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Research paper

One-Parameter Finite Mixture Distributions: Which Flexible Distribution to Use?

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Abstract

Single-parameter distributions hold significant value as they serve as fundamental models for developing future distributions. This manuscript evaluates and meticulously compares the performance of four one-parameter univariate continuous distributions. We do this by conducting a comprehensive numerical study to assess the performance of the Akash distribution against the performances of the Komal, New XLindley, and XGamma distributions. The density functions of the considered distributions can be expressed as a convex amalgamation of exponential and gamma distributions. Various structural properties of these distributions are discussed, including explicit expressions for the hazard rate, reversed hazard rate, cumulative hazard rate, moments, and quantile functions. We offer numerical examples to show the flexibility of each model. These distributions are carefully assessed and analyzed in terms of information criteria, providing valuable insights into the practical application of the distributions in statistical modeling and data analysis contexts. The Akash distribution is a strong candidate for applications in environmental science, biomedical science, and reliability engineering datasets, compared to other one-parameter distributions reviewed in this article.

Keywords: *Biomedical Science; Akash Distribution; Data Analysis; Mean Residual Function; Information Criterion.*

1. Introduction

A random variable T is said to have a finite mixture distribution if the probability density function (PDF) satisfies the following

$$f(t) = \sum_{j=1}^{m} q_i \, f_i(t) \tag{1}$$

where $q_1, q_2, \ldots, q_m$ delineate the mixing proportion and each $f_i(t)$ articulate a PDF. The mixing proportions are non-negative and sum to unity.

New distributions have been formed by merging a finite number of distributions with a mixed proportion. The derived distributions are utilized to frequently simulate a broad spectrum of random occurrences, in particular, to account for the heterogeneity of the undetected data. Mixed distribution models have been popularized by statisticians and data modelers.

Single-parameter distributions hold significant value as they serve as fundamental models for developing future distributions. The exponential and gamma distributions are popular in modeling stochastic lifespan datasets. The gamma distribution generalizes the exponential, Erlang, and chi-square distributions. The exponential distribution, indexed by a single parameter, is well-known for describing the time between events in a Poisson process. It is a continuous memoryless distribution and has a constant failure rate. As a nested model of the gamma distribution, the exponential distribution can be deployed in modeling waiting times data. The exponential distribution is more popular than the gamma distribution because the survival function of the gamma distribution is intractable and requires numerical integration. In addition, since the gamma distribution cannot be represented in a quantile function, no closed-form expressions exist for its value-at-risk and expected shortfall risk measures.

It is rare to come across an event with a constant failure rate. The exponential distribution does not offer sufficient fit to some engineering data due to its applied or theoretical perspectives. Numerous extensions or modifications of both distributions are documented in the body of statistical knowledge ([1]).

The proposition of novel lifetime distributions or modified lifetime distributions has become a time-honored fashion in several applied domains. However, each of these lifetime models has high and low points over one another due to the number of parameters involved, shape, hazard rate function, and mean residual life function, etc. A little discussion concerning the potential applications of a single-parameter distribution can be found in the statistics literature, but it seems that a comprehensive exploration of their potential applications across diverse fields of knowledge has not been attempted. The fundamental motivation for this study in practice is to select a one-parameter

model that offers consistently better parametric fit than other one-parameter generated models constructed based on a convex combination of exponential and gamma distributions.
This manuscript delves into the applications and comparisons of one-parameter mixture distributions derived from a unique convex amalgamation of exponential and gamma distributions via a special mixture formulation, under a real-life scenario. This is in tandem with the parsimony rule, which asserts that the best model requires fewer assumptions and parameters. Distribution theory analysts prefer to study models with fewer parameters and less complexity. It should be mentioned that this article will provide another option to a practitioner for the purposes of data analytics. The study excludes convex combinations of other distributions. No scholar, at least not known to the author, has studied this problem yet. To this end, we carry out a comprehensive numerical study to model real data from varied applied sectors. The fits of the distributions are compared using well-known information criteria. The manuscript is organized as follows. Some distributions derived as convex combinations of exponential and gamma distributions are reviewed and their reliability measures discussed in Section 2. Six real applications to demonstrate the utility of the distributions are presented in Section 3. Finally, Section 4 concludes the manuscript, and Section 5 offers future research directions.

2. Some Flexible One-Parameter Distributions

We review some scholarly scientific publications within peer-reviewed journals of one-parameter distributions derived from convex amalgamation of exponential and gamma distributions in the sequel.

2.1. Lindley distribution

In the earlier decades, statistical scientists have expressed great interest in constructing novel expanded distributions by induction of extra parameter(s) to the parent distribution against the intuition of model parsimony. The main incentive is to expand the boundaries of flexibility to model varied characteristics of data sets. Generally, researchers have paid attention to convex combinations of exponential and gamma distributions. The Lindley (Li) distribution was pioneered by Lindley [2] as counter example of fiducial statistics. Scholars Ghitany et al. [3] demonstrated via a numerical example that the failure rate function of the Li distribution does not exhibit a constant hazard rate, signaling its flexibility over the exponential distribution. A detailed analysis of its application showed the superiority of the Li distribution over the exponential distribution for the waiting times of bank customers. Shanker et al. [4] have worked on comparative studies on modeling of lifetime data using exponential and Lindley distributions. Highlighting their findings, the authors uncovered that there are many lifetime data sets where the Li distribution fit is sub-optimal as compared to the exponential distribution.
During a brief span of time, two-parameter and three-parameter Li distributions have been documented in the body of distribution theory. Shanker and Mishra [5] obtained a two-parameter distribution and named it the quasi Lindley distribution (QLiD) and discussed its several statistical characteristics, estimation of parameters, and goodness of fit. The Li distribution is nested in the QLiD for a specific parameter setting. Shanker et al. [6] have explained numerous desirable characteristics of QLiD along with its applications for modeling various lifetime data and articulated that it outpaces its competitors in most of the datasets considered. The exponential, Li, and gamma distributions are nested in a distribution constructed by research scientists Zakerzadeh and Dolati [7]. This distribution was named the three-parameter generalized Lindley distribution (TPGLiD). The discussion about its mathematical properties, estimation of unknown parameters, and applications can be found in Zakerzadeh and Dolati [7]. Not long ago, Shanker et al. [8] provided and delineated a novel distribution referred to as a three-parameter Li distribution (ATPLiD). Several earlier distributions are particular cases for specific parameter settings. Various mathematical and statistical properties of ATPLiD have been derived and discussed. The estimation of unknown parameters of ATPLiD has been approached by maximum likelihood estimates, and its goodness of fit compared with TPGLiD proposed by Zakerzadeh and Dolati [7].
The Li distribution has been recognized as an appropriate model to analyze lifetime data, in particular, applications modeling stress-strength reliability. Nonetheless, there are scenarios where the Li distribution may not be suitable from theoretical or applied perspectives. The Li distribution with probability density function (PDF) and cumulative distribution function (CDF)

$$c(t) = \frac{\phi^2}{\phi+1}(t+1)e^{-\phi t}, \phi, t > 0 \tag{2}$$

$$C(t) = 1 - \frac{\phi t+\phi+1}{\phi+1}e^{-\phi t} \tag{3}$$

The Li distribution can be denoted as a two-component mixture distribution with PDF.

$$c(t) = \omega c_E(t) + (1-\omega)c_G(t)$$

where $c_E(t) = \phi e^{-\phi t}$ and $c_G(t) = \phi^2 t e^{-\phi t}$ are PDFs of exponential and gamma distributions, in that order, and $\omega = \phi(\phi+1)^{-1}$ is the mixing proportion of distributions.
Proof.

$$c(t) = \frac{\phi}{1+\phi}Exp(\phi) + \frac{1}{1+\phi}Gamma(2,\phi)$$

$$c(t) = \frac{\phi}{1+\phi}\phi e^{-\phi t} + \frac{1}{1+\phi}\frac{\phi^2}{\Gamma(2)}t^{2-1}e^{-\phi t}$$

$$c(t) = \frac{\phi}{1+\phi}\phi e^{-\phi t} + \frac{1}{1+\phi}\phi^2 t e^{-\phi t}$$

$$c(t) = \frac{\phi^2}{1+\phi}\{1+t\}e^{-\phi t}$$

This completes the proof.
The failure rate function is an important quantity characterizing life phenomena. The hazard rate function of the Li distribution is determined as

$$h(t) = \frac{\phi^2(t+1)}{(\phi+\phi t+1)} \tag{4}$$

Mean residual life function.
In survival and reliability studies, the Mean Residual Life (MRL) function is an important metric for measuring the expected remaining time of a system, given that it has already survived for a specified duration. The MRL function of the Lindley distribution, denoted by $\mathbf{m}(t)$, is given by

$$\mathbf{m}(t) = \frac{\phi+\phi t+2}{(\phi+\phi t+1)\phi} \tag{5}$$

2.2. Akash distribution

The core reason for proposing new models is to circumvent the hurdles that are present in already available probability distributions in statistical literature. Shanker [9] recommended a single-parameter continuous distribution, christened Akash distribution. This one-parameter distribution was derived as a mixture of exponential and gamma distributions. A detailed study about its various statistical and mathematical characteristics, estimation of the parameter, and applications displaying the dominance of the Akash distribution over other competitors for engineering and medical science data has been presented.
Despite these advancements, the Akash distribution's suitability is restricted by its upside-down bathtub failure rate, which is very difficult to justify in real-life problems. Also, because it is indexed by a single scale parameter and no shape parameter, its modeling abilities may be challenged. The introduction of shape parameter(s) has been proved useful in exploring tail properties of generated distributions. It is worth mentioning that this distribution is bereft of physical motivation in modeling lifetime data. Moreover, the inferences for this distribution were based on classical procedures. Bayes and E-Bayes inferences can be used.
The Akash distribution with a scale parameter is defined by the PDF.

$$c(t) = \frac{\varphi^3}{\varphi^2+2}(t^2+1)e^{-\varphi t} \tag{6}$$

The CDF of the Akash distribution is determined as

$$C(t) = 1 - \left[1 + \frac{(\varphi t)^2+2\varphi t}{\varphi^2+2}\right]e^{-\varphi t},\ t,\varphi > 0 \tag{7}$$

The one-parameter Akash distribution PDF can be readily written as a two-component mixture of exponential and gamma distributions, i.e.

$$c(t) = \omega c_E(t) + (1-\omega)c_G(t)$$

where

$$\omega = \frac{\varphi^2}{\varphi^2+2},\ c_E(t) = \varphi e^{-\varphi t},\ c_G(t) = \frac{1}{2}\varphi^2 t^2 e^{-\varphi t}$$

Proof.

$$c(t) = \frac{\varphi^2}{\varphi^2+2}Exp(\varphi) + \frac{2}{\varphi^2+2}Gamma(3,\varphi)$$

$$c(t) = \frac{\varphi^2}{\varphi^2+2}\varphi e^{-\varphi t} + \frac{2}{\varphi^2+2}\frac{\varphi^3 t^{3-1}}{\Gamma(3)}e^{-\varphi t}$$

$$c(t) = \frac{\varphi^2}{\varphi^2+2}\varphi e^{-\varphi t} + \frac{1}{\varphi^2+2}\varphi^3 t^2 e^{-\varphi t}$$

$$c(t) = \frac{\varphi^3}{\varphi^2+2}\{1+t^2\}e^{-\varphi t}$$

For a given probability $u \in (0,1)$, we obtain the quantile function of T, say $Q(u)$, by investing $C(t) = u$ in equation (7). Then, for each u, we have to compute numerically the nonlinear equation given by

$$\varphi t - log\left[1 + \frac{(\varphi t)^2+2\varphi t}{\varphi^2+2}\right] + log\{1-u\} = 0$$

Consequently, a random number can be generated based on the above equation.
The hazard rate function is given by

$$h(t) = \frac{\varphi^3(t^2+1)}{\varphi^2t^2+\varphi^2+2\varphi t+2} \quad (8)$$

The mean residual life function takes the form.

$$m(t) = \frac{\varphi^2t^2+6+(4+\varphi)\varphi}{[\varphi^2t^2+\varphi^2+2\varphi t+2]\varphi} \quad (9)$$

The mean residual life function of the Akash distribution is a decreasing function of $t$as well as φ indicating a superior flexibility compared to Lindley and exponential distributions.
The kthraw moments of the Akash distribution are given by

$$\mu_k^{'} = \frac{\langle(k+1)(k+2)+\varphi^2\rangle k!}{(2+\varphi^2)\varphi^k} \quad (10)$$

Proof.
By definition, $\mu_k^{'} = \int_0^\infty t^k \frac{\varphi^3}{\varphi^2+2}(t^2+1)e^{-\varphi t}dt$

$$\mu_k^{'} = \frac{\varphi^2}{\varphi^2+2}\left\{\varphi\int_0^\infty t^{k+2}\, e^{-\varphi t}dt + \varphi\int_0^\infty t^k\, e^{-\varphi t}dt\right\}$$

Let $u = \varphi t$, $t = 0, u = 0$, $t = \infty$, $u = \infty$, $du = \varphi dt$, $t = \frac{u}{\varphi}$

$$\mu_k^{'} = \frac{\varphi^2}{\varphi^2+2}\left\{\left(\frac{1}{\varphi}\right)^{k+2}\int_0^\infty u^{k+2}\, e^{-u}du + \left(\frac{1}{\varphi}\right)^k\int_0^\infty u^k\, e^{-u}du\right\}$$

$$= \frac{\varphi^2}{\varphi^2+2}\left\{\left(\frac{1}{\varphi}\right)^{k+2}\Gamma(k+3) + \left(\frac{1}{\varphi}\right)^k\Gamma(k+1)\right\}$$

$$= \frac{1}{\varphi^k(\varphi^2+2)}\{\Gamma(k+3) + \varphi^2\Gamma(k+1)\}$$

$$\Gamma(k+3) = (k+1)(k+2)k!$$

$$\mu_k^{'} = \frac{k!}{\varphi^k(\varphi^2+2)}\{(k+1)(k+2)+\varphi^2\}$$

This completes the proof.
Using the equation defined in (10), the first four raw moments can be obtained by putting. $k = 1, 2, 3$and 4.
It is important to mention that, when the moments of a distribution do not exist, alternative measures for the skewness and kurtosis, based on quantile functions, are sometimes more appropriate. The lack of symmetry of a unimodal distribution is expressed via measures of skewness. Ekström and Jammalamadaka [10] proposed a general measure of skewness based on the quantile, which encompasses the measures by Bowley [11] and David and Johnson [12] as particular cases. Table 1 lists the mean, variance, skewness and kurtosis of the Akash distribution for selected values of φ. These values were determined numerically using R. However, alternative software including Mathematica, MATLAB, etc. can be used to carry out the computation.

Table 1: Mean, Variance, Skewness and Kurtosis of Akash distribution

φ	Mean	Variance	Skewness	Kurtosis
0.2	14.8039	75.9419	1.1341	4.9503
0.3	9.7129	34.2078	1.1146	4.8975
0.4	7.1296	19.5387	1.0962	4.8387
0.5	5.5555	12.6913	1.0839	4.7849
1.0	2.3333	3.2222	1.1654	4.8335
1.5	1.2941	1.3056	1.3881	5.4727
2.0	0.8333	0.6388	1.6137	6.3913
2.5	0.5939	0.3551	1.7935	7.3101
3.0	0.45454	0.2176	1.9224	8.0899

In Table 1, we provide numerical values for the mean, variance, skewness, and kurtosis of the Akash distribution, for some chosen values of φ to demonstrate its effects on these descriptive statistics. Table 1 shows that the mean and variance are decreasing functions of φ. It is noted that the skewness and kurtosis of the Akash distribution decrease to a point and then start increasing for an increase in the values of φ. Thus, a few chosen values of the parameter influenced these measures. From Table 1, the Akash distribution is right-tailed and leptokurtic. That is, the mass of the distribution is concentrated on the right.

2.3. Komal distribution

In a noteworthy illuminating article, Shanker [13] uncovered a distribution known as the Komal distribution. This distribution, indexed by a single parameter, was proposed in the context of non-Bayesian statistics. The Komal distribution is delineated by a scale parameter, which can regulate data variability. The motivation of the author stemmed from the fact that existing one-parameter distributions are unsuitable due to the nature of the distribution and the stochastic nature of lifetime data. This endeavor leverages convex amalgamations of exponential and gamma distributions. The academic elucidated with utmost clarity the remarkable statistical properties and approached the parameter estimation using the maximum likelihood and method of moments techniques. The failure times of 20 electric bulbs were considered in the application. The importance of this distribution resonates in both theoretical and practical applications. Despite its theoretical and practical applications, the Komal distribution exhibits certain shortcomings, including a lack of a shape parameter. It is highly handicapped in modeling highly skewed and kurtotic data. This structural constraint may hinder its capability in sufficiently capturing the diverse and often complex traits observed in a broad spectrum of real-world applied domains. This notable vacuum in statistical literature serves as sufficient incentive underpinning this study. It is worth accentuating that the versatility of this distribution allows for its expansion and exploration via the incorporation of a diverse array of more generalized families, extending the scope of its applicability and relevance. The Komal distribution has been modified, extended, and generalized, along with detailed applications in reliability and other sectors of knowledge, by an array of research scientists. Shanker et al. [14] defined and studied a novel distribution known as the weighted Komal distribution. Some reliability measures as well as moment-based statistics were explored. The parameter estimation was approached by the maximum likelihood method. To demonstrate the applicability and usefulness of the proposed model, two examples of observed real lifespan datasets were the subject of consideration.

In a scientific report, a novel two-parameter model was developed via power transformation methodology on the one-parameter Komal distribution. The authors, namely Shanker et al. [15], called the distribution the power Komal distribution (PKD). This distribution is nested in the Komal distribution as a particular case. As an attractive feature, the nature of the probability density function (PDF) of PKD has been studied and presented. PKD can generate positively skewed, decreasing, negatively skewed, symmetric, leptokurtic, as well as platykurtic distributions for varying values of the parameters, tailored for the analysis and modeling of data emanating from reliability engineering. In addition, the visual representation of the hazard rate function of PKD reveals is monotonically increasing, very quickly and slowly in nature. Essential elegant expressions investigated and discussed encompass the first four raw moments, variance, and mean residual life function. The researchers have also made an attempt to discuss stochastic ordering, a concept which quantifies how one random variable is smaller than another. The inferences from the distribution are interrogated. In particular, the parameters are ascertained using the efficient method, namely maximum likelihood estimation. Using the failure time windshields dataset and comparisons with other distributions, the effectiveness of PKD has been demonstrated and visually presented.

Exponential-Komal distribution is analytically obtained and articulated by research scientists Ray and Shanker [16]. Key statistical properties were obtained. Leveraging the merits and capabilities of the maximum likelihood estimation technique, the distribution's parameter was estimated. The goodness of fit of the constructed distribution was tested via a real univariate dataset to promote its use, and the fit has been quite satisfactory over its competitors. The shortcoming of this proposed compound distribution is that it doesn't have a moment-generating function or moments.

A very important distribution to evaluate radiation data and prevalence and characteristics of disability in the Kingdom of Saudi Arabia has been created by statistical scientists Ahmadini and El-Saeed [17]. Leveraging decreasing or increasing hazard functions, practitioners can have a great deal of flexibility when developing statistical models that amalgamate disability with radiation data. Numerous key statistical properties are obtained. Bayesian and frequentist estimation techniques were utilized to estimate the parameters of the model. Three real-data sets were used to evaluate the effectiveness of the recommended distribution.

The probability density function (PDF) of the Komal distribution is expressed as

$$c(t) = \frac{(1+\lambda+t)}{\lambda^{-2}(\lambda^2+\lambda+1)e^{\lambda t}}, t > 0, \lambda > 0 \tag{11}$$

The cumulative distribution function (CDF) is presented as

$$C(t) = 1 - \frac{\Delta_\lambda(t)}{e^{\lambda t}}, t > 0, \lambda > 0 \tag{12}$$

where $\Delta_\lambda(t) = \left(1 + \frac{\lambda t}{\lambda^2+\lambda+1}\right)$

Komal distribution can be denoted as a two-component mixture distribution with PDF.

$$c(t) = \omega_1 c_E(t) + (1-\omega_1)c_G(t)$$

where $\omega_1 = \frac{\lambda(\lambda+1)}{\lambda^2+\lambda+1}$, $c_E(t) = \lambda e^{-\lambda t}$ and $c_G(t) = \lambda t e^{-\lambda t}$

Reliability Analysis

This part delves in depth into the Komal distribution (KD) reliability characteristics, including reliability function, hazard rate function, reverse hazard, second rate of failure, cumulative hazard rate H(z), Mills ratio (MR), and mean residual life (MRL).

Reliability function

The reliability function is articulated and analytically derived as the complement of the related probability model's CDF.

$$R(t) = \frac{\Delta_\lambda(t)}{e^{\lambda t}}$$

Failure rate function

This is the rate at which failures occur at a particular point in time, assuming the system has survived up to that point, and is represented by the hazard function, sometimes referred to as the failure rate. Numerically, it is defined as

$$hr(t) = \frac{c(t)}{1 - C(t)}$$

Thus, the failure rate function of the Komal distribution is derived as

$$hr(t) = \frac{\lambda^2(t+\lambda+1)}{\lambda t+1+\lambda+\lambda^2} \tag{13}$$

Several shapes of the failure rate of the KD distribution are graphically demonstrated by the authors, for varying values of the parameter. λ. It is observed that the hazard rate function of the KD is monotonically non-decreasing. Besides, as the value of the parameterλ increases, the failure rate function increases sharply, indicating a higher risk across values of t.
Reverse Hazard rate and second rate of failure
The reverse hazard rate and the second rate of failure (srf) of the KD can be expressed using well-recognized formulas in that order, such as

$$rh(t) = \frac{c(t)}{C(t)}$$

Thus, for the KD, the reverse hazard rate is obtained as follows.

$$rh(t) = \frac{\lambda^2(t+1+\lambda)e^{-\lambda t}}{[1+\lambda+\lambda^2]\left\{1-\left[1+\frac{\lambda t}{1+\lambda+\lambda^2}\right]e^{-\lambda t}\right\}} \tag{14}$$

and

$$srf(t) = ln\left(\frac{e^{\lambda}\left[1+\frac{\lambda t}{1+\lambda+\lambda^2}\right]}{\left[1+\frac{\lambda(t+1)}{1+\lambda+\lambda^2}\right]}\right) \tag{15}$$

Cumulative hazard rate and Mills ratio
The cumulative hazard rate $H(t)$ and Mill ratio $M(t)$ of the KD can be elegantly represented via well-established mathematical formulas and expressed as

$$H(t) = -\,ln(S(t))$$

$$H(t) = \lambda t - ln\left\{\left[1+\frac{\lambda t}{\lambda^2+\lambda+1}\right]\right\}$$

and

$$M(t) = \frac{S(t)}{c(t)}$$

$$M(t) = \frac{(\lambda^2+\lambda+1)\left[1+\frac{\lambda t}{1+\lambda+\lambda^2}\right]}{(1+\lambda+t)\lambda^2} \tag{16}$$

Mean residual life function.
In survival and reliability studies, the Mean Residual Life (MRL) function is an important metric for measuring the expected remaining time of a system, given that it has already survived for a specified duration. The MRL function of the Komal distribution, denoted by $\mathbf{m}(t)$, is given by

$$\mathbf{m}(t) = \frac{\lambda^2+\lambda+\lambda t+2}{(\lambda^2+\lambda+\lambda t+1)\lambda} \tag{17}$$

The behavior of the MRL function of KD was investigated by the authors. They uncovered that the mean residual life function of the Komal distribution is monotonically non-increasing for selected values of the parameter.

2.4. The new XLindley distribution

Highlighting the need for more adaptable distributions, researchers Chouia and Zeghdoudi [18] formulated, studied, and implemented a new distribution known as the XLindley distribution. This was done in the context of fiducial statistics. This distribution is a special finite mixture of exponential and Lindley distributions. The XLindley distribution has a decreasing density function, while its failure rate function is increasing. This distribution is not competent to handle datasets with non-monotonic failure rate function behaviors. Therefore, the need arises for introducing distributions that can adequately deal with a high degree of skewness and kurtosis as well as improve the goodness-of-fit in empirical distributions significantly. In search for some modification of the XLindley distribution, Khodja et al [19] created a new version designated as the new XLindley (NXL) distribution. This distribution, indexed by a single parameter, is derived from a unique convex amalgamation of exponential and gamma distributions via a special mixture formulation. The research extensively investigated the mathematical properties of the NXL distribution. Besides the maximum likelihood estimation method, the study employed 9 various estimation methods in estimating the parameter of the NXL distribution. Numerical Monte Carlo simulation research was carried out to determine the optimal method for estimating the parameter of the NXL distribution. The metrics to evaluate and contrast the effectiveness of these estimators were based on average bias (BIAS), mean squared errors (MSE), and mean relative errors (MRE). Based on the outcomes of the simulation, the scholars identified the maximum product of the spacing estimation method as the best for estimating the NXL distribution parameter. Some actuarial measures of NXL distribution were discussed and derived mathematically. Two real-life engineering datasets were deployed to demonstrate the usefulness and applicability of NXL distribution. The NXL distribution exhibited a superior fit relative to its competitive models. Prodhani and Shanker [20] provided a comprehensive mathematical and statistical treatment of the NXL distribution.
The cumulative distribution function of the NXL distribution is given by

$$C(t) = 1-(0.5\beta t+1)e^{-\beta t} \tag{18}$$

The corresponding probability density function takes the form.

$$c(t) = 0.5\beta(\beta t + 1)e^{-\beta t} \quad (19)$$

While its probability density function can accommodate data sets with decreasing shapes, the failure rate function has an increasing shape as reflected through graphs. The NXL distribution cannot be used to model lifespan data emanating from machine life cycles and human mortality, as they exhibit non-monotonic failure rate functions. Besides, the modeling power of the NXL distribution is restricted because of its single parameter.
The survival and hazard rate functions for the NXL distribution are, respectively, defined as follows.

$$S(t) = (0.5\beta t + 1)e^{-\beta t} \quad (20)$$

and

$$h(t) = \frac{\beta+\beta^2 t}{\beta t+2} \quad (21)$$

The PDF and hazard rate function plots for some choices of parameters revealed that the PDF is decreasing while the hazard rate function has an increasing shape.
Reverse Hazard rate and second rate of failure
The reverse hazard rate and the second rate of failure (srf) of the NXL distribution can be expressed using well-recognized formulas in that order, such as

$$rh(t) = \frac{c(t)}{C(t)}$$

$$rh(t) = \frac{0.5\beta(\beta t+1)e^{-\beta t}}{1-(0.5\beta t+1)e^{-\beta t}} \quad (22)$$

and

$$srf(t) = ln\left\{\frac{(0.5\beta t+1)e^{\beta}}{(0.5\beta(t+1)+1)}\right\} \quad (23)$$

Cumulative hazard rate and Mills ratio
The cumulative hazard rate $H(t)$ and Mill ratio $M(t)$ of the NXL distribution can be elegantly represented via well-established mathematical formulas and expressed as

$$H(t) = -\,ln(S(t))$$

$$H(t) = -\,ln\{(0.5\beta t + 1)e^{-\beta t}\}$$

and

$$M(t) = \frac{S(t)}{c(t)}$$

$$M(t) = \frac{\beta t+2}{\beta^2 t+\beta} \quad (24)$$

Moments
The kth of the NXL distribution is given by

$$\mu_k^{'} = \frac{1}{2\beta^k}\{\Gamma(k+1) + \Gamma(k+2)\}$$

Proof.
We know

$$\mu_k^{'} = E(T^k) = \int_0^\infty t^k f(t)dt$$

$$= \int_0^\infty t^k\, 0.5\beta(\beta t + 1)e^{-\beta t}dt$$

$$= 0.5\beta^2 \int_0^\infty t^{k+1}\, e^{-\beta t}dt + 0.5\beta \int_0^\infty t^k\, e^{-\beta t}dt$$

Let $u = \beta t$, $t = 0$, $u = 0$ and $t = \infty$, $u = \infty$, $du = \beta dt$, $t = \frac{u}{\beta}$

$$= 0.5\beta \int_0^\infty \left(\frac{u}{\beta}\right)^{k+1} e^{-u}du + 0.5 \int_0^\infty \left(\frac{u}{\beta}\right)^k e^{-u}du$$

$$= \frac{0.5}{\beta^k} \int_0^\infty u^{(k+2)-1} e^{-u} du + \frac{0.5}{\beta^k} \int_0^\infty u^{(k+1)-1} e^{-u} du$$

$$= \frac{0.5}{\beta^k} \langle \Gamma(k+1) + \Gamma(k+2) \rangle$$

This completes the proof.
The first four non-central moments are

$$\mu_1' = E(T) = \frac{0.5}{\beta} \langle \Gamma(2) + \Gamma(3) \rangle = \frac{3}{2\beta}$$

$$\mu_2' = E(T^2) = \frac{4}{\beta^2}$$

$$\mu_3' = E(T^3) = \frac{15}{\beta^3}$$

$$\mu_4' = E(T^4) = \frac{72}{\beta^4}$$

In particular, the mean, variance, coefficient of variation, coefficient of skewness, as well as coefficient of kurtosis, are defined as

$$\mu_1' = mean = \frac{3}{2\beta}$$

$$Var = E(T^2) - \{E(T)\}^2 = \frac{7}{4\beta^2}$$

$$Skewness = \frac{E(T^3)}{\{Var\}^{\frac{3}{2}}} = \frac{\frac{15}{\beta^3}}{\left(\frac{7}{4\beta^2}\right)^{\frac{3}{2}}} = 6.4793$$

$$Coefficient \text{ of variation} = \frac{\sqrt{Var}}{E(T)} = \frac{\sqrt{\frac{7}{4\beta^2}}}{\frac{3}{2\beta}} = \frac{\sqrt{7}}{3}$$

$$Kurtosis = \frac{E(T^4)}{\{Var\}^2} = \frac{\frac{72}{\beta^4}}{\left(\frac{7}{4\beta^2}\right)} = 23.5102$$

An empirical study was undertaken to ascertain the effect of change in the values of β. Table 2 provides the mean, variance, skewness and kurtosis of the NXL distribution for different values of β. It is discernible from Table 2 that the mean and the variance of the NXL distribution decline for an increase in the value. β. However, an increase in the value of β does not influence the coefficients of skewness and kurtosis.

Table 2: Mean, Variance, Skewness and Kurtosis of NXL distribution

β	Mean	Variance	Skewness	Kurtosis
0.2	7.5000	43.7500	1.6198	6.7959
0.3	5.0000	19.4444	1.6198	6.7959
0.4	3.7500	10.9375	1.6198	6.7959
0.5	3.0000	7.0000	1.6198	6.7959
1.0	1.5000	1.7500	1.6198	6.7959
1.5	1.0000	0.7778	1.6198	6.7959
2.0	0.7500	0.4375	1.6198	6.7959
2.5	0.6000	0.2800	1.6198	6.7959
3.0	0.5000	0.1944	1.6198	6.7959

For skewed unimodal distributions, the three most common measures of central tendency, the mode, the median and the mean, do not coincide. One can see, from Table 1, that the NXL distribution is right-skewed and leptokurtic. Therefore, the NXL distribution is flexible and can be used in modelling skewed data.

2.5. The XGamma distribution

New probability distributions, formulated based on a finite amalgamation of probability distributions, play a pivotal role in analyzing real-life phenomena. Moreover, finite mixture densities have a diverse spectrum of usage in modeling various data spanning from medicine, biology, finance and economics, ecology, and beyond ([21]). An extension of the gamma distribution, known as the XGamma, also exists in the literature. The crafting of this distribution was pioneered and elucidated by researchers Sen et al. [22]. The distribution derived its

name from the special amalgamation of exponential and gamma distributions. The XGamma distribution is quite effective in modeling data as a replacement for the exponential distribution. Some of the basic structural properties of the new model and its parameter estimation were derived and studied using maximum likelihood and moment methodologies of inference. The authors argued that the XGamma is a member of the exponential family of distributions and therefore has superior power of flexibility relative to the exponential distribution. Pain management is still a challenging task confronting anesthesiologists. Modeling of relief times of patients who were receiving an analgesic was approached by the XGamma distribution. The XGamma provided a decent distributional fit relative to the exponential distribution. The XGamma distribution has garnered considerable attention from researchers and practitioners. In a quest to make the distribution a good candidate for modeling different data sets, the XGamma distribution has been adopted by various research scientists as a baseline to generate new distributions from generalized classes. In the literature, these generalizations, modifications, and extensions are as follows. Yadav et al. [23] defined and studied the inverted form of the XGamma distribution. The research paper delves into the investigation of the statistical properties of the distribution, alongside its prospective utility across a myriad of sectors. The recommended model is effective in modeling positively skewed data. In contributing their widows might into statistical distribution theory and practice, the authors Sen and Chandra [24] proposed a new distribution indexed by two parameters. The novel distribution is named the quasi XGamma distribution. The quasi XGamma distribution is obtained by mixing the exponential distribution with mean $\frac{1}{\varphi}$ and the gamma distribution with shape parameter 3 and scale parameter φ. The new distribution generalizes not only XGamma but also another new distribution. Alomair et al. [25] formulated the power quasi XGamma distribution. This model exhibits more flexibility owing to its variable hazard function rate shapes. The point estimation of the distribution parameters is discussed through five estimation techniques. It has been mentioned in the literature of distribution theory that the incorporation of a new parameter can be performed either using the available generator or by developing a new technique for generating a new improved distribution relative to the conventional parent model. Following the spirit of the former, Yousof et al. [26] defined and studied a new applied probability model, which is an extension of the XGamma distribution. The estimation of parameters of the distribution was approached by multiple techniques, and simulation analysis was performed to identify the optimal technique. Five actuarial robust risk tools were deployed to explain the risk exposure in the context of data on insurance claims and reinsurance revenue. The authors implemented a peaks over a random threshold (PORT) analysis using the value-at-risk metric to assess extreme financial insurance peaks. The peak over a random threshold value-at-risk (PORT-VaR) provides a versatile tool for actuaries and risk analysts, ensuring regulatory compliance within the insurance industry Alizadeh et al. [27].
Denote by $XG(\delta)$ a single-parameter XGamma distribution. If the lifetime random variable, X, of an experimental unit(s) , adheres to $XG(\delta)$ then the distribution function is expressed as

$$C(t;\ \delta) = 1 - \left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right) exp(-\delta t),\ t,\delta > 0 \tag{25}$$

The associated PDF (probability density function) is given by

$$c(t;\ \delta) = \frac{\delta^2}{(1+\delta)}\left(1+\frac{1}{2}(\delta t^2)\right) exp(-\delta t),\ t,\delta > 0 \tag{26}$$

where δ is a shape parameter.
The PDF can also be expressed as a form of the following mixture.

$$c(t;\ \delta) = p_1 g_1(t) + p_2 g_2(t)$$

where

$g_1(t) = Exp(\delta),\ g_2(t) = Gamma(3,\delta),\ p_1 = \frac{\delta}{1+\delta}$ and $p_2 = \frac{1}{1+\delta}$

Proof.

$c(t;\ \delta) = p_1 g_1(t) + p_2 g_2(t)$

$$c(t;\ \delta) = \frac{\delta}{1+\delta} Exp(\delta) + \frac{1}{1+\delta} Gamma(3,\delta)$$

$$c(t;\ \delta) = \frac{\delta}{1+\delta}\delta e^{-\delta t} + \frac{1}{1+\delta}\frac{\delta^3}{\Gamma(3)} t^{3-1} e^{-\delta t}$$

$$c(t;\ \delta) = \frac{\delta}{1+\delta}\delta e^{-\delta t} + \frac{1}{1+\delta}\frac{\delta^3}{2} t^2 e^{-\delta t}$$

$c(t;\ \delta) = \frac{\delta^2}{1+\delta}\left\{1+\frac{\delta t^2}{2}\right\} e^{-\delta t}$

Reliability practitioners often use reliability characteristics of any lifetime model as the cornerstone for evaluating the capacity of any electronic system. The survival and failure rate functions, respectively, are

$$S(t;\ \delta) = \left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right) exp(-\delta t) \tag{27}$$

$$h(t;\ \delta) = \left(\frac{1+\frac{1}{2}(\delta t^2)}{1+\delta(1+t)+0.5(\delta t)^2}\right)\delta^2 \tag{28}$$

The reverse hazard rate function expression takes the form.

$$rh(t;\ \delta)=\frac{\frac{\delta^2}{(1+\delta)}\left(1+\frac{1}{2}(\delta t^2)\right)exp(-\delta t)}{1-\left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right)e^{-\delta t}} \tag{29}$$

The cumulative hazard rate is articulated subsequently as

$$H(t;\ \delta)=\delta t-ln\left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right) \tag{30}$$

The second rate function is determined as

$$srf(t;\ \delta)=\frac{\left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right)exp(\delta)}{\left(\frac{1+\delta(2+t)+0.5(\delta\{t+1\})^2}{1+\delta}\right)} \tag{31}$$

$$M(t;\ \delta)=\frac{\left(\frac{1+\delta(1+t)+0.5(\delta t)^2}{1+\delta}\right)}{\delta^2(1+\delta)^{-1}[1+0.5(\delta t^2)]} \tag{32}$$

3. Real Data Analysis

In this segment, we fit the four distributions, namely, NXL distribution, Komal distribution, Akash distribution, and XGaama distribution, to real data sets and also compare the fitted distributions. This comparison is expected to provide another option for a practitioner to use for data analysis purposes.

In what follows, we deploy approaches and procedures of ubiquitous methods to determine the most suitable model for the data sets. We can use some of the information theoretic criteria to select a model that provides a consistently better fit. These analytic metrics encompass the Akaike information criterion, Schwarz criterion, and the corrected Akaike information criterion. For these selection techniques, the superior distribution is the one with the minimum test statistic.

We discussed the parameter algorithm utilized to estimate the parameter of each model in fitting the various datasets. A barrage of parameter estimation methodologies has been documented in statistics literature to estimate the parameter(s) of distributions. Amongst these, the maximum likelihood received increased interest from devoted statisticians. Following this spirit, we utilized the maximum likelihood technique.

The maximum likelihood estimate of the parameter for each distribution is estimated using the bbmle package in R ([28]). The initial value of the parameter for each fitted distribution employed in the optimization is obtained using the Generalized Simulated Annealing (GenSA) package in R. This complementary tool to, rather than a substitute for, other widely used R packages for optimization was developed by Xiang et al. [29]. This has been successfully used for solving a non-convex portfolio optimization problem in finance and the Thomson problem in physics. It is important to mention that the development of GenSA was motivated by the assertion that no one method is the best for all optimization problems. The maximum likelihood estimates and corresponding standard errors in parentheses, estimated Akaike Information Criteria (AIC), corrected Akaike Information Criteria, Hannan-Quin information criteria (HQIC) and Bayesian Information Criteria (BIC) are reported in Tables 4, 5, 6, 7, 8 and 9. We present the summary of preliminary statistics of the datasets considered in Table 3. The second, third and fourth columns of Table 3 indicate the mean, median and standard deviations, while the fifth and sixth give kurtosis and skewness, in that order. It is observed that all the datasets are positively skewed. Further, data sets I, III and V are platykurtic, while data sets II and IV are leptokurtic.

Table 3: Some Summary Measures of the Data Sets

Data set	Mean	median	Standard deviation	Kurtosis	Skewness
I	3.61	3.49	0.80	2.02	0.29
II	5.85	4.45	4.61	3.58	1.04
III	1.90	1.70	0.70	2.92	1.72
IV	10.49	8.32	8.84	4.86	1.38
V	30.81	29.90	7.25	2.29	0.41

Total Time on Test (TTT) plot is a standard graphical method for verifying the shape of the failure rate function for the considered dataset. Aarset [30] mentioned that the hazard rate function is decreasing (increasing) if the TTT plot is convex (concave). A bathtub-shaped hazard is obtained when first convex followed by concave. The hazard rate function is upside down, otherwise known as unimodal, if the TTT plot commences with concave above and convex below the 45^0line. The failure rate is constant if the shape of the TTT plot is straight diagonal. The TTT plot of a given dataset displays a decreasing-increasing-decreasing (DID) hazard rate function when it first commences convex below the45^0line, followed by concave above the 45^0line and finally convex below the 45^0line. Let T denote a positive random variable, representing survival time. The empirical TTT curve is constructed by plotting

$$T\left(\frac{r}{n}\right)=\frac{\{\sum_{i=1}^{r}Y_{1:n}+(n-r)Y_{r:n}\}}{\sum_{i=1}^{n}Y_{i:n}}\text{ against }\frac{r}{n},$$

$r=1,2,3,\dots,n$ and $Y_{i:n}(i=1,2,3,\dots,n)$ are order statistics of the sample.

Data set I: The first data set measures the acidity of rainfalls for forty days in the state of Minnesota. This data is documented in Ross [31] and the values are 3.71, 4.23, 4.16, 2.98, 3.23, 4.67, 3.99, 5.04, 4.55, 3.24, 2.80, 3.44, 3.27, 2.66, 2.95, 4.70, 5.12, 3.77, 3.12, 2.38, 4.57, 3.88, 2.97, 3.70, 2.53, 2.67, 4.12, 4.80, 3.55, 3.86, 2.51, 3.33, 3.85, 2.35, 3.12, 4.39, 5.09, 3.38,2.73,3.07.

Table 4: Estimation And Information Criteria Results for First Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	0.6635694 (0.066637)	168.1453	168.2505	169.8341	168.7559
Komal (λ)	0.4385906 (0.049168)	171.1621	171.2674	172.851	171.7728
NewXLindley (β)	0.44568 (0.060104)	174.8766	174.9818	176.5655	175.4872
Akash(γ)	0.7172721 (0.062988)	154.4461	154.5514	156.135	155.0568

Data set II: This dataset, obtained from Patil and Rao [32], represents the distances from the transect line for the 68 stakes detected in walking $L = 100m$ and searching $w = 20m$on each of the lines. The observations are subsequently given as 2.0, 0.5, 10.4, 3.6, 0.9, 1.0, 3.4, 2.9, 8.2, 6.5, 5.7, 3.0, 4.0, 0.1, 11.8, 14.2, 2.4, 1.6, 13.3, 6.5, 8.3, 4.9, 1.5, 18.6, 0.4, 0.4, 0.2, 11.6, 3.2, 7.1, 10.7, 3.9, 6.1, 6.4, 3.8, 15.2, 3.5, 3.1, 7.9, 18.2, 10.1, 4.4, 1.3, 13.7, 6.3, 3.6, 9.0, 7.7, 4.9, 9.1, 3.3, 8.5, 6.1, 0.4, 9.3, 0.5, 1.2, 1.7, 4.5, 3.1, 3.1, 6.6, 4.4, 5.0, 3.2, 7.7, 18.2, 4.1.

Table 5: Estimation And Information Criteria Results for Second Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	0.4107479 (0.03154314)	373.5355	373.5961	375.755	374.415
Komal (λ)	0.2948028 (0.025170)	374.2961	374.3567	376.5156	375.1756
NewXLindley (β)	0.2585518 (0.00123)	374.7177	374.7784	376.9373	375.5972
Akash(γ)	0.4775827 (0.03256)	382.0654	382.126	384.2849	382.9449

Environmental Data Sets

The ML estimates of the parameters and their associated standard errors in parentheses and model selection criteria for fitted distributions are reported in Tables 4 and 5. Given that it possesses the minimum values of AIC, CAIC, BIC and HQIC, the Akash distribution satisfactorily fits the data set I as shown in Table 4. On the other hand, the XGamma distribution offers the best fit to the transect line data set as indicated in Table 5.

Data set III: The time-to-pain relief (in minutes) data set on 20 observations of patients receiving analgesics. Pain assessment and management pose significant challenges to numerous anesthesiologists in less endowed countries where monitors and experts are deficient. Analgesics (pain killers) play a critical role in relieving this pain. Measuring time-to-pain-relief is critical for evaluating the efficacy of analgesics. The data on relief times (in hours) of 20 patients receiving an analgesic are given by 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6,2.0. The data were obtained from Gross and Clark [33]. It has also been recently used by Alamgir et al. [34].

Table 6: Estimation And Information Criteria Results for Third Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	1.107468 (0.16943)	65.01649	65.23871	66.01222	65.21087
Komal (λ)	0.740371 (0.12074)	64.35929	64.58152	65.35503	64.55367
NewXLindley (β)	0.842221 (0.16117)	63.21309	63.43531	64.20882	63.40747
Akash(γ)	1.156923 (0.14555)	61.52261	61.74483	62.51834	61.71699

Analgesic data set

Table 6 gives the ML estimates (standard errors in parentheses) and goodness of fit statistics for the Analgesic data (third data set). It is observed that Akash distribution outshines all its competitors as it records the lowest values for all the metric measures.

Data set IV: The subsequent data set regarding failure times of 20 electric bulbs explicated by the scholars Murthy et al. [35] is considered and the observations encompasses 1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92.

Survival data set

Table 7 lists the ML estimates and discrimination statistics for the failure times of 20 electric bulbs (fourth data set). It is noted that the new XLindley distribution has the lowest value of all the statistics. It therefore suggests that the new XLindley distribution provides a sufficient fit to the empirical data.

Table 7: Estimation And Information Criteria Results for Fourth Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	0.2445704 (0.033720)	135.8057	136.028	136.8015	136.0001
Komal (λ)	0.1745502 (0.027502)	135.3335	135.5557	136.3292	135.5278
NewXLindley (β)	0.1440403 (0.028176)	135.1183	135.3405	136.1141	135.3127
Akash(γ)	0.2786616 (0.035566)	140.4705	140.6928	141.4663	140.6649

Data Set V: The fifth data set is the strength data of glass of the aircraft window documented in Fuller et al [36]. The observations are as follows: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Table 8: Estimation and Information Criteria Results for Fifth Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	0.09375 (0.00981)	246.5469	246.6849	247.9809	247.0144
Komal (λ)	0.062826 (0.00797)	256.0901	256.228	257.524	256.5575
NewXLindley (β)	0.05222 (0.00799)	268.9397	269.0776	270.3736	269.4071
Akash(γ)	0.097062 (0.01004)	242.6818	242.8197	244.1157	243.1492

Engineering data set

The ML estimates of the parameters and their corresponding standard errors in parentheses and model selection criteria for all competitive distributions are presented in Table 8. As a rule of thumb, the probability distribution with the lowest values of the information criteria is the best-suited model for analyzing the considered dataset. Based on the numerical illustration in Table 8, it is seen that the Akash distribution has proven its dominance and applicability in the area of statistical modeling of strength data of glass of the aircraft window.

Table 9: Estimation and Information Criteria Results for Sixth Data

Model	$\hat{\delta}(SE)$	AIC	CAIC	BIC	HQIC
XGamma (δ)	0.064750 (0.006964)	354.6967	354.8258	356.1932	355.2003
Komal (λ)	0.0478369 (0.005882)	339.4509	339.58	340.9474	339.9544
New XLindley (β)	0.0347278 (0.00539)	317.2746	317.4036	318.7711	317.7781
Akash(γ)	0.073256 (0.00735)	385.0491	385.1781	386.5456	385.5526

Sixth Data

The sixth dataset represents the survival times of 33 patients (in weeks) with the condition acute myelogenous leukemia. The observations are as follows 156, 65, 100, 16, 134, 108, 4, 121, 39, 56, 143, 26, 1, 22, 1, 65, 5, 56, 17, 65, 7, 22, 16, 3, 2, 4, 3, 4, 3, 8, 30, 43, 4. The data were taken from Affify et al. [37].

Table 9 lists the MLEs of and the various information criteria of the fitted models along with their standard errors. From Table 9, we notice that the New XLindley distribution shows huge flexibility in modeling the survival times of the patients. The outcomes indicate that the New XLindley distribution is the leading candidate among competing models, making it a preferred option for statistical modeling of the considered dataset.

Seventh Data

The data represents the failure times of 20 electric bulbs discussed in Murthy et. al. [35]. The considered observations are as follows:
1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92.

Table 10: Estimation and Goodness-of-Fit Results for Seventh Data

Model	$\hat{\delta}(SE)$	AIC	$-2LogL$	K-S	p-value
XGamma (δ)	0.2446	135.81	133.81	0.1337	0.822
Komal (λ)	0.1746	135.33	133.33	01125	0.938
New XLindley (β)	0.1440	135.12	133.12	0.1100	0.947
Akash(γ)	0.2787	140.47	138.47	0.1485	0.716

The values of maximum likelihood estimates of the parameters, $-2LogL$, Kolmogorov-Smirnov (K-S) with its p-values and AIC for the considered models for the considered observations have been calculated and displayed in Table 10.

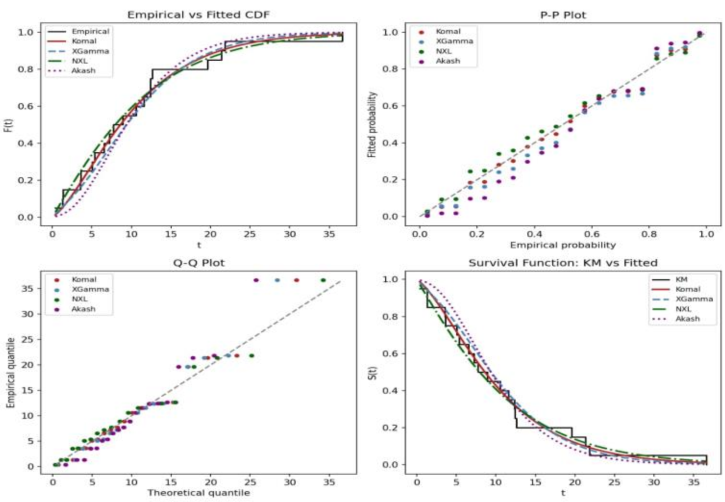

Fig. 1: Plots of Empirical, Pp, QQ and KM.

The visual plots of the considered models for the given dataset have been presented in Figure 1. The outcomes in Table 10 and the figure show that the NXL distribution offers the best fit relative to the competitors and therefore can be considered as the most suitable lifetime distribution for modeling lifetime data emanating from engineering settings. The NXL distribution provides another option for a practitioner to use for data analysis purposes.

4. Conclusions

This research article studied well–established and widely used one-parameter distributions that emerged from the convex amalgamation of exponential and gamma distributions, which is the main goal of this paper. It is important to mention that this work excludes two or more-parameter distributions obtained from the convex combination of exponential and gamma distributions. The reliability characteristics of each distribution have been presented. The visual representations of the PDF as well as the failure functions have been discussed. Furthermore, widespread methods for selecting the most efficient model have been explicated. The distributions modeling analyses have been investigated based on numerous applied real data sets to demonstrate their fitting abilities. This endeavor leverages recently introduced applied one-parameter probability distributions, namely Akash distribution, Komal distribution, XGamma distribution, and new XLindley distribution. These inquiries have explored a wide array of data sets in the context of environmental science, biomedical science, reliability engineering and survival studies. Of all the varied avenues of inquiry into these data sets, the Akash distribution holds particular significance as it offers a reasonable parametric fit to environmental science, biomedical science, and reliability engineering datasets. Therefore, practitioners are encouraged to use the Akash distribution in order to get effective outcomes when it comes to lifetime data.

5. Future Projects

The scope of possible future projects would encompass: (1) to propose more new extensions of these distributions which have not been attempted; (2) to apply the new extensions into a real-life data sets; (3) to estimate the parameters of the new extensions using both non-Bayesian, Bayesian and E-Bayesian techniques; (4) to review the multi variate, matrix variate and complex variate of the generalized distribution; and (5) to propose multivariate extensions, censored likelihood inference, regression versions and Bayesian hierarchical extensions.

Data Availability

The datasets considered have been provided in the main body of this manuscript.

Conflicts of Interest

The author hereby declares no conflict of interest.

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