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Research paper

A New Probability Distribution with Applications to Relief Times, Fiber Stress and Aircraft Failure Data

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Abstract

In this article, a new flexible statistical model is developed, known as the Khalil New Generalized Fréchet (KNG-Fréchet) distribution, which extends the classical Fréchet distribution to better capture complex data behaviors. Several statistical properties of the proposed distribution, including its quantile function, moments, moment generating function, and characteristic function, are thoroughly explored. Parameter estimation is performed using the maximum likelihood estimation (MLE) method. A simulation study is conducted to evaluate the performance of the MLEs in terms of bias and mean squared error (MSE). To validate the effectiveness of the proposed distribution in real-world scenarios, the proposed distribution is applied to three real-life datasets and compared with existing models using goodness-of-fit criteria. The results demonstrate that the proposed model provides a good fit compared to existing alternative distributions, consistently outperforming them in terms of goodness-of-fit measures, parameter stability, and overall flexibility when applied to different datasets and sample sizes.

Keywords: *Fréchet Distribution; Moments; Moment Generating Function; Maximum Likelihood Estimation; Simulation Study; Goodness of Fit.*

1. Introduction

The Fréchet distribution, named after Maurice Fréchet, is a type of probability distribution that models extreme events. The Fréchet distribution has a heavy right tail, making it suitable for modeling extreme events. The Fréchet distribution is max-stable, meaning that the maximum of a sample of independent and identically distributed random variables follows the same distribution. The cumulative distribution function (cdf) and probability density function (pdf) of the Fréchet distribution are given by

$$F(x) = exp\left(-x^{-\delta}\right); \delta > 0, x > 0a \tag{1}$$

And

$$f(x) = \delta x^{-1-\delta}\, exp\left(-x^{-\delta}\right) \delta > 0, x > 0 \tag{2}$$

Recent research work on Fréchet distribution. Ramos et.al [1] introduce The Fréchet distribution: estimation and application, Afify et.al [2] introduce The Weibull Fréchet distribution and its applications, Nadarajah et.al [3] introduce The exponentiated Fréchet distribution, Barreto-Souza et.al [4] introduce Some results for beta Fréchet distribution, Mahmoud et.al [5] introduce On the transmuted Fréchet distribution, Krishna et.al [6] introduce The Marshall-Olkin Fréchet distribution, Yousof et.al [7] introduce On six-parameter Fréchet distribution: properties and applications, Shafiq et.al [8] introduce A new modified Kies Fréchet distribution: Applications of mortality rate of Covid-19, Suleiman et.al [9] introduce A novel extension of the Fréchet distribution: statistical properties and application to groundwater pollutant concentrations, Elbatal et.al [10] introduce Transmuted exponentiated Fréchet distribution: properties and applications, Phaphan et.al [11] study the Properties and maximum likelihood estimation of the novel mixture of Fréchet distribution, Abbas et.al [12] introduce Bayesian Analysis of Three-Parameter Fréchet Distribution with Medical Applications, Korkmaz et.al [13] Some theoretical and computational aspects of the odd Lindley Fréchet distribution, Gómez et.al [14] introduce An extension of the Fréchet distribution and applications, Oguntunde et.al [15] introduce The Gompertz Fréchet distribution: properties and applications. As compared to these distributions, the proposed distribution employs extra shape parameters to achieve greater flexibility in terms of accommodating skewness and kurtosis. In contrast to the mixing approach, the proposed distribution maintains the simplicity of being a one-component distribution, whereas, in comparison to the Bayesian method, it avoids the specification of priors and computation burdens.
Despite the demonstrated flexibility of the proposed model for right-skewed and heavy-tailed data, the existing literature lacks explicit identification of scenarios where such generalized Fréchet-type distributions are most effective. In particular, these models are especially useful in reliability engineering and risk modeling contexts, where data often exhibit extreme variability and heavy tails. The proposed

model is particularly advantageous when existing classical distributions fail to adequately capture tail behavior while maintaining computational simplicity.
This study develops a new generalization of the Fréchet distribution. The new distribution is called the Khalil new generalized Fréchet distribution. To derive statistical properties of the new distribution. To estimate the parameters of the new distribution using the maximum likelihood estimation method. To access the performance of the parameter estimates using simulation. To check the potential of the new distribution using information criteria, i.e., AIC, BIC, CAIC, and HQIC. To apply the new distribution to real data.
The paper is classified as follows: in Section 2, we proposed a new distribution, plots of the probability density function, cumulative density function. In Section 3, we derive some important statistical properties, including the quartile function, moments, moment generating function, and characteristic function. In Section 4, the maximum likelihood estimation method is used to estimate the parameters of the new distribution. Section 5 represents a simulation study. Section 6 represents the application of the new distribution, and Section 7 finally concludes the paper.

2. The New Distribution

Salahuddin et al (2021) proposed a new family of probability distribution called the Khalil new generalized (KNG) family of distribution. The cumulative distribution function (cdf) of the KNG family is given as;

$$G(x) = \frac{exp\left(-\alpha\left(F(x)\right)^{\beta}\right)-1}{exp(-\alpha)-1}, \alpha, \beta > 0 \tag{3}$$

Where $F(x)$is the cdf of the baseline distribution. The probability density function (pdf), survival function (sf), and hazard function (hf) are defined as;

$$g(x) = \frac{\alpha\beta\, exp\left(-\alpha\left(F(x)\right)^{\beta}\right)\left(F(x)\right)^{\beta-1}}{1-exp(-\alpha)}, \alpha, \beta > 0 \tag{4}$$

$$S(x) = \frac{\alpha\beta\, exp\left(-\alpha\left(F(x)\right)^{\beta}\right)-exp(-\alpha)}{exp(-\alpha)-1} \tag{5}$$

$$H(x) = \frac{\alpha\beta\, exp\left(-\alpha\left(F(x)\right)^{\beta}\right)\left(F(x)\right)^{\beta-1}}{exp\left(-\alpha\left(F(x)\right)^{\beta}\right)-exp(-\alpha)} \tag{6}$$

Now using the cdf of the Fréchet distribution. The new model, termed the Khalil new generalized Fréchet (KNGF) distribution. The cdf and pdf of the KNGF distribution are defined as

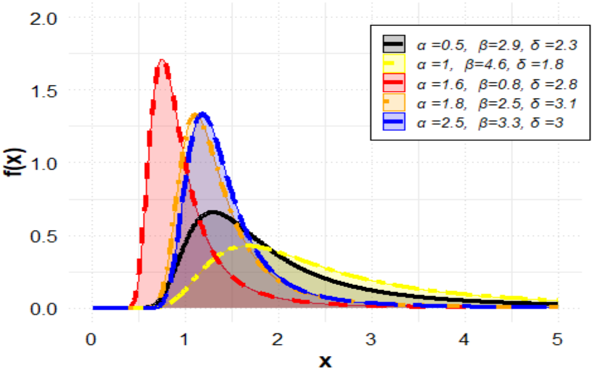
$$G(x) = \frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)-1}{exp(-\alpha)-1}, \alpha, \beta, \delta > 0, x > 0 \tag{7}$$

And the corresponding PDF

$$g(x) = \frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)\left(exp(-x^{-\delta})\right)^{\beta}x^{-1-\delta}\alpha\beta\delta}{1-\exp(-\alpha)}, \alpha, \beta, \delta, x > 0 \tag{8}$$

Figure 1 represents the plots for the PDF and CDF of the KNGF distribution under various parameter settings.

(A)



(B)

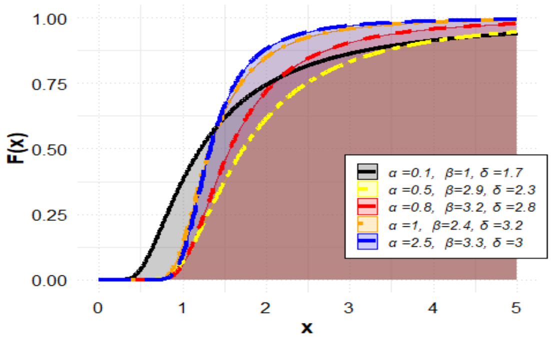


Fig. 1: Plots for the CDF and PDF.

The survival function (sf) and hazard function (hf) of the KNGF distribution are defined as

$$S(x) = 1 - \frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)-1}{exp(-\alpha)-1} \tag{9}$$

And

$$H(x) = \frac{\frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)\left(exp\left(-x^{-\delta}\right)\right)^{\beta}x^{-1-\delta}\alpha\beta\delta}{1-\exp(-\alpha)}}{1-\frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)-1}{exp(-\alpha)-1}} \tag{10}$$

Figure 2 represents the plots for the survival function and hazard function of the KNGF distribution under various parameter settings.

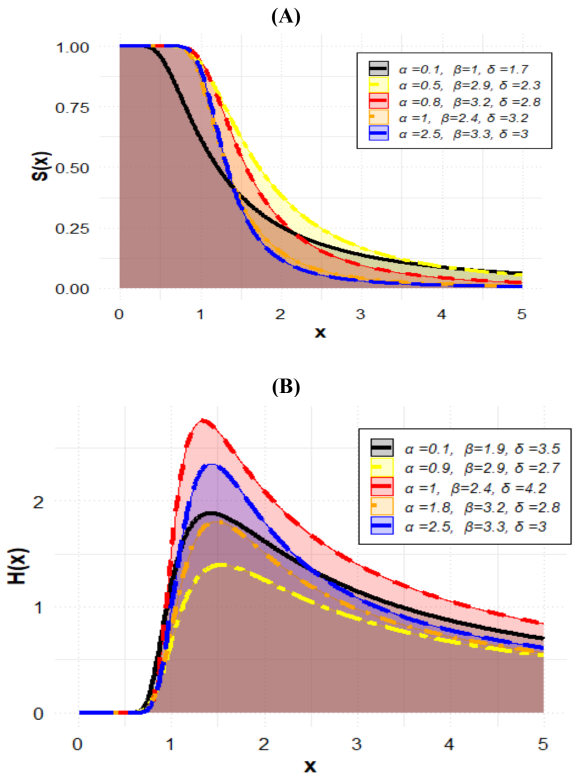


Fig. 2: Plots for the SF and HF.

Table 1: Descriptive Statistics of the KNGF Distribution for Β = 0.2 and Δ = 1.0.

α	Mean	Standard Deviation	Skewness	Kurtosis
0.1	1.523413	13.69433	36.67544	1785.513
0.5	1.283522	12.34281	40.66791	2196.618
1.0	1.025557	10.73953	46.69658	2898.681
1.3	0.892554	9.830287	51.00117	3458.626
2.0	0.640900	7.892450	63.45466	5359.382
2.7	0.4603551	6.229358	80.30611	8592.903
3.0	0.4009152	5.602349	89.25008	10618.44

The numerical findings in Table 1 display the descriptive statistics of the KNGF distribution for several values of the α. From the results, it is clear that increasing the values of α the mean and standard deviation decrease, while the skewness and kurtosis increasing by increasing the values of α showing that the proposed model is skewed to the right.

3. Statistical Properties

In this section, several statistical properties of the KNGF distribution are derived, including the quantile function, moments, moment generating function, characteristics function.

3.1. Quantile function

The quantile function for the KNGF distribution is found by simplifying eq (5) for x.

$$x = G(u)$$
$$u = G^{-1}(x)$$
$$u = \frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right) - 1}{exp(-\alpha) - 1}$$

$$u(exp(-\alpha) - 1) + 1 = \exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\alpha\right)$$

Taking the natural log on both sides

$$log[u(exp(-\alpha) - 1) + 1] = -\alpha\, exp\left(-\beta x^{-\delta}\right)$$

$$\frac{log\left[\frac{log[u(exp(-\alpha)-1)+1]}{-\alpha}\right]}{-\beta} = x^{-\delta}$$

$$\left\{\frac{log\left[\frac{log[u(exp(-\alpha)-1)+1]}{-\alpha}\right]}{-\beta}\right\}^{-\frac{1}{\delta}} = x \quad (11)$$

The quantile function for the KNGF distribution is defined in eq (11), can be used for the purpose of simulation and is used to obtain the quantiles, deciles, and percentiles.

3.2. Moments

Let a random variable. x follows the KNGF distribution, then the r^{th} moment is defined as

$$\mu_r' = E(x^r) = \int_0^\infty x^r g(x)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta}\right)\left(\exp(-x^{-\delta})\right)^{\beta} x^{-\delta-1}\alpha\beta\delta}{1-\exp(-\alpha)}dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\exp\left(-e^{-\beta x^{-\delta}}\right)e^{-\beta x^{-\delta}} x^{-\delta-1}\alpha\beta\delta}{1-\exp(-\alpha)}dx$$

Using the substantiation method

$$u = -e^{-\beta x^{-\delta}}, -du = e^{-\beta x^{-\delta}}\delta\beta x^{-\delta-1}$$

$$\log(u) = -\beta x^{-\delta}, -\log(u) = \beta x^{-\delta}, -\frac{1}{\beta}\log(u) = x^{-\delta}, x = \left(-\frac{1}{\beta}\log(u)\right)^{-\frac{1}{\delta}}$$

The new ranges for the variable u is

if $x = 0$, then, $u = \infty$

if $x = \infty$, then, $u = 0$

The integral reduces to

$$\mu_r' = \frac{\alpha}{1-e^{\alpha}}\int_\infty^0 \left(-\frac{1}{\beta}\log(u)\right)^{-\frac{r}{\delta}} - du$$

$\mu_r' = \frac{\alpha}{1-e^{\alpha}} \int_0^{\infty} \left(-\frac{1}{\beta}\log(u)\right)^{-\frac{r}{\delta}} du$

$\mu_r' = \frac{\alpha\beta^{-\frac{r}{s}}}{1-e^{\alpha}} \int_0^{\infty} (-\log(u))^{-\frac{r}{\delta}} du$

Again, by the substitution method

$t = -\log(u), u = e^{-t}, du = -e^{-t}dt$

The new ranges for the variable u is

$\text{if } u = 0, \text{then}, t = \infty$

$\text{if } u = \infty, \text{then}, t = 0$

$\mu_r' = \frac{\alpha\beta^{-\frac{r}{s}}}{1-e^{\alpha}} \int_{\infty}^{0} t^{-\frac{r}{s}} - e^{-t}dt$

$\mu_r' = \frac{\alpha\beta^{-\frac{r}{s}}}{1-e^{\alpha}} \int_0^{\infty} t^{-\frac{r}{s}} e^{-t}dt$

Using the gamma function

$\mu_r' = \frac{\alpha\beta^{-\frac{r}{s}}}{1-e^{\alpha}} \Gamma\left(1-\frac{r}{s}\right), \frac{r}{s} < 1$ (12)

For $r = 1$ The mean of the KNGF distribution is given by

$\mu_1' = \frac{\alpha\beta^{-\frac{1}{s}}}{1-e^{\alpha}} \Gamma\left(1-\frac{1}{s}\right)$ (13)

For $r = 2$ The second raw moment of the KNGF distribution is given by

$\mu_2' = \frac{\alpha\beta^{-\frac{2}{s}}}{1-e^{\alpha}} \Gamma\left(1-\frac{2}{s}\right)$ (14)

The variance of the KNGF distribution is given as

$\mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2' = \frac{\alpha\beta^{-\frac{2}{s}}}{1-e^{\alpha}} \Gamma\left(1-\frac{2}{s}\right) - \left[\frac{\alpha\beta^{-\frac{1}{s}}}{1-e^{\alpha}} \Gamma\left(1-\frac{1}{s}\right)\right]^2$ (15)

3.3. Moment generating function

The moment generating function of the KNGF distribution is acquired using

$$M_x(t) = E(exp(tx)) = \int_0^{\infty} exp(tx)\, g(x)dx$$

Utilizing Taylor's series $exp(tx) = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$In the above integral, the moment generating function is obtained as follows:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{k!} \int_0^{\infty} x^r g(x)dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \mu_r'$$

$M_x(t) = \sum_{r=0}^{\infty} \frac{\alpha\beta^{-\frac{r}{s}} t^r}{(1-e^{\alpha})r!} \Gamma\left(1-\frac{r}{s}\right), \frac{r}{s} < 1$ (16)

3.4. Characteristics generating function

The characteristic generating function of the KNGF distribution is acquired using

$$\phi_x(t) = E(exp(itx)) = \int_0^{\infty} exp(itx)\, g(x)dx$$

Utilizing Taylor's series $exp(itx) = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!}$In the above integral, the moment generating function is obtained as follows:

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{k!} \int_0^{\infty} x^r g(x)dx$$

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r^{'}$$

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{\alpha\beta^{-\frac{r}{s}}(it)^r}{(1-e^{\alpha})r!} \Gamma\left(1-\frac{r}{s}\right), \frac{r}{s} < 1 \quad (17)$$

4. Estimation

The parameters of the KNGF distribution are unknown, and they are estimated using the observed data. For that purpose, the usual method, maximum likelihood estimation (MLE), is used to estimate the unknown parameters of the KNGF distribution. Let $x_1, x_2, x_3, \ldots\ldots, x_n$be n random sample from the KNGF distribution, then the likelihood function is

$$L = \prod_{i=1}^{n} \left[\frac{\exp\left(-\left(e^{-x^{-\delta}}\right)^{\beta} \alpha\right)\left(exp(-x^{-\delta})\right)^{\beta} x^{-1-\delta}\alpha\beta\delta}{1-\exp(-\alpha)} \right] \quad (18)$$

The loglikelihood function is given by

$$log\, L = n\, log(\alpha\beta\delta) - \alpha \sum_{i=1}^{n} e^{-\beta x_i^{-\delta}} + \sum_{i=1}^{n} x_i^{-1-\delta} - \beta \sum_{i=1}^{n} x_i^{-\delta} - n\, log(1-e^{-\alpha}) \quad (19)$$

For parameter estimation, the above non-linear equation is simplified by taking first partial derivatives with respect to each unknown parameter and equating them to zero.

$$\frac{\partial\, log\, L}{\partial \alpha} = n\left(\frac{1}{1-\exp(\alpha)} + \frac{1}{\alpha}\right) - \sum_{i=1}^{n} \exp\left(-\beta x_i^{-\delta}\right) = 0 \quad (20)$$

$$\frac{\partial\, log\, L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^{-\delta} - \alpha \sum_{i=1}^{n} - \exp\left(-\beta x_i^{-\delta}\right) x_i^{-\delta} = 0 \quad (21)$$

$$\frac{\partial\, log\, L}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} - log(x_i)\, x_i^{-1-\delta} - \beta \sum_{i=1}^{n} - log(x_i)\, x_i^{-\delta} - \alpha\beta \sum_{i=1}^{n} \exp\left(-\beta x_i^{-\delta}\right) log(x_i)\, x_i^{-\delta} = 0 \quad (22)$$

On solving the equations (20), (21), and (22), the MLEs of the unknown parameters i-e $\alpha, \beta$and δ are acquired. The analytical solution of the above equations is not possible. Thus, to solve these nonlinear equations, some iterative procedures are used.
The inverse Fisher information observed covariance matrix.(FI^{-1}) defined as;

$$FI^{-1} = \begin{bmatrix} \frac{\partial^2 log\, L}{\partial \alpha^2} & \frac{\partial^2 log\, L}{\partial \alpha\, \partial \beta} & \frac{\partial^2 log\, L}{\partial \alpha\, \partial \delta} \\ \frac{\partial^2 log\, L}{\partial \beta\, \partial \alpha} & \frac{\partial^2 log\, L}{\partial \beta^2} & \frac{\partial^2 log\, L}{\partial \beta\, \partial \delta} \\ \frac{\partial^2 log\, L}{\partial \delta\, \partial \alpha} & \frac{\partial^2 log\, L}{\partial \delta\, \partial \beta} & \frac{\partial^2 log\, L}{\partial \delta^2} \end{bmatrix}^{-1} \quad (23)$$

$$FI^{-1} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\beta}) & cov(\hat{\alpha}, \hat{\delta}) \\ cov(\hat{\beta}, \hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta}, \hat{\delta}) \\ cov(\hat{\delta}, \hat{\alpha}) & cov(\hat{\delta}, \hat{\beta}) & var(\hat{\delta}) \end{bmatrix}^{-1} \quad (24)$$

$$\frac{\partial\, log\, L}{\partial \alpha^2} = n\left(\frac{e^{\alpha}}{(1-e^{\alpha})^2} - \frac{1}{\alpha^2}\right) \quad (25)$$

$$\frac{\partial\, log\, L}{\partial \beta^2} = \frac{n}{\beta^2} + \alpha \sum_{i=1}^{n} \exp\left(-\beta x_i^{-\delta}\right) x_i^{-2\delta} \quad (26)$$

$$\frac{\partial\, log\, L}{\partial \delta^2} = \sum_{i=1}^{n} log(x_i)\, x_i^{-\delta} + \alpha \sum_{i=1}^{n} \left(-\exp\left(-\beta x_i^{-\delta}\right)\beta\, log(x_i)\, x_i^{-2\delta} + \exp\left(-\beta x_i^{-\delta}\right) log(x_i)\, x_i^{-\delta}\right) \quad (27)$$

$$\frac{\partial\, log\, L}{\partial \alpha\, \partial \beta} = \sum_{i=1}^{n} \exp\left(-\beta x_i^{-\delta}\right) x_i^{-\delta} \quad (28)$$

$$\frac{\partial\, log\, L}{\partial \alpha\, \partial \delta} = -\beta \sum_{i=1}^{n} \exp\left(-\beta x_i^{-\delta}\right) log(x_i)\, x_i^{-\delta} \quad (29)$$

$$\frac{\partial\, log\, L}{\partial \beta\, \partial \delta} = -\sum_{i=1}^{n} - log(x_i)\, x_i^{-\delta} - \alpha \sum_{i=1}^{n} \left(\exp\left(-\beta x_i^{-\delta}\right)\beta\, log(x_i)\, x_i^{-2\delta} + \exp\left(-\beta x_i^{-\delta}\right) log(x_i)\, x_i^{-\delta}\right) \quad (30)$$

The asymptotic confidence interval can be constructed for unknown parameters of the KNGF distribution with the assumption that the MLEs of the unknown parameters are approximately normal with mean.(α, β, δ). Hence, the asymptotic confidence interval $(1-\theta)100\%$ for the unknown parameters can be obtained as;

$$\hat{\alpha} \pm Z_{\frac{\theta}{2}}\sqrt{\mathrm{var}(\hat{\alpha})}\,,\hat{\beta} \pm Z_{\frac{\theta}{2}}\sqrt{\mathrm{var}(\hat{\beta})}\,,\hat{\delta} \pm Z_{\frac{\theta}{2}}\sqrt{\mathrm{var}(\hat{\delta})}$$

Where $Z_{\frac{\theta}{2}}$ is the upper critical value from the standard normal distribution.

5. Simulation

This section offers a simulation study for the parameters of the KNGF distribution. A simulation study is performed to assess the performance of the MLEs. For this purpose, data are simulated from the KNGF distribution using the R programming language. A total of 1000 samples are generated for each sample size. n = 20, 70, 150, 300, 450, and 600. The parameter values are considered under different configurations, namely$(\alpha = 0.5, \beta = 2.1, \delta = 2.0)$, $(\alpha = 4.5, \beta = 2.7, \delta = 5.0)$, and $(\alpha = 3.0, \beta = 4.0, \delta = 7.0)$. Random samples are generated using the quantile function of the proposed distribution. The MLEs of the parameters are obtained via numerical optimization using the BFGS algorithm implemented in the optim function in R. The Biases and Mean square error (MSEs) are calculated using the following expressions

$$\mathrm{Bias} = \frac{1}{w}\sum_{i=1}^{w}(\hat{\gamma}_i - \gamma)\,, \mathrm{MSE} = \frac{1}{w}\sum_{i=1}^{w}(\hat{\gamma}_i - \gamma)^2$$

The results of the simulation study are revealed in Tables 2, 3, and 4.

Table 2: Simulation Results for $\alpha = 0.5$, $\beta = 2.1$ and $\delta = 2.0$

Parameters	n	MLE	Bias	MSEs
$\hat{\alpha}$	20	1.16877	0.66877	44.72535
$\hat{\beta}$		2.42473	0.32473	10.54504
$\hat{\delta}$		2.08728	0.08728	0.761845
$\hat{\alpha}$	70	0.83481	0.33481	11.21002
$\hat{\beta}$		2.25021	0.15021	2.256426
$\hat{\delta}$		1.92892	-0.07108	0.505256
$\hat{\alpha}$	150	0.36360	-0.13639	1.860419
$\hat{\beta}$		2.10983	0.00982	0.009659
$\hat{\delta}$		2.01097	0.01096	0.012020
$\hat{\alpha}$	300	0.59033	0.09033	0.816025
$\hat{\beta}$		2.13334	0.03334	0.111174
$\hat{\delta}$		1.96250	-0.03749	0.140623
$\hat{\alpha}$	450	0.46810	-0.03189	0.101737
$\hat{\beta}$		2.09378	-0.00621	0.003864
$\hat{\delta}$		1.98206	-0.01793	0.032174
$\hat{\alpha}$	600	0.51614	0.016141	0.026054
$\hat{\beta}$		2.08063	-0.01937	0.037540
$\hat{\delta}$		2.01474	0.014735	0.021715

Table 3: Simulation Results for $\alpha = 4.5$, $\beta = 2.7$ and $\delta = 5.0$

Parameters	n	MLE	Bias	MSEs
$\hat{\alpha}$	20	4.98886	0.48886	23.89895
$\hat{\beta}$		2.42888	-0.271111	7.35014
$\hat{\delta}$		6.63342	1.63342	266.8077
$\hat{\alpha}$	70	2.89705	-1.60294	256.9425
$\hat{\beta}$		2.12862	-0.571375	32.64699
$\hat{\delta}$		6.77371	1.773716	314.6069
$\hat{\alpha}$	150	3.44217	-1.05782	111.8997
$\hat{\beta}$		2.32232	-0.37767	14.26399
$\hat{\delta}$		6.14281	1.14281	130.602
$\hat{\alpha}$	300	3.75647	-0.743521	55.28243
$\hat{\beta}$		2.45574	-0.24425	5.96600
$\hat{\delta}$		5.69890	0.69899	48.85995
$\hat{\alpha}$	450	3.78691	-0.713083	50.84885
$\hat{\beta}$		2.48857	-0.211430	4.470279
$\hat{\delta}$		5.59544	0.595445	35.45547
$\hat{\alpha}$	600	4.30184	-0.198157	3.926641
$\hat{\beta}$		2.63339	-0.0666042	0.443613
$\hat{\delta}$		5.20423	0.2042343	4.171164

Table 4: Simulation Results for $\alpha = 3.0$, $\beta = 4.0$ and $\delta = 7.0$

Parameters	n	MLE	Bias	MSEs
$\hat{\alpha}$	20	2.783192	-0.216807	4.70056
$\hat{\beta}$		3.687209	-0.312791	9.78381
$\hat{\delta}$		8.791078	1.791078	320.7960
$\hat{\alpha}$	70	1.545582	-1.45441	211.5331
$\hat{\beta}$		3.458325	-0.541674	29.34114
$\hat{\delta}$		9.075004	2.075004	430.5643
$\hat{\alpha}$	150	2.061098	-0.938902	88.1536
$\hat{\beta}$		3.608493	-0.391506	15.3277
$\hat{\delta}$		8.308742	1.308742	171.2805
$\hat{\alpha}$	300	1.880515	-1.11948	125.3248
$\hat{\beta}$		3.524407	-0.475593	22.61888
$\hat{\delta}$		7.977473	0.977473	95.54537
$\hat{\alpha}$	450	2.671884	-0.328116	10.7660
$\hat{\beta}$		3.883415	-0.116585	1.35919
$\hat{\delta}$		7.454513	0.454512	20.65819
$\hat{\alpha}$	600	2.635635	-0.3643646	13.27616
$\hat{\beta}$		3.859197	-0.1408027	1.982541
$\hat{\delta}$		7.515206	0.515206	26.54372

The results of the simulation study indicate that, as the sample size increases, the parameter estimates approach their true values. Moreover, it is observed that both the Bias and the mean-squared error (MSE) decrease with increasing sample size. This behavior confirms the consistency and efficiency of the maximum likelihood estimators. Therefore, it can be concluded that the maximum likelihood estimation method provides reliable and accurate parameter estimates for the KNGF distribution, particularly for moderate to large sample sizes.

6. Applications

In this section, three datasets are used to judge the performance of the advised model. The first dataset consists of 20 observations, which are used by Gross and Clark [17]. The data is about the relief times of twenty patients receiving painkillers. The observations of the dataset are: 1.1, 1.9, 1.7,1.4, 1.3, 1.7, 2.7, 4.1, 1.8, 1.5, 1.8, 1.6, 2.2, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The second dataset consists of 66 observations, which are used by Cordeiro and Lemonte [18] and Al-Aqtash et al. [19]. The dataset is about the breaking stress of carbon fibers of 50 mm length. The observations of the dataset are: 0.39, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.97, 0.85, 1.08, 1.25,3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56 ,4.42, 4.70, 4.90.

The third dataset consists of 84 observations, which are used by Tahir et al. [20]. The dataset is about the failure times of 84 aircraft windshields. The observations of the dataset are: 3.467, 1.866, 2.385, 0.040, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 2.038, 2.82, 3.0, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 1.568, 2.194, 3.103, 4.376, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625 ,1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 3.443, 0.301, 1.876, 2.481, 4.663.

The summary statistics for the three datasets are revealed in Tables 5, 6, and 7. Additionally, the graphical representation of data 1 and 2 is presented in Figures 3, 4, and 5.

Table 5: Summary Statistics for the First Data

α	Mean	Q_1	Q_3	Standard Deviation	Skewness	Kurtosis	Min	Max
1.9	1.7	13.69433	36.67544	1785.513	1.71975	5.92411	1.1	4.1

Table 6: Summary Statistics for the Second Data

α	Mean	Q_1	Q_3	Standard Deviation	Skewness	Kurtosis	Min	Max
2.75955	2.835	2.1775	3.2775	0.891452	-0.13146	3.22305	0.39	4.9

Table 7: Summary Statistics for the Third Data

α	Mean	Q_1	Q_3	Standard Deviation	Skewness	Kurtosis	Min	Max
2.562624	2.385	1.866	3.376	1.113172	0.086535	2.36543	0.04	4.663

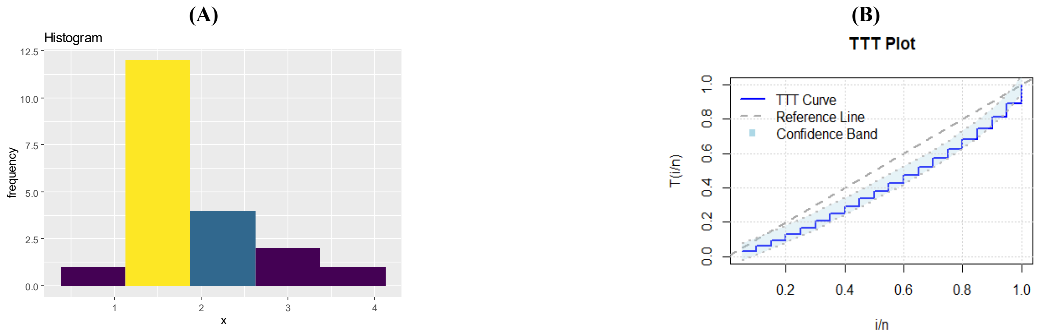


Fig. 3: A). Histogram B). TTT Plot for the First Dataset.

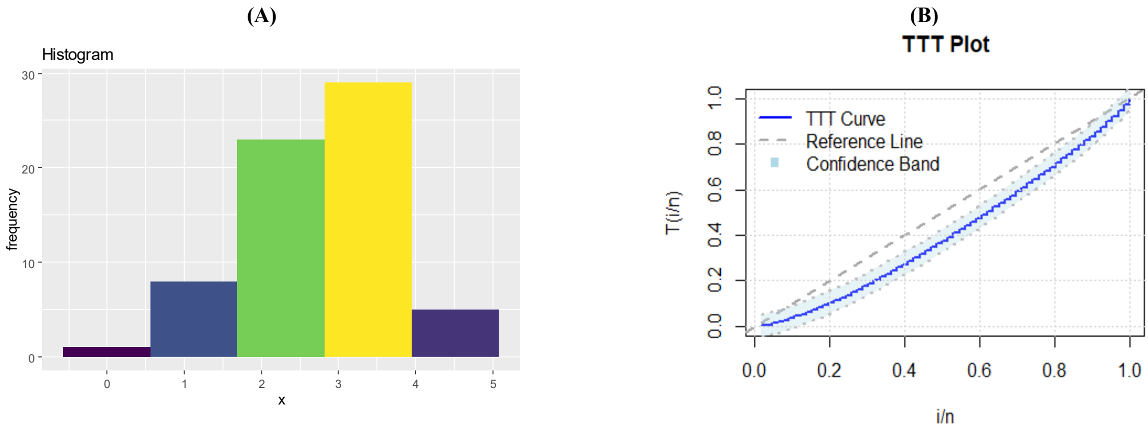


Fig. 4: A). Histogram B). TTT Plot for the Second Dataset.

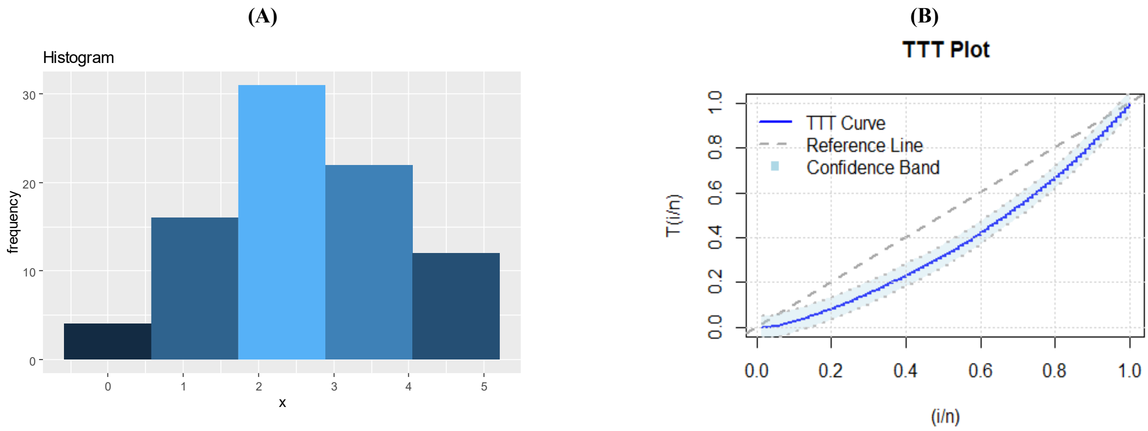


Fig. 5: A). Histogram B). TTT Plot for the Third Dataset.

The performance of the KNGF distribution is compared against several alternative distributions, namely, Fréchet distribution, Generalized Exponential distribution (GExp), Exponential distribution (Exp), and Rayleigh distribution, with the following probability density functions.

- Fréchet distribution.

$$f(x) = \frac{\alpha\left(\frac{x}{\beta}\right)^{-1-\alpha} \exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right)}{\beta}, \alpha, \beta > 0, x \geq 0$$

- Generalized Exponential distribution.

$$f(x) = \alpha\beta\, exp(-\alpha x)\,(1 - exp(-\alpha x))^{\beta-1}, \alpha, \beta > 0, x \geq 0$$

- Exponential distribution.

$$f(x) = \alpha\, exp(-\alpha x)\,, \alpha > 0, x \geq 0$$

- Rayleigh distribution.

$$f(x) = \frac{x}{\alpha^2} \exp\left(-\frac{x^2}{2\alpha^2}\right), \alpha > 0, x \geq 0$$

To evaluate and compare model performance using both the Kolmogorov-Smirnov (K-S) statistic with its p-value, and a range of model selection criteria, including AIC, CAIC, HQIC, Cramér-von Mises (CVM), and Anderson-Darling (AD) statistics.
Tables 7, 8, and 9 present the MLEs and K-S statistics values for all the distributions considered in comparison with the KNGF distribution. Tables 10, 11, and 12 contain the results of the goodness of fit, which is calculated through the package "Adequacy Model" in R software. The fit is considered good if the values of AIC, BIC, CAIC, HQIC, MLEs, K-S CVM, and AD values are small and its p-value is large.

Table 7: Parameter Estimates, K-S, and P-Value for the First Data

Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	K-S	p-value
KNGF	0.5350164	1.9286341	1.8594864	0.26901	0.1106
Fréchet	1.999754	-	-	0.45336	0.00053
GExp	1.9789648	0.7483032	-	0.31851	0.03457
Exp	0.512839	-	-	0.43114	0.00118
Rayleigh	1.133532	-	-	0.87273	1.174e-13

Table 8: Parameter Estimates, K-S, and P-Value for the Second Data

Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	K-S	p-value
KNGF	0.3647861	1.9812408	1.3416491	0.3203	2.628e-06
Fréchet	1.321148	-	-	0.50059	8.62e-15
GExp	1.9596055	0.5153334	-	0.64248	8.52e-16
Exp	0.3623802	-	-	0.35812	8.893e-08
Rayleigh	1.106871	-	-	0.72622	2.2e-16

Table 9: Parameter Estimates, K-S, and P-Value for the Third Data

Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	K-S	p-value
KNGF	1.093699	1.824239	0.766038	0.30182	3.761e-07
Fréchet	0.8689528	-	-	0.35593	8.865e-10
GExp	0.8319503	0.9141369	-	0.64358	2.2e-16
Exp	0.3901439	-	-	0.30312	4.292e-17
Rayleigh	1.499764	-	-	0.77368	2.2e-16

Table 10: Goodness of Fit Results for the First Data

Distributions	AIC	BIC	CAIC	HQIC	CVM	AD
KNGF	54.61034	57.59754	56.11034	55.19348	0.04789732	0.2655167
Fréchet	61.62909	62.62483	61.85132	61.82347	0.0535492	0.2951602
GExp	58.88595	60.87742	59.59184	59.27471	0.115825	0.642055
Exp	68.71317	69.70891	68.9354	68.90755	0.1204495	0.6683046
Rayleigh	62.7998	63.7955	62.022	61.9941	0.1841339	0.429642

Table 11: Goodness of Fit Results for the Second Data

Distributions	AIC	BIC	CAIC	HQIC	CVM	AD
KNGF	266.0854	272.6544	266.4725	268.6811	0.8161324	4.60123
Fréchet	302.554	304.7437	302.6165	303.4193	0.7200359	4.069681
GExp	333.0383	337.4176	333.2288	334.7688	0.6600824	4.411489
Exp	267.9887	270.1784	268.0512	268.8539	0.2462802	1.333595
Rayleigh	582.875	585.0647	582.9375	583.7402	0.1099257	3.590727

Table 12: Goodness of Fit Results for the Third Data

Distributions	AIC	BIC	CAIC	HQIC	CVM	AD
KNGF	395.2191	402.5471	395.5154	398.1666	1.700912	9.602453
Fréchet	403.3032	405.7458	403.3514	404.2857	1.722272	9.710955
GExp	442.5899	447.4752	442.7363	444.5549	0.1206058	1.083375
Exp	431.9754	434.418	432.0236	432.9579	0.1664689	1.397576
Rayleigh	1494.725	1497.168	1494.773	1495.708	0.05040832	0.4420363

The K–S test statistic is relatively small for the KNGF distribution across datasets 1, 2, and 3, as presented in Tables 7–9. However, the corresponding p-values are small for datasets 2 and 3, indicating rejection of the null hypothesis at conventional significance levels. This suggests that, although the KNGF distribution may not provide an adequate absolute fit for datasets 2 and 3, it performs comparatively better than the competing distributions based on the K–S statistic. Furthermore, the goodness-of-fit results reported in Tables 10–12 show that the KNGF distribution generally attains the smallest values of AIC, BIC, CAIC, and HQIC, supporting its relatively better performance in terms of information-based criteria. However, a closer examination of other goodness-of-fit measures, such as the Cramér–von Mises (CVM) and Anderson–Darling (AD) statistics, reveals that alternative models may perform better for certain datasets, particularly datasets 2 and 3, where smaller values are observed for some competing distributions. This indicates that the superiority of the KNGF distribution is not uniform across all criteria and that model performance depends on the selected goodness-of-fit measures. It is also noteworthy that several competing distributions exhibit similarly small p-values, indicating that none of the models provides a perfect fit to the data. Finally, Figures 6–8 illustrate the fitted probability density functions (PDFs) and cumulative distribution functions (CDFs) for datasets 1, 2, and 3, respectively, demonstrating that the KNGF distribution captures the overall pattern of the data reasonably well.

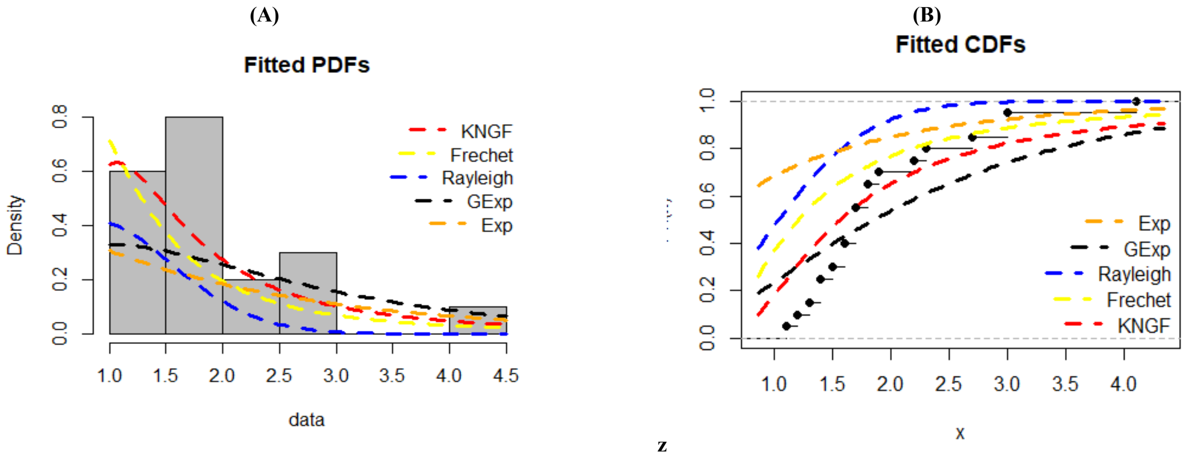


Fig. 6: Fitted Pdfs and CDFS for the First Data.

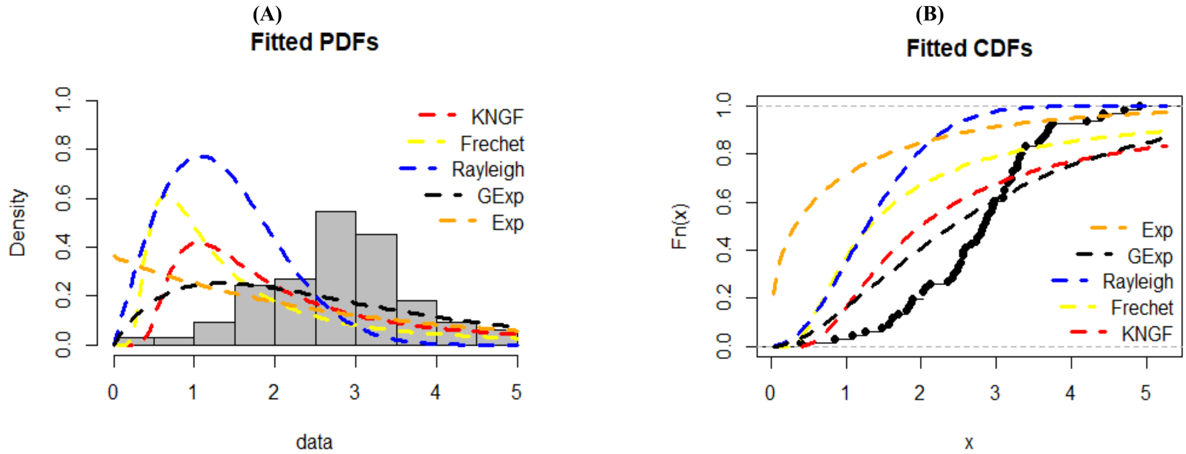


Fig. 7: Fitted Pdfs and CDFS for the Second Data.

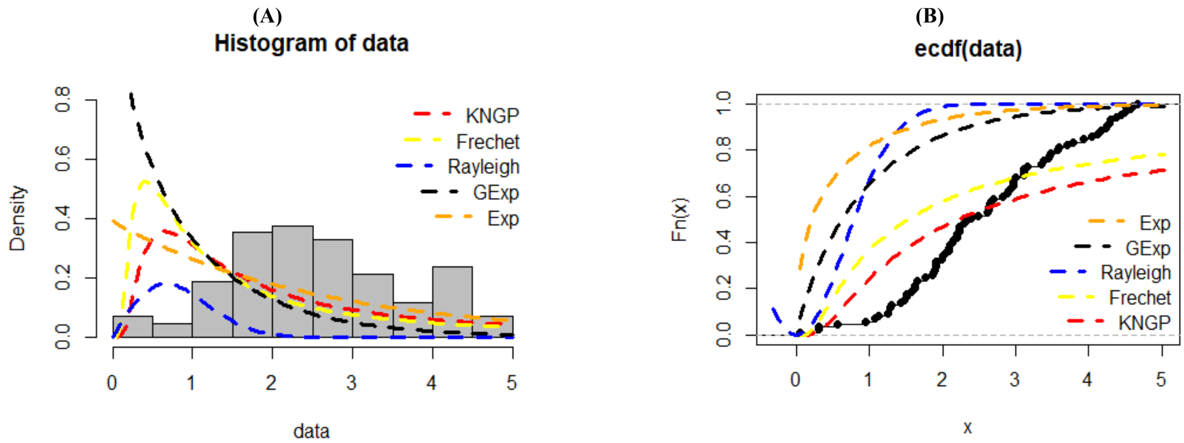


Fig. 8: Fitted Pdfs and CDFS for the Third Data.

7. Conclusion

In this article, a new extension of Fréchet distribution is introduce name as KNGF, by using a method of Khalil's new generalized family. Several statistical properties of the proposed distribution are derived and discussed. Particularly, expressions for the quantile function, moments, moment generating function, and characteristic function. The parameters of the purposed distribution are approximately estimated using the technique of MLE. A simulation study is also carried out for MLE estimates in order to present the consistency of the parameter estimates. The usefulness of the proposed distribution is best demonstrated with the help of three datasets. KNGF distribution shows the best results compared to other competing distributions.

Despite the good flexibility and efficiency of the model in capturing skewed and heavy-tailed data, it is necessary to point out several potential weaknesses of the model as well. For example, similarly to other multi-parameter generalized distributions, problems associated with the identifiability of parameters may appear in some situations, especially if the parameters are highly correlated or the sample size is relatively small. At the same time, although the model is computationally feasible in terms of maximizing the likelihood, the complexity of the likelihood function may affect the convergence in some situations. In addition, the results produced by the model can be sensitive to the choice of initial values, which can be important for the stability of the procedure.

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Conflict of Interest

The authors declare no conflicts of interest.

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