendable if and only if k* = 2

Theorem 2.5. *Let G be a completely extendable (p, q_1) graph of order k, and H be a (p, q_2) graph with deficiency number k^2 after the k^{th} extension. Then $G \vee H$ is completely extendable.*

Proof. From Theorem 1.3,

$$q_1 = \frac{p(p-1)}{2} - \frac{k(k+1)}{2}$$

$$q_2 = \frac{p(p-1)}{2} - \frac{k(k+1)}{2} - k^2$$

Number of edges of ($G \vee H$) = $p(p-1) - k(k+1) - k^2 + p^2$

$$= \frac{(2p(2p-1))}{2} - \frac{2k(2k+1)}{2}$$

Hence, $G \vee H$ is completely extendable.

3. Conclusion

Recent developments in graph extension have significantly advanced both the theoretical framework and practical utility of this concept across dynamic and evolving systems. Traditionally defined as the process of incrementally adding edges or vertices to a graph while maintaining specific structural properties, graph extension is now widely applied in areas such as communication networks, transport infrastructure, and biological systems. In this paper, we study the graph extension of the join of two graphs. Also, we get that the graph $p_n \vee P_2$ is completely extendable. If G is a completely extendable graph, then $G \vee P_2$ and $G \vee K_n$ are completely extendable, and the order of extension is the same as that of G. Finally, if the order of extension of G is k, then $G \vee G$ is completely extendable if and only if $k = 2$
The graph extension is a novel area of research and helps to model real-world problems. The characterization of graph extension is done in terms of the number of edges. The characterization of extension in terms of the adjacency matrix is an open problem. Also, this study can be extended to the corona product as well as the box product of two graphs.

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