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Research paper

Modeling Time-to-Pain-Relief of Analgesic Drug: A Three-Shape-Parameter Modified Exponential Distribution Approach

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Abstract

In this study, the three-shape parameter modified exponential, also called the Kumaraswamy Modified Fréchet Exponential, is developed to model time to pain relief. Some important properties of this new model are derived: These include the moments, moment generating function, inequality measures, and mean residual life. The parameter estimates are computed under the maximum likelihood estimator. A simulation study is carried out to investigate the behavior of parameters under the maximum likelihood estimation method. The quantile function is used to generate random numbers for the simulation study. We applied the Monte-Carlo simulation technique. The results show that the parameters with their average bias (AB) and root mean square error (RMSE) decreased alongside increasing sample sizes: 50, 150, 250, 500, and 1000. The results show that the three-shape parameter modified exponential behaves well with the maximum likelihood estimator. The three-shape parameter modified exponential model is applied to time-to-pain relief data of patients who received analgesic (pain killer medication). Smaller Akaike and Bayesian Information criteria values are achieved better than all the competitive models; thus, the three-shape parameter modified exponential model demonstrates a better fit for the data used.

Keywords: *Probability Distributions; Probability Generators; Kumaraswamy Modified Fréchet Generator; Exponential; Shape Parameter; Model Fitting; Analgesic Data; Simulation.*

1. Introduction

Pain relief is essential in clinical administration, and Analgesics(drugs) play a pivotal role in relieving this pain. Analgesics include non-opioids, opioids, and adjuvant drugs, each varying in potency, mechanism, and onset of action. Measuring time-to-pain-relief is critical for evaluating the efficacy of analgesics, and survival models provide an appropriate framework for analyzing such data in clinical studies [5]. Time-to-pain relief models leverage the modification of classical models [22]. Nonetheless, several of these modifications are realized by applying distribution generator techniques (DGT). Some of DGT used for modifications are for example; the exponentiated generator [5], [20], [17], [18], [21], [25], [32]; Beta [26]; the odd log logistics generated generator [19]; quadratic rank transmuted map generator [24]; the Modified Fréchet generator [17]; The Topp Leone odd Lindley-G family [14]; The Marshall-Olkin alpha power family [15]; The Weibull-G Poisson family [16]; The odd Dagum family [12]; Complementary Generalized Power Weibull Power Series generator [23] and the odd log logistic Lindley-G family of distributions [11]. Some modifications have aimed at making the classical exponential distribution more flexible and relevant to model several data sets [7 - 10], [13]. Even as there have been several modifications, many of these modifications have no adequate shape parameters to regulate distribution shifts and also to model data of high kurtosis and skewness as presented by analgesic data. In many instances, modifications have centered on enhancing scale parameters [2 - 4], [6], [15], [19], [20], [22]. The purpose of the modification in this paper is to enrich the classical exponential distributions with adequate shape parameters. The modification is significant because dataset like patients' analgesic [31], [32] displays high skewness and kurtosis, and hence the need to model it with distributions with adequate shape parameters. The exponential distribution has a scale parameter which is able to regulates data variability. However, it is highly deficient in modeling highly skewed and kurtotic data.

To improve upon the exponential distribution, we first derive the Kumaraswamy modified Fréchet DGT and then fuse the exponential distribution. The Kumaraswamy is combined with the modified Fréchet generator [32] to give the Kumaraswamy modified Fréchet generator. The modified Fréchet generator is defined with one shape parameter, where $C(x)$ is the cumulative distribution function (CDF) of the baseline distribution and is the probability density function (PDF) of the baseline distribution. The CDF and PDF of the modified Fréchet generator are given as;

$$G_{MFC}(x)=\frac{e^{-(C(x))^{\alpha}-1}}{(e^{-1}-1)},\ x>0, \tag{1}$$

And

$$g_{MFC}(x)=\frac{\alpha c(x)(C(x))^{\alpha-1}e^{-(C(x))^{\alpha}}}{(1-e^{-1})},\ x>0. \tag{2}$$

The Kumaraswamy distribution [27], with two shape parameters $\beta>0$ and $\gamma>0$ is defined on $0\le x\le 1$ The CDF and PDF are respectively given as; $G_K(x)=1-(1-x^{\gamma})^{\beta}$ and $g_K(x)=\gamma\beta x^{\gamma-1}(1-x^{\gamma})^{\beta-1}$. Rewriting the Kumaraswamy distribution in the form of a generator with the condition that $0\le x\le 1$ takes on the form $0\le G(x;\psi)\le 1$ with $G(x;\psi)$ being any initial baseline distribution. CDF and PDF of Kumaraswamy then become;

$$G_K(x)=1-(1-(G(x;\psi))^{\gamma})^{\beta}, \tag{3}$$

And

$$g_K(x)=\gamma\beta(g(x;\psi))^{\gamma-1}(1-(G(x;\psi))^{\gamma})^{\beta-1},\ \gamma>0,\beta>0. \tag{4}$$

Given that $C(x;\psi)$ is the CDF of any classical baseline distribution. Further, the CDF of the Kumaraswamy modified Fréchet generator is derived by inserting the modified Fréchet generator into the formed Kumaraswamy. The CDF of Kumaraswamy modified Fréchet generator is presented as;

$$F(x)=1-\left(1-\left(\frac{e^{(C(x;\psi))^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma}\right)^{\beta},\alpha,\beta,\gamma,\psi>0,x\in \tag{5}$$

Further, the three-shape parameter modified exponential is finally derived. This is achieved by inserting the exponential distribution into the Kumaraswamy modified Fréchet generator. Given that the CDF of the exponential distribution is;

$$C(x)=1-e^{-\lambda x},\lambda>0,x\in \quad . \tag{6}$$

With $\lambda>0$ As a scale parameter, the three-shape parameter modified exponential (TSPME) distribution presents a CDF as;

$$F(x)=1-\left(1-\left(\frac{e^{-(1-e^{-\lambda x})^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma}\right)^{\beta},\alpha,\beta,\gamma,\lambda>0,x\in . \tag{7}$$

By differentiating the CDF of TSPME, we obtain the PDF of the TSPME distribution as;

$$f(x)=\alpha\beta\gamma\lambda e^{-\left(x\lambda+(e^{-1}-1)^{\alpha}(e^{-1}-1)^{-1}(1-e^{-\lambda x})^{\alpha-1}\left(\frac{e^{-(1-e^{-\lambda x})^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma-1}\right)}\left(1-\left(\frac{e^{-(1-e^{-\lambda x})^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma}\right)^{\beta-1}, \tag{8}$$

$\alpha,\beta,\gamma,\lambda>0,x\in$. The shape parameters are α,β,γ and the scale parameter is λ .

Plots of the PDF of TSPME are shown in Figure 1. The PDF theoretically shows flexibility for skewness and variability ahead of empirical applications. Further, the PDF shows symmetry, right-skewed, and partially left-skewed. The strong skewness behaviors promise adequate power in modeling data with high skew characteristics. For example, data like patients receiving analgesics.

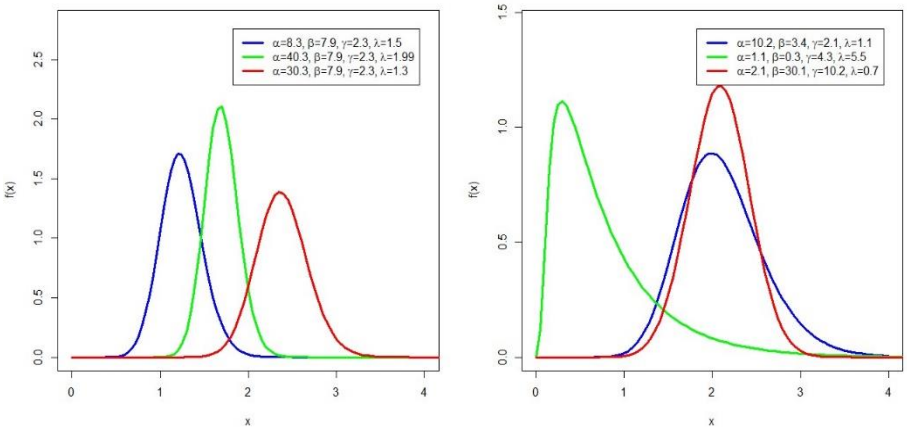

Fig. 1: PDF Plots of the TSPME.

The hazard function of TSPME is given as;

$$H(x)=\frac{\alpha\beta\gamma\lambda e^{\left(x\lambda+\left(e^{-1}-1\right)^{\alpha}\left(e^{-1}-1\right)^{-1}\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{-1}}}{1-1\left(1-\left(\frac{e^{-\left(1-\ell^{-\lambda x}\right)^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma}\right)^{\beta}\left(\frac{e^{-\left(1-\ell^{-\lambda x}\right)^{\alpha}}-1}{e^{-1}-1}\right)^{1-\gamma}\left(1-\left(\frac{e^{-\left(1-\ell^{-\lambda x}\right)^{\alpha}}-1}{e^{-1}-1}\right)^{\gamma}\right)^{1-\beta}} \tag{9}$$

$,\alpha,\beta,\gamma,\lambda>0,x\in$

The hazard plot shown in Figure 2 demonstrates bathtub shapes and upside-down bathtub shapes of failure rates. In the processes of studying the risk of events, such as deaths or failures, the hazard function will play a pivotal role. For example, the hazard function can serve in predicting survival times, service reliability, comparing experimental treatments or groups [2], [3]. The hazard function can further serve in scheduling and help in evaluating model fitness in medical and other fields.

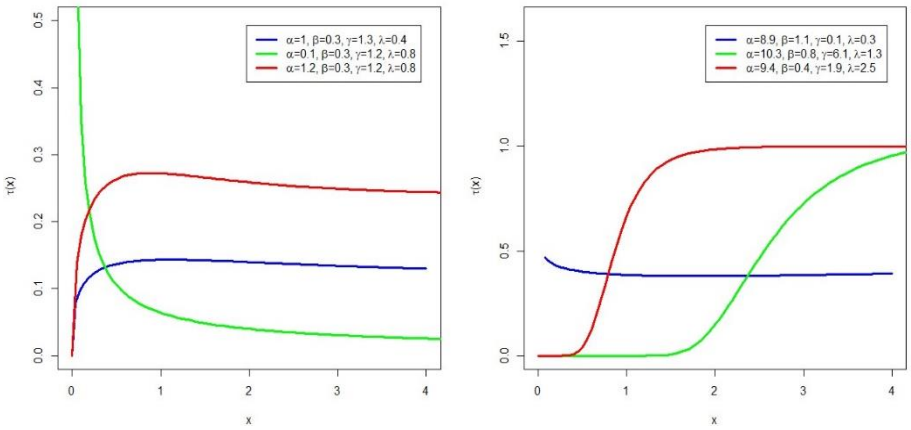

Fig. 2: Hazard Plots of the TSPME.

Interpretation of the Three Shape Parameters with Implications for Clinical Applications: Although the primary objective of this study is the development and estimation of a flexible time-to-pain-relief model, useful insight into the roles of the shape parameters can be obtained from the mathematical structure of the proposed distribution. The interpretation provided here is therefore theoretical and based on the influence of the parameters on the probability density and hazard functions. The first shape parameter α determines the behavior of the probability density function at early time points, regulating the amount of probability mass near the origin [30]. Variations in this parameter control whether the distribution favors shorter or longer times to pain relief, thereby allowing the model to accommodate early or delayed event occurrences within a time-to-event framework. The second shape parameter β influences the form and evolution of the hazard function, affecting whether the hazard rate is increasing, decreasing, or exhibits more complex behavior over time [23]. This parameter provides flexibility in modeling situations where the likelihood of pain relief changes as time progresses, a feature commonly observed in survival and reliability studies [31]. The third shape parameter γ governs the tail behavior of the distribution, impacting the probability of relatively long pain-relief times. By adjusting the heaviness of the tail, this parameter allows the model to capture variability and heterogeneity in response duration without imposing restrictive assumptions on the underlying process. Overall, the inclusion of multiple shape parameters enhances the adaptability of the modified exponential distribution, enabling it to represent a wide range of time-to-pain-relief patterns through its density and hazard structures. These interpretations are derived from the mathematical structure of the distribution and are not intended to represent direct physiological or pharmacodynamic mechanisms.

2. Statistical Properties

2.1 Quantile function

The Quantile Function of TSPME supports the simulation study by generating random numbers. It also serves as another way of describing the shape of a distribution.
Proposition 1. The quantile function of the TSPME is,

$$Q(x)=\frac{\ln(1-u)}{\lambda}\left(-\ln\left(1+\left(1-(1-u)^{\frac{1}{\beta}}\right)^{\frac{1}{\gamma}}\left(e^{-1}-1\right)\right)\right)^{\frac{1}{\alpha}}. \tag{10}$$

Proof. A quantile function is given $x_u=F^{-1}(u)$ by definition, so if $x_u=Q_x(u)$ is considered with some manipulations, then the Quantile function of the TSPME becomes,

$$Q_x(u)=\frac{-\ln(1-u)}{\lambda}\left(-\ln\left(1+\left(1-(1-u)^{\frac{1}{\beta}}\right)^{\frac{1}{\gamma}}\left(e^{-1}-1\right)\right)\right)^{\frac{1}{\alpha}}.$$

2.2. Mixture representation of PDF

Using binomial and Taylor series expansions, we obtained the mixture representation of the PDF.

$$f(x)=\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{jkmvwsdpl\tau^{1+\tau_s\tau}} \tag{11}$$

$$\xi_{jkmvwsdpl\tau^{1+\tau_s\tau}}=\frac{(-1)^{j+k+s+d+l}\left(1+k+m-\gamma\right)^{v}\left(1-l-s\right)^{\tau}}{v!w!\tau!\left(1-e^{-1}\right)\left(e^{-1}-1\right)^{-1-\gamma+\gamma j}}\binom{\beta-1}{j}\binom{\gamma j}{k}\binom{\gamma-1}{m}\binom{\alpha+\alpha v-1}{s}\binom{w}{d}\binom{d}{p}\binom{ap}{l}.$$

2.3. Moments

Finding the skewness, kurtosis, variance, and mean of a given probability distribution borrows moment techniques; therefore, the a need for TSPME moments propositions.
Proposition 2. The rth non-central moment of the TSPME is given as;

$$u^{r}=E\left(X^{r}\right)=\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\int_{-\infty}^{\infty}\lambda^{1+\tau}x^{1+r}dx \tag{12}$$

Proof. A moment is defined as $u^{r}=E\left(X^{r}\right)=\int_{-\infty}^{\infty}x^{r}f(x)dx$ .So, incorporating the mixture representation function gives,

$$u^{r}=E\left(X^{r}\right)=\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\int_{-\infty}^{\infty}\lambda^{1+\tau}x^{1+r}dx$$

2.4. Moment generating function

Proposition 3. The moment generating function of the TSPME is;

$$M_{x}(t)=\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\frac{t^{r}}{r!}\int_{-\infty}^{\infty}\lambda^{1+\tau}x^{1+r}dx. \tag{13}$$

Proof. The moment generating function is defined as. Further, based on the Taylor series. $M_{x}(t)=E\left[\sum_{r=0}^{\infty}\frac{t^{r}x^{r}}{r!}\right]=\sum_{r=0}^{\infty}\frac{t^{r}}{r!}u^{r}$ We obtain the generating function of TSPME as;

$$M_{x}(t)=\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\frac{t^{r}}{r!}\int_{-\infty}^{\infty}\lambda^{1+\tau}x^{1+\tau}dx.$$

2.5. Inequality measures

In income analysis and stress studies, inequality measures play vital roles. The Lorenz [29] and Bonferroni [28] curves are given for TSPME.
Proposition 4: The Lorenz curve Lq(x) of TSPME is;

$$Lq=\frac{\alpha\beta\gamma}{u}\int_{-\infty}^{t}\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\lambda^{1+\tau}x^{1+\tau}dx \tag{14}$$

Proof. By definition, $Lq=\frac{1}{u}\int_{-\infty}^{t}xf(x)dx$

$$Lq=\frac{\alpha\beta\gamma}{u}\int_{-\infty}^{t}\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\lambda^{1+\tau}x^{1+\tau}dx$$

Proposition 5. The Bonferroni curve Bq(x) is,

$$Bq=\frac{\alpha\beta\gamma}{uF(x)}\int_{-\infty}^{t}\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\lambda^{1+\tau}x^{1+\tau}dx \tag{15}$$

Proof. By definition,

$$Bq=\frac{Lq}{F(x)}$$

$$Bq=\frac{\alpha\beta\gamma}{uF(x)}\int_{-\infty}^{t}\alpha\beta\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty}\xi_{j_{kmvwsdpl\tau}}\lambda^{1+\tau}x^{1+\tau}dx$$

2.6. Mean residual life

In a given time t, a system will continue to work at such a time if $X_t = X - t / X > t$ In stress testing and life expectancy, the mean residual life plays a major role [30].
Proposition 6. The mean residual life of TSPME is,

$$m(t) = E(X - t / X > t) = \frac{-\mu}{1-F(t)}\left(\int_{-\infty}^{t} \alpha\beta\gamma \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty} \xi_{j_{kmwsdpl\tau}} \lambda^{1+\tau} x^{1+r} dx\right) - t \ . \tag{16}$$

Proof. By definition,

$$m(t) = E(X - t / X > t) = \frac{\int_{-\infty}^{t} (x-t) f(x) dx}{1-F(t)},$$

Substituting the mixture representation into the definition and simplifying, we get,

$$m(t) = E(X - t / X > t) = \frac{-\mu}{1-F(t)}\left(\int_{-\infty}^{t} \alpha\beta\gamma \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\tau=0}^{\infty} \xi_{j_{kmwsdpl\tau}} \lambda^{1+\tau} x^{1+r} dx\right) - t$$

2.7. Maximum likelihood estimates

We go through the following steps of the maximum likelihood estimation technique to get the estimation functions:
For a sample size(n), x_1, x_2, $x_3 \ldots x_n$, the likelihood function $L(\alpha,\beta,\gamma,\lambda) = \prod_{i=1}^{n} f(x_i;\alpha,\beta,\gamma,\lambda)$ Based on the PDF of TSPME becomes:

$$\begin{aligned}\ell(\alpha,\beta,\gamma,\lambda) = \sum_{i=1}^{n}[&\ln\alpha + \ln\beta + \ln\lambda - \lambda x_i + (\alpha-1)\ln\left(e^{-\lambda x_i}-1\right) + (\beta-1)\ln\left(1-e^{-\lambda x_i}\right) \\ &+ (\gamma-1)\ln\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right) + (\beta-1)\ln\left(1-\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma}\right)].\end{aligned} \tag{17}$$

Solving $\frac{\partial\ell}{\partial\alpha} = 0$, $\frac{\partial\ell}{\partial\beta} = 0$, $\frac{\partial\ell}{\partial\gamma} = 0$ and $\frac{\partial\ell}{\partial\lambda} = 0$, helps us obtain functions for the respective parameters α, β, γ and λ as;

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(e^{-\lambda x_i}-1\right) = 0. \tag{18}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{n}\ln\left[1-\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma}\right] = 0. \tag{19}$$

$$\frac{\partial\ell}{\partial\gamma} = \sum_{i=1}^{n}\ln\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right) - (\beta-1)\sum_{i=1}^{n}\frac{\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma}\ln\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)}{1-\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma}} = 0. \tag{20}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}[&\frac{1}{\lambda} - x_i - (\alpha-1)\frac{x_i e^{-\lambda x_i}}{e^{-\lambda x_i}-1} + (\beta-1)\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} + (\gamma-1)\left(\frac{-x_i e^{-\lambda x_i}}{e^{-\lambda x_i}-1} + \frac{e^{-\lambda}}{e^{-\lambda}-1}\right) - \\ &(\beta-1)\frac{\gamma\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma-1}\left[\frac{-x_i e^{-\lambda x_i}(e^{-\lambda}-1) + e^{-\lambda}(e^{-\lambda x_i}-1)}{(e^{-\lambda}-1)^2}\right]}{1-\left(\frac{e^{-\lambda x_i}-1}{e^{-\lambda}-1}\right)^{\gamma}}]\end{aligned} \tag{21}$$

Due to the complex nature of the functions, however, just as in the work of [4], [6], [15], and [19], computers aid in solving the solutions for the various parameters.

3. Simulation

A simulation study was carried out to study the behavior of parameters under the maximum likelihood estimation method. The quantile function was used to generate random numbers. We applied the Monte-Carlo simulation technique. The process was done for 1000 iterations on four different sets of selected initial parameter values. The results in Table 1 show that average bias (AB) and root mean square error (RMSE) decreased with increasing sample sizes: 50, 150, 250, 500, and 1000. The results show that TSPME parameters behave well with the maximum likelihood estimators.

Table 1: Simulation Results of TSPME Under Maximum Likelihood Estimation

	Population Parameter					AB				RMSE			
	α	β	γ	λ	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
I	8.3	2.2	2.4	2.1	50	-0.1047	0.1125	1.5048	0.7294	0.1479	0.2010	3.1243	0.9182
					150	-0.1320	0.0933	1.368	0.6463	0.1465	0.1458	3.0353	0.7569
					250	-0.1502	0.0881	1.1601	0.5749	0.1445	0.1234	2.8833	0.6538
					500	-0.2502	0.0681	1.0502	0.5448	0.1345	0.1134	2.6813	0.6441
					1000	-0.500	0.0081	0.0601	0.0447	0.0344	0.0133	1.6712	0.0442
II	5.3	2.2	2.4	2.1	50	-0.1537	0.1135	1.5008	0.7294	0.1439	0.2030	3.1343	0.9132
					150	-0.1620	0.0433	1.468	0.6462	0.1463	0.2058	2.0353	0.7569
					250	-0.1702	0.0383	1.3601	0.5748	0.1443	0.1234	1.8833	0.5538
					500	-0.2802	0.0283	1.2502	0.5449	0.1343	0.0134	1.6813	0.4441
					1000	-0.6120	0.0281	0.0600	0.0448	0.0342	0.0123	0.6712	0.0442
III	5.3	3.2	2.4	2.1	50	-0.1247	0.1135	1.5028	0.7264	0.1469	0.2000	3.1343	0.4152
					150	-0.1320	0.0933	1.467	0.6464	0.1462	0.1558	2.0353	0.3569
					250	-0.1402	0.0681	1.0600	0.5744	0.1442	0.1434	1.8833	0.2538
					500	-0.1702	0.0483	1.0002	0.5447	0.1343	0.1234	0.6813	0.2441
					1000	-0.5010	0.0283	0.0001	0.0446	0.0341	0.0033	0.0712	0.2442
IV	5.3	3.2	3.4	1.1	50	-0.1247	0.1225	1.4048	0.6294	0.1579	0.3010	3.3243	0.8182
					150	-0.1250	0.1933	1.3682	0.6462	0.1461	0.1458	3.0053	0.6569
					250	-0.1252	0.1881	1.1600	0.5743	0.1443	0.1234	2.0831	0.5538
					500	-0.2502	0.1682	1.0501	0.5442	0.1341	0.1134	2.0810	0.5441
					1000	-0.4000	0.1082	0.0603	0.04422	0.0343	0.0023	1.0712	0.4442

4. Application

The TSPME was applied to data to study the model fitness in response to the shape parameters' contributions to improve the flexibility of the exponential model.

Data Set: In this study, we use a time-to-pain relief (in minutes) data set on 20 observations of patients receiving analgesics. The data set was studied by [31] and [32] and given as: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6,2.0.

The descriptive statistics in Table 2 about the data show positive skewness and high kurtosis. The data has a smaller standard deviation (SD). The smaller SD, in other words, represents smaller variability. This means that the scale parameter of a model will have little role to play, especially in controlling the spread in the data. Nonetheless, higher kurtosis and skewness require models with robust shape parameters to keep the kurtosis and skewness under control. Efficient control of kurtosis and skewness reduces errors generated from models under application [32].

Table 2: Descriptive Statistics of the Data of Patients Receiving Analgesics

Kurtosis	Skewness	Standard Deviation	Mean	Minimum	Maximum
2.92	1.72	0.70	1.9	1.1	4.1

The Akaike information criterion (AIC) and Bayesian information criterion (BIC) are used to check for model fitness. The maximum likelihood estimates of the fitted model parameters were achieved by maximizing the log-likelihood function using the subroutine mle2 of the bbmle package in R software [18], [23]. The study focuses on comparing the TSPME model with its sub-models: thus, the Modified Fréchet Exponential (KME), Kumaraswamy Exponential (KE) [21], and Exponential (E) [1]. Table 3 shows all the shapes, $\hat{\beta}, \hat{\lambda}$ *and* $\hat{\gamma}$ parameter estimates of TSPME to be significant at an alpha of 0.05 under the maximum likelihood estimator, except for the scale parameter $\hat{\alpha}$.

Table 3: Parameter Estimates

Model	Parameter	Estimate	Std. Error	z-value	Pr(z)
TSPME	$\hat{\alpha}$	43.3245	310.9772	0.1393	0.8892
	$\hat{\beta}$	0.3234	0.3271	0.9889	3.227×10^{-05} ***
	$\hat{\lambda}$	4.9219	4.0357	1.219	2.226×10^{-03} ***
	$\hat{\gamma}$	24.8549	221.7099	0.1121	9.107×10^{-06} ***
MFE	$\hat{\alpha}$	32.3298	20.5465	1.5735	0.1156
	$\hat{\lambda}$	1.9984	0.4047	4.9371	7.93×10^{-05} ***
KE	$\hat{\beta}$	0.00262	0.0042	0.6241	0.5325
	$\hat{\lambda}$	6.4047	1.6400	3.905	9.416×10^{-05} ***

	$\hat{\gamma}$	0.0205	0.0060	3.411	0.0006
E	$\hat{\alpha}$	0.5263	0.1176	4.4721	7.744×10^{-06} ***

We observe that TSPME in Table 4 is better than its competitive models. This is because AIC and BIC give smaller values than MFE, KE, and E.

Table 4: Model Selection

Model	AIC	BIC
TSPME	35.9023	37.8938
MFE	38.8638	42.8467
KE	45.0106	47.9978
E	67.6741	68.669

In furtherance of confirming model fitness, the histogram and density plots in Figure 3 based on the analgesic data show that the TSPME mimics the data more closely than the sub-models, especially better than the exponential model.

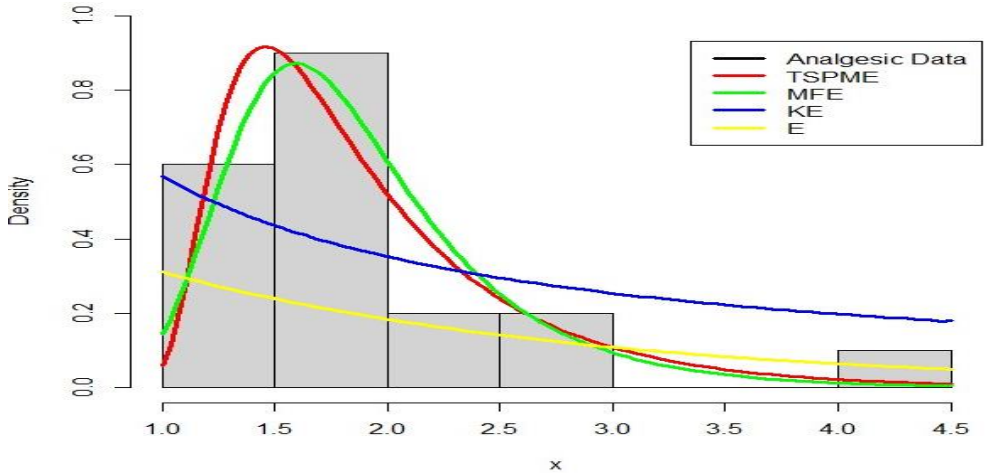

Fig. 3: Histogram and Estimated Densities of Patients Receiving Analgesics.

5. Conclusion

The improvement of the exponential distribution is achieved in this study. This is where we proposed a three-parameter modified exponential, which we also call it Kumaraswamy Modified Fréchet Exponential. Key properties relevant to the study are developed. These are: moments, moment generating function, inequality measures, and mean residual life. The parameter estimates were computed under Maximum likelihood estimation. The three-shape parameter modified exponential performed better than its competitive models. In the future, the model can be applied to several datasets beyond a sample of 20 patients to study some of the statistical properties, and also, the bivariate and regression aspects of the model can be developed. Issues of censored data, Bayesian estimation, and competing risks frameworks can also be focused on in future studies.

Data Availability

The dataset used [31] and [32] has been provided in the main body of this paper.

Conflicts of Interest

The authors hereby declare no conflict of interest

Ethical Statement

This material is the authors' own original work, which has not been previously published elsewhere.

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