International Journal of Advanced Statistics and Probability, 5 (1) (2017) 17-22

International Journal of Advanced Statistics and Probability

Website: www.sciencepubco.com/index.php/IJASP
doi: 10.14419/ijasp.v5i1.7410
Research paper

Mathematical properties of the Kumaraswamy-Lindley distribution and its applications

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Abstract

In this paper, a composite distribution of Kumaraswamy and Lindley distributions namely, Kumaraswamy-Lindley Kum-L distribution is introduced and studied. The Kum-L distribution generalizes sub-models for some widely known distributions. Some mathematical properties of the Kum-L such as hazard function, quantile function, moments, moment generating function and order statistics are obtained. Estimation of parameters for the Kum-L using maximum likelihood estimation and least square estimation techniques are provided. To illustrate the usefulness of the proposed distribution, simulation study and real data example are used.

Keywords: *Kumaraswamy Distribution; Lindley Distribution; Maximum Likelihood Estimation; Hazard Function; Order Statistics.*

1. Introduction

In recent years, there have been different approaches used to generate new family of distributions. These approaches include exponentiated, generalized, mixed and compounded distributions. They aim to make the base distribution more flexible and more convenient to analyze complex real data sets. Furthermore, the new distribution generalizes sub-models for some widely known distributions.

In literature studies, the Kumaraswamy distribution has not been common used and not seem to be very familiar to statisticians. However, background and genesis of Kumaraswamy distribution have been provided by Jones (2009) [9],and he made a comparison between the two-parameter family of distributions on (0, 1). Jones showed that thisdistribution has its genesis in terms of uniform order statistics, and has particularly straightforward distribution and quantile functions, which do not depend on special functions (and hence afford very easy random variety generation).

The commutative distribution function CDF and probability density function pdf for generalized Kumaraswamy distribution have been proposed by Cordeiro et al. (2011) [5] respectively as follows:

$$F(x)=1-\left[1-\left(G(x)\right)^{\alpha}\right]^{\beta} \tag{1.1}$$

And

$$f(x)=\alpha\,\beta\,g(x)\left(G(x)\right)^{\alpha-1}\left[1-\left(G(x)\right)^{\alpha}\right]^{\beta-1} \tag{1.2}$$

Lindley (1958) [11] obtainedexponential distribution with scale parameter θ mixture of gamma distribution with shape and scale parameters respectably $(2,\theta)$ (2, θ) named, Lindley distribution. The CDF and pdf for Lindley distribution respectively are:

$$G(x)=1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.3}$$

And

$$g(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.4}$$

Some properties of Lindley distribution have been discussed by Ghitany et al. (2008) [8]. They showed that, in some applications, the pdf which in (1.4) provides a goodness of fit model than the exponential distribution. Lindley distribution has been transmuted via the quadratic rank transmutation map, weighted, exponentiated and modified by many authors such Nadarajah et al. (2011) [13], Bakouch et al.Ghitany et al. (2011) [7]. Merovci and Elbatal (2014) [12], Ghitany et al. (2013) [6], Sharma et al. (2015) [15] and among others.

Now, substituting (1.3) in (1.1) and (1.4) in (1.2)then, the CDF and pdf of the Kum-L distribution respectively are given by

$$F(x)=1-\left[1-\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{\alpha}\right]^{\beta} \quad ;x>0,\ \theta>0 \tag{1.5}$$

And

$$f(x)=\alpha\beta\left(\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\right)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{\alpha-1} \times\left[1-\left[1-\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{\alpha}\right]^{\beta-1}\right] \quad ;x>0,\ \theta>0 \tag{1.6}$$

Using the expansion $(1\text{-}z)^{\beta} = \sum_{j=0}^{\infty}(\text{-}1)^{j}\binom{\beta\text{-}1}{j}z^{j}$, then, (1.5) and (1.6) can be written respectively as:

$$F(x) = 1 - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta}{j}\binom{\alpha j}{k} \times \left[\frac{\theta+1+\theta x}{\theta+1}\right]^{k}(1+x)e^{-\theta k x} \quad (1.7)$$

And

$$f(x) = \alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k} \times \binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k} \times \left[\frac{\theta+1+\theta x}{\theta+1}\right]^{k}(1+x)e^{-\theta(k+1)x} \quad (1.8)$$

The survival function $R(x)$ and hazard rate function $h(x)$ respectively are:

$$R(x) = \left[1-\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right\}^{\alpha}\right]^{\beta}$$

$; x > 0, \alpha, \beta, \theta > 0$ (1.9)

And

$$h(x) = \frac{f(x)}{R(x)} = \alpha\beta\left(\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\right) \times \frac{\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right\}^{\alpha-1}}{1-\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right\}^{\alpha}} \quad (1.10)$$

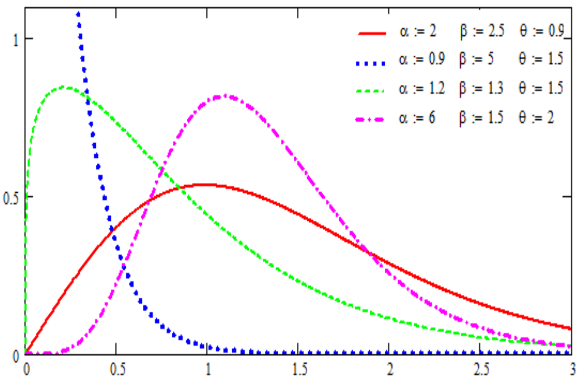

Fig. 1.1: Different Forms for Pdf of The Kum-L Distribution with Various Values of the Parameters (α, β, θ) Respectively.

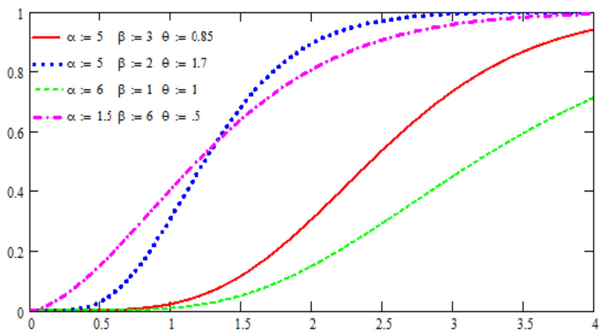

Fig. 1.2: Different Forms for CDF of the Kum-L Distribution with Various Values of the Parameters (α, β, θ) Respectively.

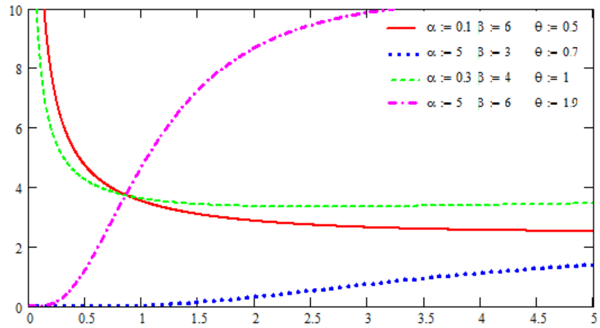

Fig. 1.3: Different Forms for H(X) of the Kum-L Distribution with Various Values of the Parameters (α, β, θ) Respectively.

Figures (1.1) shows that the carve of the pdf for the Kum-L distribution sometimes is positive skewed, decreasing and approximately symmetric respectively for different values of the parameters (α, β, θ).

The remainder of this paper is divided into four sections. Section 2 concerns with obtain some statistical properties of the Kum-L distribution such as hazard function, quintile function, moments, moment generating function and order statistics. Insection 3, estimation methods of parameters for the Kum-L distribution, using maximum likelihood estimation and least square estimation techniques are provided. Section 4 concerns with application of numerical study and real data set to illustratetheoretical results in previous sections. Finally, Section 5 presents our conclusions.

2. Statistical properties

2.1. Quantile function

The quantile function $Q(x)$ of the Kum-L distribution can be computed by inverting the CDF as follows:

$$\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x} = 1 - \sqrt[\alpha]{\sqrt[\beta]{(1-u)}} \quad (2.1.1)$$

$$, 0 < u < 1.$$

Numerically solution can be used to solve (2.1.1) in x.

The quantiles of the Kum-L distribution can be obtained by setting $u = (0.25, 0.50, 0.75)$ in the previous equation. Also, based on (2.1.1), the skewness Sk and kurtosis K for the Kum-L distribution respectively are

$$K = \frac{Q(7/8)-2\,Q\left(\frac{5}{8}\right)}{Q(6/8)-Q(2/8)} + \frac{Q(3/8)-Q(3/8)}{Q(6/8)-Q(2/8)} \quad (2.1.2)$$

2.2. Moments

Theorem 1: *The r^{th} non-central moments of the Kum-L random variable X, are given by*

$$\acute{\mu}_r = E(X^r) = \alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)\binom{\beta-1}{j} \times \binom{\alpha j+\alpha-1}{k}\binom{k}{i}\left(\frac{\theta}{\theta+1}\right)^{i}\frac{\Gamma(r+i+1)}{[\theta(k+1)]^{(r+i+1)}} \times \frac{\Gamma(r+i+2)}{[\theta(k+1)]^{(r+i+2)}} \quad (2.2.1)$$

Proof

$$\acute{\mu}_r = E(X^r) = \int_0^{\infty} x^r f(x)dx\acute{\mu}_r = E(X^r) = \int_0^{\infty} x^r f(x)dx$$

$$= \alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}$$

$$\times\binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k}$$

$$\times\int_0^\infty x^r\left[\frac{\theta+1+\theta x}{\theta+1}\right]^k(1+x)e^{-\theta(k+1)x}dx \quad (2.2.2)$$

The expansion $\left[\frac{\theta+1+\theta x}{\theta+1}\right]^k$ can be written as $\left[1+\frac{\theta x}{\theta+1}\right]^k$. Now, by using the binomial expansion

$$\left(1+\frac{\theta x}{\theta+1}\right)^k=\sum_{i=0}^k\binom{k}{i}\left(\frac{\theta}{\theta+1}\right)^i x^i\ ,|\theta|>1\ .$$

We have

$$\acute{\mu}_r=\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^{j+k}\left(\frac{\theta}{\theta+1}\right)^I$$

$$\times\binom{k}{i}\binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k}\times\int_0^\infty x^{r+i}(1+x)e^{-\theta(k+1)x}dx \quad (2.2.3)$$

And,

$$\int_0^\infty x^{r+i}(1+x)e^{-\theta(k+1)x}dx=\int_0^\infty x^{r+i}e^{-\theta(k+1)x}dx$$

$$+\int_0^\infty x^{r+i+1}(1+\ x)e^{-\theta(k+1)x}dx$$

Let $y=\theta(k+1)x$, then, $x=\frac{y}{\theta(k+1)}$ and $dx=\frac{dy}{\theta(k+1)}$.

So that, the value of the first and two integrations respectively are: $\frac{\Gamma(r+i+1)}{[\theta(k+1)]^{(r+i+1)}}\ and\ \frac{\Gamma(r+i+2)}{[\theta(k+1)]^{(r+i+2)}}$, which complete the proof.
The mean and the variance of the Kum-L distribution are given respectively by

$$\mu=E(x)\ =\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)S_{ijk}\frac{\Gamma(i+2)}{[\theta(k+1)]^{(r+2)}}$$

$$\times\frac{\Gamma(i+3)}{[\theta(k+1)]^{(r+3)}} \quad (2.2.4)$$

And

$$V(x)\ =S_{ijk}\frac{\Gamma(i+3)}{[\theta(k+1)]^{(r+3)}}\times\frac{\Gamma(i+4)}{[\theta(k+1)]^{(r+4)}}$$

$$-\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\left[S_{ijk}\frac{\Gamma(i+2)}{[\theta(k+1)]^{(r+2)}}\times\frac{\Gamma(i+3)}{[\theta(k+1)]^{(r+3)}}\right]^2 \quad (2.2.5)$$

Where,

$$S_{ijk}=\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{i=0}^k(-1)^{j+k}\binom{k}{i}\binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k}\left(\frac{\theta}{\theta+1}\right)^i.$$

Using the relation between the central and non-central moments which is

$$\mu_r=E\big(x-E(x)\big)^r=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\acute{\mu}_i\mu^{r-i}$$

Then, the central moments μ_r are given.

2.3. Moment generating function

$$M_x(t)=\int_0^\infty e^{tx}f(x)dx=\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)S_{ijk}$$

$$\times\left[\int_0^\infty x^ie^{-x[\theta(k+1)-t]}dx+\int_0^\infty x^ie^{-x[\theta(k+1)-t]}dx\right] \quad (2.3.1)$$

$$=\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)S_{ijk}\frac{\Gamma(i+1)}{[\theta(k+1)-t]^{(i+1)}}\times\frac{\Gamma(i+2)}{[\theta(k+1)-t]^{(i+2)}} \quad (2.3.2)$$

Where,

$$S_{ijk}=\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{i=0}^k(-1)^{j+k}\binom{k}{i}\binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k}\left(\frac{\theta}{\theta+1}\right)^i.$$

2.4. Order statistics

Let $X_1,X_2,\cdots,X_n$ the observed values of a sample of size n. Let $X_{(1)},X_{(2)},\cdots,X_{(n)}$ denote the ordered values of $X_1,X_2,\cdots,X_n$. Then, $X_{(1)},X_{(2)},\cdots,X_{(n)}$ are called the order statistics. The pdf of the order statistics $X_{(r)}$ is given by

$$f_X(x)=\frac{n!}{(r-1)(n-r)}$$

$$\times[F(x)]^r[R(x)]^{n-r}f(x); \quad (2.4.1)$$

Where

$r=1,2,\cdots,n.$

Now, using Equations (1.5), (1.6) and (1.10) in (2.4.1), we have

$$f_X(x)=\frac{n!}{(k-1)(n-k)}$$

$$\left[1-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta}{j}\binom{\alpha j}{k}\left[\frac{\theta+1+\theta x}{\theta+1}\right]^r(1+x)e^{-\theta kx}\right]^k$$

$$\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta}{j}\binom{\alpha j}{k}\left[\frac{\theta+1+\theta x}{\theta+1}\right]^k(1+x)e^{-\theta kx}\right]^{n-r}$$

$$\times\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\sum_{j=0}^\infty\sum_{k=0}^\infty(-1)^{j+k}\binom{\beta-1}{j}\binom{\alpha j+\alpha-1}{k}$$

$$\times\left[\frac{\theta+1+\theta x}{\theta+1}\right]^k(1+x)e^{-\theta(k+1)x};\ x\in R \quad (2.4.2)$$

3. Parameter estimation methods

3.1. Maximum likelihood estimation

The likelihood and log likelihood functions of the Kum-L distribution based on samples $X_1,X_2,\cdots,X_n$ are given respectively by

$$L(X;\ \alpha,\beta,\theta)=(\alpha\beta)^n\left(\frac{\theta^2}{\theta+1}\right)^n\left(1+\sum_{i=1}^n x_i\right)$$

$$\times e^{-\theta\sum_{i=1}^n x_i}\prod_{i=1}^n\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right\}^{\alpha-1} \quad (3.1.1)$$

$$\times\prod_{i=1}^n\left[1-\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right\}^{\alpha}\right]^{\beta-1}$$

And,

$$\mathcal{L}(X;\ \alpha,\beta,\theta) = n ln\,\alpha + n ln\,\beta + 2n ln\,\theta - n ln(\theta+1)$$

$$+\sum_{i=1}^{n} ln(1+x_i) - \theta\sum_{i=1}^{n} x_i$$

$$+\ (\alpha-1)\sum_{i=1}^{n} ln\left[1-\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\right]$$

$$+(\beta-1)\sum_{i=1}^{n} ln\left[1 - \left\{1-\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\right\}^{\alpha}\right] \quad (3.1.2)$$

The first derivative of $\mathcal{L}(X;\ \alpha,\beta,\theta)$ with respect to α,β and θ respectively are

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left(1-\ ce^{-\theta x_i}\right)\ -(\beta-1)$$

$$\times\sum_{i=1}^{n}\frac{\left[1-\ ce^{-\theta x_i}\right]^{\alpha}}{1-\left\{1-\ ce^{-\theta x_i}\right\}^{\alpha}}\times ln\left(1-\ ce^{-\theta x_i}\right) \quad (3.1.3)$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left[1-\left\{1-\ ce^{-\theta x_i}\right\}^{\alpha}\right] \quad (3.1.4)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \sum_{i=1}^{n} x_i$$

$$+(\alpha-1)\sum_{i=1}^{n} x_i e^{-\theta x_i}\frac{A\theta}{1-\ ce^{-\theta x_i}}$$

$$+\alpha(\beta-1)\sum_{i=1}^{n}\left[\frac{\left(1-\ ce^{-\theta x_i}\right)^{\alpha-1}}{1-\left(1-\ ce^{-\theta x_i}\right)^{\alpha}}\right]\times x_i e^{-\theta x_i}A\theta \quad (3.1.5)$$

Where $A = \frac{\theta+2+\theta x_i+x_i}{(\theta+1)^2}$ and $c = \frac{\theta+1+\theta x_i}{\theta+1}$.

Solving these above non-linear equations after setting them to zero, then, the likelihood estimators $\hat{\alpha},\hat{\beta}$ and $\hat{\theta}$ for the parameters α,β and θ respectively can be given. So, one of the numerical analysis methods such Newton–Raphson can be used to solve this system, (see, Salem and selim [12]).
Now, to construct the asymptotic confidence intervals, the approximate variance-covariance matrix $I_{ij}(\varphi)$ can be obtained as follows

$$I_{ij}(\varphi) = \begin{bmatrix} \frac{\partial^2\mathcal{L}}{\partial\alpha^2} & \frac{\partial\mathcal{L}}{\partial\alpha\partial\beta} & \frac{\partial\mathcal{L}}{\partial\alpha\partial\theta} \\ & \frac{\partial^2\mathcal{L}}{\partial\beta^2} & \frac{\partial\mathcal{L}}{\partial\beta\partial\theta} \\ & & \frac{\partial^2\mathcal{L}}{\partial\theta^2} \end{bmatrix} \quad (3.1.6)$$

Where the elements of the main diagonal in (3.1.6) are the variances of the parameters (α,β,θ) and the off-diagonal elements are the covariance between the parameters. The second derivative of $\mathcal{L}(X;\ \alpha,\beta,\theta)$ with respect to α,β and θ respectively are given by

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2} = -\frac{n}{\alpha^2}\ -(\beta-1)$$

$$\times\sum_{i=1}^{n}\left\{\left[1-\ ce^{-\theta x_i}\right]^{\alpha}\times ln\left[1-\ ce^{-\theta x_i}\right]\right\} \quad (3.1.7)$$

$$\times\frac{1-2\left(1-\ ce^{-\theta x_i}\right)^{\alpha-1}}{1-\left(1-\ ce^{-\theta x_i}\right)^{\alpha}}$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2} = -\frac{n}{\beta^2}$$

$$\frac{\partial^2\mathcal{L}(X;\alpha,\beta,\theta)}{\partial\theta^2} = -2\frac{n}{\theta^2} + \frac{n}{(\theta+1)^2} + (\alpha-1) \quad (3.1.8)$$

$$\times\sum_{i=1}^{n}\left\{\frac{e^{-\theta x_i}}{1-\ ce^{-\theta x_i}}\left[\frac{2(1+x_i)}{(\theta+1)^2}+\frac{2(1+x_i)}{\theta+1}-2cx_i+cx_i{}^2\right]\right\}$$

$$-\frac{1}{1-\ ce^{-\theta x_i}}\left[\frac{c-1-x_i}{\theta+1}+cx_i\right]-\alpha(\beta-1)$$

$$\times\frac{d}{d\theta}\left[\sum_{i=1}^{n}\left[\frac{\left(1-\ ce^{-\theta x_i}\right)^{\alpha-1}}{1-\left(1-\ ce^{-\theta x_i}\right)^{\alpha}}\right]\times x_i e^{-\theta x_i}A\theta\right] \quad (3.1.9)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} = \sum_{i=1}^{n}\frac{\left[1-\ ce^{-\theta x_i}\right]^{\alpha}}{1-\left\{1-\ ce^{-\theta x_i}\right\}^{\alpha}} \quad (3.1.10)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\theta} = \ e^{-\theta x_i}\left\{\left[\frac{\frac{x_i}{(\theta+1)}-\frac{\theta x_i}{(\theta+1)^2}}{1-\ c}\right]-x_i\times ln(1-\ c)\right\}$$

$$-(\beta-1)\times\sum_{i=1}^{n}\left\{\frac{uvw-(uvw)^2}{u(\theta+1)[-1+(uvw)]^2}\right\}$$

$$+\frac{u(\theta^2+2\theta+2\theta^2x_i+1+2\theta x_i+(\theta x_i)^2)}{u(\theta+1)[-1+(uvw)]^2} \quad (3.1.11)$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial\theta} = \sum_{i=1}^{n}\frac{e^{-\theta x_i}\left\{\left[\frac{x_i}{\theta+1}-\frac{\theta x_i}{(\theta+1)^2}e^{-\theta x_i}\right]^{\alpha}+cx_i\right\}}{ce^{-\theta x_i}} \quad (3.1.12)$$

Where

$$u = (\theta+1+\theta x_i)^{\alpha},\ v = \left(\frac{1}{\theta+1}\right)^{\alpha},\ w = e^{(-\theta x_i)^{\alpha}},\ A = \frac{\theta+2+\theta x_i+x_i}{(\theta+1)^2}$$

and

$$c = \frac{\theta+1+\theta x_i}{\theta+1}.$$

Using (3.1.6), the confidence intervals for the parameters α,β and θ with an approximate $100(1-\in)\%$ respectively will be

$$\hat{\alpha} + Z_{\frac{\in}{2}}\times\sqrt{-E\left(\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right)}\ ,\ \hat{\beta} + Z_{\frac{\in}{2}}\times\sqrt{-E\left(\frac{\partial^2\mathcal{L}}{\partial\beta^2}\right)}$$

And

$$\hat{\theta} + Z_{\frac{\in}{2}}\times\sqrt{-E\left(\frac{\partial^2\mathcal{L}}{\partial\theta^2}\right)} \quad (3.1.13)$$

Hence $Z_{\frac{\in}{2}}$ is the upper $\in$th percentile of the standard normal distribution.

3.2. Least square estimators

Swain et al. (1988) [16] obtained the least square estimators for beta distribution via minimizing the relation that

$$Q(\tau) = \sum_{i=1}^{n}\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2 \quad (3.2.1)$$

Using this method and substituting (1.5) in (3.2.1), we have

$$Q(\tau) = \sum_{i=1}^{n}\left[1 - \left[1 - \left\{1 - \frac{\theta+1+\theta x_{(i)}}{\theta+1} e^{-\theta x_{(i)}}\right\}^{\alpha}\right]^{\beta} - \frac{i}{n+1}\right]^2 \quad (3.2.2)$$

So, the least square estimators $\breve{\alpha}, \breve{\beta} and \breve{\theta}$ of the parameters $\alpha, \beta and \theta$ for the Kum-L distribution can be obtained by differentiating (3.2.2) to the parameters $\alpha, \beta and \theta$ respectively as follows

$$\frac{\partial Q(\alpha,\beta,\theta)}{\partial \alpha} = \beta \sum_{i=1}^{n} \left\{1 - \left[1 - ce^{-\theta x_i}\right]^{\alpha}\right\}^{\beta-1} \times \left[1 - ce^{-\theta x_i}\right]^{\alpha} \times ln\left[1 - ce^{-\theta x_i}\right] \quad (3.2.2)$$

$$\frac{\partial Q(\alpha,\beta,\theta)}{\partial \beta} = \alpha \sum_{i=1}^{n} \left\{1 - \left[1 - ce^{-\theta x_i}\right]^{\alpha}\right\}^{\beta} \times ln\left[1 - \left(1 - ce^{-\theta x_i}\right)^{\alpha}\right] \quad (3.2.3)$$

$$\frac{\partial Q(\alpha,\beta,\theta)}{\partial \theta} = \alpha\beta \sum_{i=1}^{n} \left\{1 - \left[1 - ce^{-\theta x_i}\right]^{\alpha}\right\}^{\beta-1} \quad (3.2.4)$$

$$\times\left[1 - ce^{-\theta x_i}\right]^{\alpha-1}$$

$$\times e^{-\theta x_i}\left(\frac{(cx_i + 2cx_i\theta + cx_i\theta^2 + c - x_i\theta - x_i - \theta - 1)}{(\theta+1)^2}\right)$$

Where

$$c = \frac{\theta+1+\theta x_i}{\theta+1}.$$

Setting the above non-linear equations system to zero and solving them numerically, then, the least square estimators will obtained.

4. Application

Computer programs Math-CAD (2001) is used to find the numerical analysis for the last theoretical results. At different sample sizes and different initial parameter values, the performance of the maximum likelihood method is studied to estimating the parameters of the Kum-L distribution by conduct simulations. 500 samples are generated from Kum-L distribution with parameters $\alpha, \beta$and. Table (1.a) and (1.b) display some measures such Average Bias (AV) and Root Mean Square Error (RMSE) of the MLE $\hat{\emptyset}$ of the parameter$\emptyset = (\alpha, \beta, \theta)$. Also, Coverage Probability (CP) and Average Width (AV) of 95% confidence intervals of the parameter$\emptyset = (\alpha, \beta, \theta)$ are calculated. Where,

$$AV = \frac{1}{N}\sum\left(\hat{\emptyset} - \emptyset\right), RMSE = \sqrt{\frac{1}{N}\sum\left(\hat{\emptyset} - \emptyset\right)^2}$$

Table (1.A): Bias, MSE, Coverage Probability, and Average Width for Mlesof KUM-L Distribution

		$\alpha = 0.5, \beta = 0.4\ and \theta = 0.3$			
Parameter	n	Bias	MSE	CP	AW
	50	-0.00671	0.35528	0.8437	1.7264
	100	-0.00395	0.23759	0.8234	1.4397
α	200	-0.00042	0.19457	0.7918	1.1849
	400	-0.00002	0.16897	0.7347	0.9452
	500	-0.00016	0.13598	0.7193	0.6579
	50	0.89214	3.5642	0.8759	8.9421
	100	0.67235	2.0156	0.8684	7.0195
β	200	0.59243	1.1245	0.8497	6.3485
	400	0.42358	0.8567	0.8267	4.6587
	500	0.25648	0.6354	0.8009	3.3476
	50	0.00527	0.06531	0.865	0.2994
	100	0.00356	0.04589	0.8574	0.2347
θ	200	0.00085	0.03741	0.8433	0.1839
	400	0.00033	0.26984	0.8327	0.1635
	500	0.00019	0.19351	0.7967	0.3761

Table 1.B: Bias, MSE, Coverage Probability, and Average Width for Mles of KUM-L Distribution

		$\alpha = 2, \beta = 2\ and \theta = 0.5$			
Parameter	n	Bias	MSE	CP	AW
	50	-0.20347	0.97642	0.97548	6.5478
	100	-0.17397	0.84261	0.98116	5.3246
α	200	-0.05341	0.70168	0.98324	4.9573
	400	-0.03712	0.59473	0.97542	3.1198
	500	-0.00971	0.35476	0.94826	2.6249
	50	0.96511	1.74813	0.95111	8.3654
	100	0.51275	1.06124	0.94652	5.3248
β	200	0.20169	0.84271	0.94872	3.9572
	400	0.08421	0.46892	0.95476	2.4758
	500	0.03476	0.28722	0.94712	1.3654
	50	-0.00094	0.21458	0.98457	0.65112
	100	-0.00062	0.16489	0.96522	0.58716
θ	200	-0.00012	0.09147	0.95147	0.39571
	400	-0.00011	0.06429	0.95011	0.26583
	500	-0.00105	0.05411	0.95413	0.20854

The previous results indicate to that as the sample size n increases, the biases of all the parameters decrease, the RMSEs decay toward zero, CPs are quite close to 95% and AC widths Stricture.

A real data set will be analyzed as well. Table (2) shows the data which is given from Jorgensen (1982) [10]. This data represents the active repair times (h) for an airborne communication transceiver.

Table 2: Active Repair Times (H)

0.8	0.7	0.7	0.7	0.6	0.6	0.5	0.8
1.5	1.3	1.1	1	1	1	1	1.5
2.7	2.5	2.2	2	2	1.5	1.5	3
5	4.7	4.5	4	4	3.3	3	5.4
22	10.2	9	8.8	7.5	7	5.4	24.5

Three criteria statistics can be used to compare the proposed distribution results with the four selection distributions. These criteria statistics respectively are (value of the log likelihood for the estimated model that maximized) $-2\,\mathcal{L}$, (Akaike information criterion) AIC and (corrected Akaike information criterion) AICC of data, which are respectively defined as:

$$AIC = -2\mathcal{L} + 2p and AICC = -2\mathcal{L} + 2p + \frac{2p(p+1)}{n-p-1}$$

$p$is the number of parameters to be estimated. The best distribution fits quite well to data than other. Also, the four selection distributions are Exponentiated power Lindley EPL, power Lindley PL, Gamma G and Lindley L distributions. Table (3) shows that the comparison of the three mentioned criteria statistics.

Table 3: Comparison Criterion

	$-2\,\mathcal{L}$	AIC	AICC
Kum-L	191.2134	197.2134	197.8801
EPLD	192.3149	198.3149	198.9816
PLd	198.3547	202.3547	202.679
GD	201.6589	205.6589	205.9832
LD	204.8491	208.8491	208.9544

Table (4) displays the maximum likelihood estimators for the parameters to each one of the five fitted distributions.

Table 4: Maximum Likelihood Estimators for the Parameters$\alpha, \beta$and θ

	α	β	θ
Kum-L	3.1053	0.3648	33.6547
EPLD	2.9724	0.2981	31.8942
PLd	0.9437	0.8462	
GLD	0.8694		1.9523
LD	0.6842		

5. Conclusion

A new distribution named Kumaraswamy-Lindley distribution has been introduced and studied. Various mathematical and statistical properties of the Kum-L distribution such as hazard function, quintile function, moments, moment generating function and order statistics have been obtained. The maximum likelihood estimation and the least square estimation methods have also been discussed for estimating the parameters of the Kum-L distribution. An example of real data set, and numerical study have been presented to show the applications and goodness of fit of Kum-L distribution over on exponential power Lindley, power Lindley, gamma and Lindley distributions. The Kum-L distribution has been fitted active repair times for an airborne communication transceiver data better than the four selection distributions.

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