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Research paper

The Khalil New Generalized Rayleigh (KNGR) Distribution: Statistical Properties, Estimation, and Applications

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Abstract

This study introduces the Khalil New Generalized Rayleigh (KNG-R) distribution, a novel and flexible lifetime model derived by applying the Khalil New Generalized Family (KNGF) generator to the Rayleigh distribution. The proposed model extends the Ray-leigh family by incorporating additional parameters that enhance its ability to capture diverse data behaviors such as increasing, creasing, and bathtub-shaped hazard rates. Key statistical properties, including the probability density function, cumulative distribution function, moments, moment-generating function, entropy, and quantile function, were derived. Additionally, parameter estimation was conducted using the maximum likelihood method, and a Monte Carlo simulation confirms the consistency and efficiency of the estimators. The KNG-R distribution was applied to three real datasets and outperformed competing distributions, including the Weibull–Rayleigh, TIIEHL-PLo, SMR, and Rayleigh models, based on log-likelihood, AIC, BIC, and AICc criteria. The results demonstrate that the KNG-R is a powerful and adaptable distribution suitable for modeling complex lifetime and reliability data.

Keywords: *KNG-R Distribution; Lifetime Modeling; Reliability Analysis; Maximum Likelihood Estimation; Simulation Study.*

1. Introduction

The development of new probability distributions has long been an essential pursuit in statistical modeling, reliability analysis, and lifetime data analysis, and as such, classical distributions such as exponential, Rayleigh, Weibull, and Pareto have served as foundational tools for modeling random phenomena in engineering, biological sciences, economics, and other fields. However, the increasing complexity of modern data, characterized by non-monotonic hazard rates, skewed patterns, and heavy tails, often renders these traditional models inadequate. To address these limitations, researchers have developed generalized families of distributions that introduce additional parameters to enhance flexibility, tail behavior, and model adaptability including Beta Generated-G Family [1], Kumaraswamy Generated (Kw-G) Family [2], Exponentiated Generated (Exp-G) Family [3], Marshall Olkin Extended (MO) Family [4], Transformed Transformer (TT) Family [5], Kumaraswamy odd log-logistic-G [6], Weibull Generated (W-G) Family [7], Generalized Topp Leone -G Family [8], Gamma Generated (GG) Family [9].
One major advancement in this area is the Khalil New Generalized Family (KNGF) proposed by [10]. The KNGF expands existing lifetime models through a transformation of a baseline cumulative distribution function (CDF), enabling the derivation of more flexible submodels capable of describing both monotonic and non-monotonic hazard functions. The KNGF was shown to effectively generalize various well-known distributions, such as the Weibull, Pareto, and Lindley, and its submodel, the Khalil New Generalized Pareto (KNGP) distribution, demonstrated significant flexibility in handling diverse hazard rate behaviors. The appeal of the KNGF approach lies in its capacity to generate new distributions that provide better fits to empirical data while maintaining analytical tractability in terms of moments, quantile functions, and parameter estimation. Let $G(x;\theta)$ Be the baseline distribution's CDF, then the Khalil New Generalized Family generator is obtained as:

$$F(x,\alpha,\beta,\theta)=\frac{1-\exp\left[-\alpha(G(x;\theta))^{\beta}\right]}{1-e^{-\alpha}},\ \alpha>0,\ \beta>0\ \ x>\Re \tag{1}$$

And its PDF is given by;

$$f(x,\alpha,\beta,\theta)=\frac{\alpha\beta g(x;\theta)(G(x;\theta))^{\beta-1}}{1-e^{-\alpha}}\exp\left[-\alpha(G(x;\theta))^{\beta}\right] \tag{2}$$

Similarly, Baaqeel et al. [12] recently introduced the Weibull–Rayleigh (WR) distribution by blending properties of the Weibull and Rayleigh models. Essentially, the WR distribution achieved improved flexibility in modeling lifetime and reliability data using the inverse of the exponentiated Weibull hazard rate function as a generator. The model also captured a wide range of distributional shapes, including symmetric, right-skewed, left-skewed, and inverse J-shaped forms. Furthermore, the authors illustrated its superiority in fitting real-world data sets and in detecting change points in data structures, showcasing the model's practical utility in applied statistical inference. Let a random variable. X Follow the WR distribution, then its CDF and PDF are obtained as;

$$G(x)=1-e^{-\frac{1}{2\gamma^2}\left(\frac{x^{1-\rho}}{\varphi\rho}\right)^{2\tau}}\ ,\ \tau,\gamma>0 \text{ and } 0<\rho<1 \tag{3}$$

The corresponding PDF is given by

$$g(x)=\frac{\tau(1-\rho)x^{2\tau(1-\rho)-1}}{\gamma^2(\varphi\rho)^{2\tau}}e^{-\frac{1}{2\gamma^2}\left(\frac{x^{1-\rho}}{\varphi\rho}\right)^{2\tau}}\ ,\ x>0 \tag{4}$$

Building on these developments, the current study introduces a Khalil New Generalized Family Rayleigh (KNG–R) distribution as a new and flexible member of the KNGF family [10]. The proposed KNG–R model is derived by applying the KNGF generator [10] to the baseline Rayleigh distribution. The inclusion of additional parameters allows the new model to adjust for skewness and tail behavior, making it capable of modeling data patterns with increasing, decreasing, or bathtub-shaped hazard rate features often observed in reliability and survival datasets. The Rayleigh distribution serves as a suitable baseline because of its simplicity and historical use in engineering, signal processing, and environmental modeling. However, its conventional form lacks the flexibility to represent complex life-testing data; the KNG–R extension resolves this limitation through the KNGF transformation.

The proposed KNG–R distribution generalizes several existing models as special cases and provides a unified structure capable of modeling diverse real-world phenomena. Its statistical properties, including the probability density function (PDF), cumulative distribution function (CDF), hazard function, moments, moment-generating function (MGF), and entropy measures, are derived to highlight its analytical versatility. Furthermore, the study explores estimation methods and examines the model's behavior under different parameter conditions, thereby extending the scope of applications of the KNGF framework to Rayleigh-type data.

2. The Khalil New Generalized Rayleigh (KNGR) Distribution

In this section, we introduce the Khalil New Generalized Rayleigh (KNGR) distribution obtained from the Khalil New Generalized Family (KNGF) using the Rayleigh distribution as the baseline distribution. The CDF and PDF of the Rayleigh distribution are given by;

$$F(x)=1-e^{-\frac{x^2}{2\sigma^2}}\ ,\ x\geq 0,\sigma>0 \tag{5}$$

$$f(x)=\frac{x}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}} \tag{6}$$

Respectively, substituting them into the KNGF, we get the CDF, PDF, and hazard function of the KNG-R distribution as;

$$F(x)=\frac{1-\exp\left[-\alpha\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta}\right]}{1-e^{-\alpha}} \tag{7}$$

$$f(x)=\frac{\alpha\beta\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta-1}\exp\left[-\alpha\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta}\right]}{1-e^{-\alpha}} \tag{8}$$

$$h(x)=\frac{\frac{\alpha\beta}{1-e^{-\alpha}}\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta-1}\exp\left[-\alpha\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta}\right]}{\exp\left[-\alpha\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta}\right]-e^{-\alpha}};x>0 \tag{9}$$

Where: $\sigma>0: scale, \alpha>0: shape, \beta>0: Shape$. The parameter α primarily controls the tail thickness, influencing skewness and kurtosis, while β Modifies the scale and overall spread of the distribution. Together, these parameters allow the KNG-R to capture increasing, decreasing, and bathtub-shaped hazard rates. The CDF, PDF, and hazard plots of KNG-R for some chosen parameters are shown below (fig1-3)

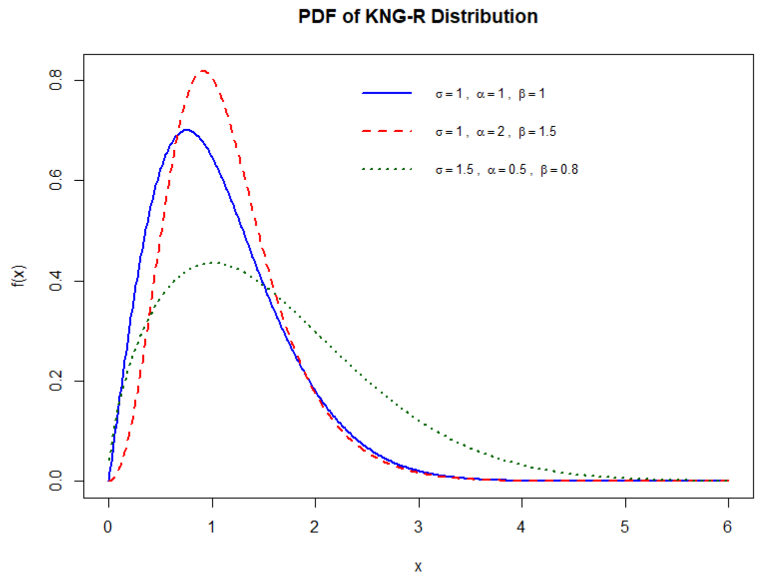

Fig. 1: PDF Plot of KNG-R.

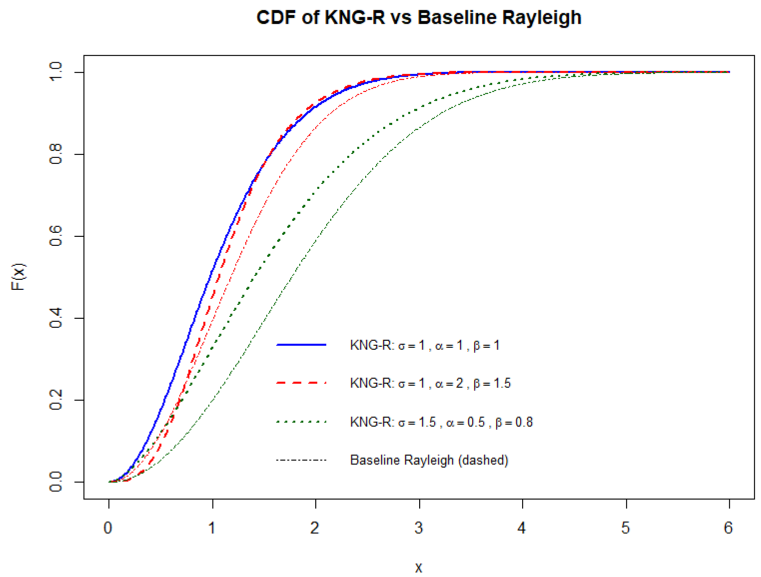

Fig. 2: CDF Plot of KNG-R.

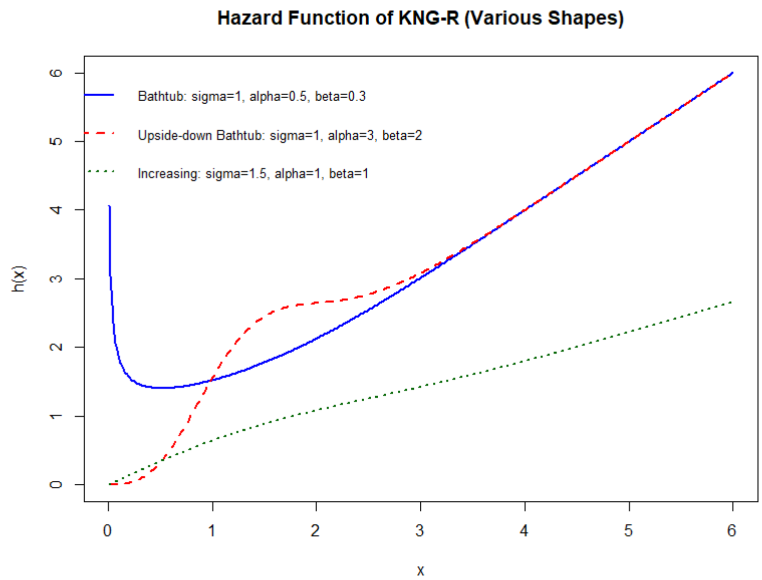

Fig. 3: Hazard Plot of KNG-R.

3. Statistical Properties

3.1. Moments

Position: If a random variable X follows the KNG-R distribution, then the r^{th} Moment of the KNGR distribution follows the form:

$$\mu_r^{'}=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\frac{1}{2}\cdot\left(\frac{1+l}{2\sigma^2}\right)^{-\frac{r+2}{2}}\Gamma\left(\frac{r+2}{2}\right) \tag{10}$$

Proof: By definition $\mu_r^{'}=E\left[W^r\right]=\int_{-\infty}^{\infty}w^r p(x)dx$ [18;19], hence substituting (8) yields;

$$\mu_r^{'}=E\left[X^r\right]=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\int_0^{\infty}x^{r+1}\cdot e^{-\frac{x^2}{2\sigma^2}}\cdot\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta-1}\exp\left[-\alpha\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)^{\beta}\right]dx \tag{11}$$

Let $t=\frac{x^2}{2\sigma^2}$ then $e^{-\frac{x^2}{2\sigma^2}}=e^{-t}$

$$\mu_r^{'}=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\int_0^{\infty}x^{r+1}\cdot e^{-t}\cdot\left(1-e^{-t}\right)^{\beta-1}\exp\left[-\alpha\left(1-e^{-t}\right)^{\beta}\right]dx$$
$$\mu_r^{'}=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\int_0^{\infty}x^{r+1}\cdot e^{-t}\cdot\left(1-e^{-t}\right)^{\beta-1}e^{-\alpha\left(1-e^{-t}\right)^{\beta}}dx \tag{12}$$

Expanding the outer exponential as a power series;

$$e^{-\alpha\left(1-e^{-t}\right)^{\beta}}=\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\left(1-e^{-t}\right)^{\beta i}$$
$$\left(1-e^{-t}\right)^{\beta-1}e^{-\alpha\left(1-e^{-t}\right)^{\beta}}=\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\left(1-e^{-t}\right)^{\beta i+\beta-1} \tag{13}$$

Using generalized binomial expansion $(1-k)^h=\sum_{l=0}^{\infty}(-1)\binom{h}{l}k^l$

$$\left(1-e^{-t}\right)^{\beta i+\beta-1}=\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}e^{-tl}\text{, then} \tag{14}$$

$$\mu_r^{'}=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\int_0^{\infty}x^{r+1}\cdot e^{-t}\cdot\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}e^{-tl}dx \tag{15}$$

$$\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\int_0^{\infty}x^{r+1}\cdot e^{-t}\cdot e^{-tl}dx$$
$$\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\int_0^{\infty}x^{r+1}\cdot e^{-t(1+l)}dx \tag{16}$$
$$\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\int_0^{\infty}x^{r+1}\cdot e^{-\frac{(1+l)x^2}{2\sigma^2}}dx$$

Using the standard gamma integral, $\int_0^{\infty}x^m e^{-cx^2}dx=\frac{1}{2}c^{-\frac{(m+1)}{2}}\Gamma\left(\frac{r+2}{2}\right)$ (17)

$$\int_0^{\infty}x^{r+1}\cdot e^{-\frac{(1+l)x^2}{2\sigma^2}}dx=\frac{1}{2}\cdot\left(\frac{1+l}{2\sigma^2}\right)^{-\frac{r+2}{2}}\Gamma\left(\frac{r+2}{2}\right) \tag{18}$$

$$\mu_r^{'}=\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\frac{1}{2}\cdot\left(\frac{1+l}{2\sigma^2}\right)^{-\frac{r+2}{2}}\Gamma\left(\frac{r+2}{2}\right) \tag{19}$$

3.2. Moment generating function

If a random variable X Follows the KNG-R, then its moment generating function (MGF) conforms to:

$$M_x(t)=E(e^{tx})=\int_0^{\infty}e^{tx}f(x)dx \tag{20}$$

Using the Taylor series on the above yields $M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(x^r)$ and hence,

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\alpha\beta}{\sigma^2\left(1-e^{-\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-\alpha)^i}{i!}\sum_{l=0}^{\infty}(-1)^l\binom{\beta i+\beta-1}{l}\frac{1}{2}\cdot\left(\frac{1+l}{2\sigma^2}\right)^{-\frac{r+2}{2}}\Gamma\left(\frac{r+2}{2}\right) \tag{21}$$

Table 1 shows the first six moments of the KNG-R distribution with fixed values of $\sigma=1$ for different α and β Values. From the Table, it can be observed that at a fixed $\sigma=1$ the inclusion of α and β creates an impact on μ As the moment increases with increasing α and β Values.

Table 1: The First Six Moments of the KNG-R Is Obtained Below with $\sigma=1$

α	β	μ_1'	μ_2'	μ_3'	μ_4'	μ_5'	μ_6'
0.5	0.5	0.7996	1.0421	1.7535	3.4916	7.8710	19.5629
	1.0	1.1627	1.7578	3.1794	6.5820	15.1714	38.2122
	2.0	1.5355	2.7156	5.3964	11.8417	28.3163	73.0178
	3.0	1.7444	3.3653	7.0999	16.2354	39.9393	105.0110
	4.0	1.8864	3.8562	8.4905	20.0318	50.4083	134.7114
1.0	0.5	0.7150	0.8745	1.4166	2.7554	6.1196	15.0643
	1.0	1.0759	1.5340	2.6585	5.3379	12.0411	29.8735
	2.0	1.4536	2.4492	4.6714	9.9132	23.0861	58.3306
	3.0	1.6668	3.0816	6.2575	13.8432	33.1228	85.1458
	4.0	1.8119	3.5631	7.5686	17.2910	42.3114	110.4405
2.0	0.5	0.5658	0.5990	0.8880	1.6352	3.5095	8.4534
	1.0	0.9191	1.1532	1.8118	3.3853	7.2560	17.3730
	2.0	1.3047	1.9858	3.4590	6.7962	14.8678	35.7771
	3.0	1.5253	2.5840	4.8316	9.9201	22.2497	54.2002
	4.0	1.6760	3.0471	5.9969	12.7561	29.2701	72.2614
3.0	0.5	0.4471	0.4016	0.5348	0.9208	1.8959	4.4502
	1.0	0.7902	0.8654	1.2140	2.0779	4.1784	9.5682
	2.0	1.1805	1.6233	2.5643	4.6096	9.3455	21.1483
	3.0	1.4070	2.1900	3.7589	7.1045	14.7669	33.6625
	4.0	1.5623	2.6359	4.8018	9.4562	20.1565	46.5258
4.0	0.5	0.3577	0.2689	0.3156	0.5006	0.9795	2.2286
	1.0	0.6891	0.6593	0.8174	1.2621	2.3456	5.0781
	2.0	1.2094	1.6392	2.4785	4.1790	7.8640	16.5067
	3.0	1.3125	1.8923	2.9918	5.1937	9.9264	20.9280
	4.0	1.4714	2.3229	3.9365	7.1800	14.1512	30.2609

3.3. Entropy measures

This study employed the Rényi entropy [11,20] to measure the uncertainty as well as the variation in the dataset.
Proposition 2: If a random variable X Follows the KNG-R distribution, then its Rényi entropy is given as;

$$H_q=\frac{1}{1-q}\log\left[\begin{array}{l}\left(\frac{\alpha\beta}{1-e^{-\alpha}}\right)^q 2^{\frac{q-1}{2}}\sigma^{1-q}\sum_{i=0}^{\infty}\frac{\left(-\alpha q\right)^i}{i!}\\ \times\sum_{l=0}^{\infty}\left(-1\right)^l\binom{\beta i+q\left(\beta-1\right)}{l}\Gamma\left(\frac{q+1}{2}\right)\left(l+q\right)^{-\left(\frac{q+1}{2}\right)}\end{array}\right] \tag{22}$$

Proof

By definition, the Rényi entropy is given as $H_q=\frac{1}{q-1}\log\left[\int_{-\infty}^{\infty}g^q(x)\right]$ (23)

Let $u=F(x)=1-e^{-t}$ with $t=\frac{x^2}{2\sigma^2}$,

$$\begin{aligned}&f_{Ray}(x)=\frac{x}{\sigma^2}e^{-t},C=\frac{\alpha\beta}{1-e^{-\alpha}}\\&f_{KNG-R}(x)=Ce^{-\alpha u^\beta}u^{\beta-1}f_{Ray}(x)\end{aligned} \tag{24}$$

The Rényi entropy of order $q\left(for\, q>0, q\neq1\right)$ is obtained by;

$$H_q=\frac{1}{1-q}\log\int_0^{\infty}g(x)^q dx=\frac{1}{1-q}\log I(q) \tag{25}$$

Now we compute $I(q)$

$$\begin{aligned}
&f_{KNG-R}(x)^q = C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} f_{Ray}(x)^q \\
&I(q) = \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} f_{Ray}(x)^q dx \\
&I(q) = \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} \left(\frac{x^q}{\sigma^{2q}} e^{-tq}\right) dx \\
&I(q) = \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} \sigma^{-2q} x^q e^{-tq} dx
\end{aligned} \tag{26}$$

Converting the integral to t variable, $dt = \left(\frac{x}{\sigma^2}\right)dx \Rightarrow dx = \left(\frac{\sigma^2}{x}\right)dt$ then;

$$\begin{aligned}
I(q) &= \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} \sigma^{-2q} x^q e^{-tq} \left(\frac{\sigma^2}{x}\right) dt \\
&= \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} \sigma^{-2q} x^q e^{-tq} x^{-1} \sigma^2 dt \\
&= \int_0^\infty C^q e^{-\alpha q u^\beta} u^{q(\beta-1)} \sigma^{2-2q} x^{q-1} e^{-tq} dt \\
&= C^q \sigma^{2-2q} \int_0^\infty e^{-\alpha q u^\beta} u^{q(\beta-1)} x^{q-1} e^{-tq} dt
\end{aligned} \tag{27}$$

Now recall that $t = \frac{x^2}{2\sigma^2} \Rightarrow x^2 = 2\sigma^2 t \Rightarrow x = \sqrt{2\sigma^2 t} = \left(2\sigma^2 t\right)^{\frac{1}{2}}$, then $x^{q-1} = \left(2\sigma^2\right)^{\frac{q-1}{2}} t^{\frac{q-1}{2}}$

$$\begin{aligned}
I(q) &= C^q \sigma^{2-2q} \int_0^\infty e^{-\alpha q u^\beta} u^{q(\beta-1)} \left(2\sigma^2\right)^{\frac{q-1}{2}} t^{\frac{q-1}{2}} e^{-tq} dt \\
&= C^q \sigma^{2-2q} \left(2\sigma^2\right)^{\frac{q-1}{2}} \int_0^\infty e^{-\alpha q u^\beta} u^{q(\beta-1)} t^{\frac{q-1}{2}} e^{-tq} dt \\
&= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \int_0^\infty t^{\frac{q-1}{2}} e^{-\alpha q u^\beta} u^{q(\beta-1)} e^{-tq} dt
\end{aligned} \tag{28}$$

Using a power series for $e^{-\alpha q u^\beta}$ and binomial expansion for $u = 1 - e^{-t}$;

$$e^{-\alpha q u^\beta} = \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} u^{\beta i} \tag{29}$$

$$\begin{aligned}
I(q) &= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \int_0^\infty t^{\frac{q-1}{2}} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} u^{\beta i} u^{q(\beta-1)} e^{-tq} dt \\
&= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \int_0^\infty t^{\frac{q-1}{2}} u^{\beta i} u^{q(\beta-1)} e^{-tq} dt \\
&= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \int_0^\infty t^{\frac{q-1}{2}} u^{\beta i + q(\beta-1)} e^{-tq} dt \\
&= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \int_0^\infty t^{\frac{q-1}{2}} \sum_{l=0}^\infty \left(-1\right)^l \binom{\beta i + q(\beta-1)}{l} e^{-lt} e^{-tq} dt \\
&= C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \sum_{l=0}^\infty \left(-1\right)^l \binom{\beta i + q(\beta-1)}{l} \int_0^\infty t^{\frac{q-1}{2}} e^{-t(l+q)} dt
\end{aligned} \tag{30}$$

Now using the standard gamma integral. $\int_0^\infty t^{a-1} e^{-ct} dt = \frac{\Gamma(a)}{c^a}$ (31)

Comparing $a - 1 = \frac{q-1}{2} \Rightarrow a = \frac{q-1}{2} + 1 = \frac{q+1}{2}$ and $c = l + q$

$$\int_0^\infty t^{\frac{q+1}{2}} e^{-t(l+q)} dt = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\left(l+q\right)^{\frac{q+1}{2}}} = \Gamma\left(\frac{q+1}{2}\right)\left(l+q\right)^{-\left(\frac{q+1}{2}\right)} \tag{32}$$

$$I(q) = \left[\begin{array}{l} C^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \\ \qquad \times \sum_{l=0}^\infty \left(-1\right)^l \binom{\beta i + q(\beta-1)}{l} \Gamma\left(\frac{q+1}{2}\right)\left(l+q\right)^{-\left(\frac{q+1}{2}\right)} \end{array}\right] \tag{33}$$

Hence, the Rényi entropy becomes

$$H_q = \frac{1}{1-q} \log \left[\begin{array}{l} \left(\frac{\alpha\beta}{1-e^{-\alpha}}\right)^q 2^{\frac{q-1}{2}} \sigma^{1-q} \sum_{i=0}^\infty \frac{\left(-\alpha q\right)^i}{i!} \\ \qquad \times \sum_{l=0}^\infty \left(-1\right)^l \binom{\beta i + q(\beta-1)}{l} \Gamma\left(\frac{q+1}{2}\right)\left(l+q\right)^{-\left(\frac{q+1}{2}\right)} \end{array}\right] \tag{34}$$

3.4. Quantile function

If a random variable X follows the KNG-R distribution, then its quantile function $Q(z)$ is given as;

$$Q(z) = F^{-1}(z)$$

$$Q(z) = \sigma\sqrt{-2\log\left[1-\left(-\frac{1}{\alpha}\log\left(1-z\left(1-e^{-\alpha}\right)\right)\right)^{\frac{1}{\beta}}\right]}, 0<z<1 \tag{35}$$

When we substitute $z=0.5, z=0.75, z=0.25$ We obtain the 2nd quartile, 3rd quartile, and the first quartile, respectively. The Bowley's and Moore's Skewness and Kurtosis Plots are shown in Figure 4. It is observed that the KNG-R is sensitive to α and β with a fixed value of $\sigma=1$. The plots demonstrate a responsive, flexible change in the kurtosis and skewness.

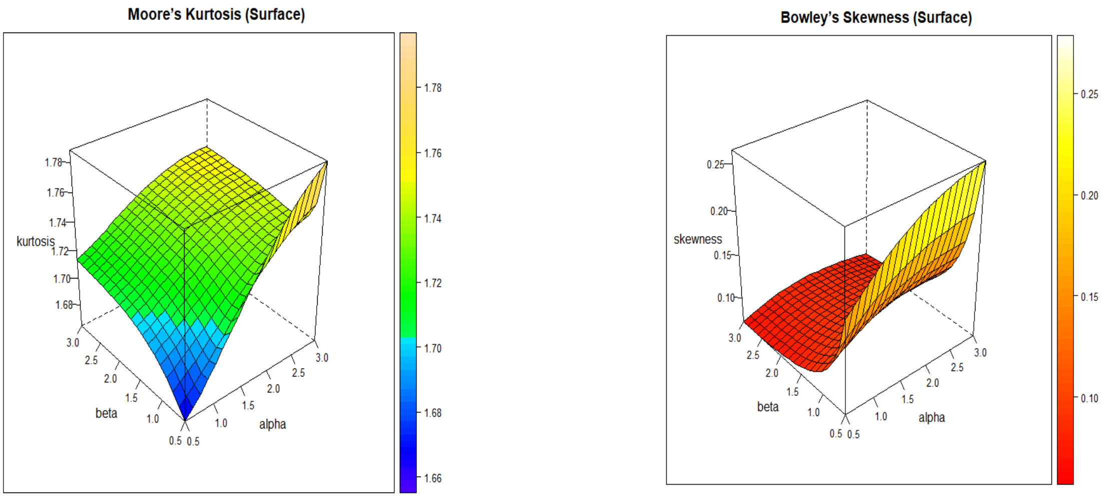

Fig. 4: Bowley's and Moore's Skewness and Kurtosis Plots.

4. Parameter Estimation

In this section, the Maximum Likelihood Estimation (MLE) technique is used to estimate the parameters of KNG-R. Let's consider a random sample. $Y=(Y_1,Y_2,Y_3,\ldots,Y_n)$ obtained from the KNG-R, given the log likelihood $\ell(\alpha,\beta,\sigma)=\prod_{i=1}^{n} f(x_i)$ Then the log likelihood of KNG-R becomes;

$$\ell(\alpha,\beta,\sigma) = n\ln C - \alpha\sum_{i=1}^{n} u_i^{\beta} + (\beta-1)\sum_{i=1}^{n}\ln(u_i) + \sum_{i=1}^{n}\ln(x_i) - 2n\ln(\sigma) - \sum_{i=1}^{n} t_i \tag{36}$$

Where $t_i=\dfrac{x_i^2}{2\sigma^2}, u_i=1-e^{-t_i}, f_{Ray}(x_i)=\dfrac{x_i}{\sigma^2}e^{-t_i}, C=\dfrac{\alpha\beta}{1-e^{-\alpha}}$ as

$$\begin{aligned}\ln f(x_i) &= \ln C - \alpha u_i^{\beta} + (\beta-1)\ln(u_i) + \ln f_{Ray}(x_i)\\ \ln f_{Ray}(x_i) &= \ln(x_i) - 2\ln(\sigma) - t_i\end{aligned} \tag{37}$$

Note: $\sum_{i=1}^{n}\ln(x_i)$ Does not depend on the parameters, so it drops out of derivatives.

Taking the partial derivative w.r.t α; only $n\ln C$ and $-\alpha\sum_{i=1}^{n}u_i^{\beta}$ depends on α so;

$$\ln C = \ln(\alpha\beta) - \ln(1-e^{-\alpha}) = \ln(\alpha) + \ln(\beta) - \ln(1-e^{-\alpha}) \tag{38}$$

$$\frac{\partial}{\partial\alpha}\ln C = \frac{1}{\alpha} - \frac{1}{1-e^{-\alpha}}\cdot e^{-\alpha} = \frac{1}{\alpha} - \frac{e^{-\alpha}}{1-e^{-\alpha}} \tag{39}$$

And

$$\frac{\partial}{\partial\alpha}\left(-\alpha\sum_{i=1}^{n}u_i^{\beta}\right) = -\sum_{i=1}^{n}u_i^{\beta} \tag{40}$$

Combining the terms; $n\ln C - \alpha\sum_{i=1}^{n}u_i^{\beta}$ becomes;

$$\frac{\partial \ell}{\partial \alpha} = n\left(\frac{1}{\alpha} - \frac{e^{-\alpha}}{1-e^{-\alpha}}\right) - \sum_{i=1}^{n} u_i^{\beta}$$
$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \frac{ne^{-\alpha}}{1-e^{-\alpha}} - \sum_{i=1}^{n} u_i^{\beta} = 0 \tag{41}$$

Taking the partial derivative w.r.t β only $n\ln C$, $-\alpha\sum_{i=1}^{n} u_i^{\beta}$, and $(\beta-1)\sum_{i=1}^{n}\ln(u_i)$ depends on β ;

$$\ln C = \ln(\alpha\beta) - \ln(1-e^{-\alpha}) = \ln(\alpha) + \ln(\beta) - \ln(1-e^{-\alpha}) \tag{42}$$

$$\frac{\partial}{\partial \beta}\ln C = \frac{1}{\beta} \; ; \; \frac{\partial}{\partial \beta}\left(-\alpha\sum_{i=1}^{n} u_i^{\beta}\right) = -\alpha\sum_{i=1}^{n} u_i^{\beta}\ln(u_i) \; ; \; \frac{\partial}{\partial \beta}\left[(\beta-1)\sum_{i=1}^{n}\ln(u_i)\right] = \sum_{i=1}^{n}\ln(u_i) \tag{43}$$

Combining the terms;

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \alpha\sum_{i=1}^{n} u_i^{\beta}\ln(u_i) + \sum_{i=1}^{n}\ln(u_i) = 0 \tag{44}$$

Taking the partial derivative w.r.t σ ; $-\alpha\sum_{i=1}^{n} u_i^{\beta}$, $(\beta-1)\sum_{i=1}^{n}\ln(u_i)$, $2n\ln(\sigma)$ and $\sum_{i=1}^{n} t_i$

$$t_i = \frac{x_i^2}{2\sigma^2} ; \frac{\partial t_i}{\partial \sigma} = \frac{x_i^2}{2}\cdot\sigma^{-2} = \frac{x_i^2}{2}\cdot -2\sigma^{-3} + \sigma^{-2}(0) = \frac{x_i^2}{1}\cdot -\frac{1}{\sigma^3} = -\frac{x_i^2}{\sigma^3} \tag{45}$$

Also

$$u_i = 1-e^{-t_i} ; \frac{\partial u_i}{\partial \sigma} = -e^{-t_i}\cdot -\frac{\partial t_i}{\partial \sigma} = \frac{\partial t_i}{\partial \sigma}e^{-t_i} = -\frac{x_i^2}{\sigma^3}e^{-t_i} \tag{46}$$

Now, differentiating each term;

$$\frac{\partial}{\partial \sigma}\left(-\alpha\sum_{i=1}^{n} u_i^{\beta}\right) = -\alpha\sum_{i=1}^{n}\beta u_i^{\beta-1}\cdot\frac{\partial u_i}{\partial \sigma} = -\alpha\sum_{i=1}^{n}\beta u_i^{\beta-1}\cdot -\frac{x_i^2}{\sigma^3}e^{-t_i} = \alpha\beta\sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}u_i^{\beta-1}e^{-t_i} \tag{47}$$

$$\frac{\partial}{\partial \sigma}\left((\beta-1)\sum_{i=1}^{n}\ln(u_i)\right) = (\beta-1)\sum_{i=1}^{n}\frac{1}{u_i}\cdot\frac{\partial u_i}{\partial \sigma} = (\beta-1)\sum_{i=1}^{n}\frac{1}{u_i}\cdot -\frac{x_i^2}{\sigma^3}e^{-t_i} = -(\beta-1)\sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}\cdot\frac{e^{-t_i}}{u_i} \tag{48}$$

$$\frac{\partial}{\partial \sigma}(-2n\ln(\sigma)) = -2n\frac{1}{\sigma} = -\frac{2n}{\sigma} \text{ and } \frac{\partial}{\partial \sigma}\left(-\sum_{i=1}^{n} t_i\right) = \sum_{i=1}^{n}\frac{x_i^2}{\sigma^3} \tag{49}$$

Now combining the terms yields;

$$\frac{\partial \ell}{\partial \sigma} = \alpha\beta\sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}u_i^{\beta-1}e^{-t_i} - (\beta-1)\sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}\cdot\frac{e^{-t_i}}{u_i} + \sum_{i=1}^{n}\frac{x_i^2}{\sigma^3} - \frac{2n}{\sigma}$$
$$\frac{\partial \ell}{\partial \sigma} = \sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}\left[\alpha\beta u_i^{\beta-1}e^{-t_i} - (\beta-1)\frac{e^{-t_i}}{u_i} + 1\right] - \frac{2n}{\sigma} = 0 \tag{50}$$

So finally,

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \frac{ne^{-\alpha}}{1-e^{-\alpha}} - \sum_{i=1}^{n}\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)_i^{\beta} = 0 \tag{51}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \alpha\sum_{i=1}^{n}\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)_i^{\beta}\ln\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right) + \sum_{i=1}^{n}\ln\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right) = 0 \tag{52}$$

$$\frac{\partial \ell}{\partial \sigma} = \sum_{i=1}^{n}\frac{x_i^2}{\sigma^3}\left[\alpha\beta\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)_i^{\beta-1}\cdot e^{-\frac{x_i^2}{2\sigma^2}} - (\beta-1)\frac{e^{-\frac{x_i^2}{2\sigma^2}}}{1-e^{-\frac{x_i^2}{2\sigma^2}}} + 1\right] - \frac{2n}{\sigma} = 0 \tag{53}$$

5. Simulation Studies

This section presents the Monte Carlo simulation conducted to examine the behavior of KNG-R as N Becomes large. Random samples were generated using the inverse transform method by applying the quantile function. $Q(u) = F^{(-1)}(u)$, where $u \sim U(0,1)$ This ensures that simulated samples accurately follow the KNG-R distribution. Therefore, 1000 independent repetitions were done for the sample sizes. $n = 30, 50, 100, 200, 500$ With two different sets of parameter values shown below:

SET 1: $\alpha = 1.5, \beta = 0.2, \sigma = 2.1$

SET 2: $\alpha = 1.58, \beta = 0.8, \sigma = 0.7$

The simulation results in Tables 2 and 3 show that the biases and RMSEs of the parameter estimates decrease as the sample size increases, confirming the consistency and asymptotic efficiency of the estimators. The relative biases are within ±5% for n ≥ 100, indicating excellent finite-sample performance.

Table 2: Simulation at $\alpha = 1.5, \beta = 0.2, \sigma = 2.1$

Sample Size	Parameter	Mean	Bias	Relative Bias	MSE	RMSE
$n = 30$	$\hat{\alpha}$	1.4820	-0.0180	-1.1987	1.5018	1.2255
	$\hat{\beta}$	0.2088	0.0088	4.379	0.0023	0.0476
	$\hat{\sigma}$	2.0003	-0.0997	-4.7455	0.599	0.774
$n = 50$	$\hat{\alpha}$	1.5779	0.0779	5.1948	1.1663	1.0799
	$\hat{\beta}$	0.2062	0.0062	3.111	0.0015	0.0381
	$\hat{\sigma}$	2.1249	0.0249	1.1855	0.4801	0.6929
$n = 100$	$\hat{\alpha}$	1.4881	-0.0119	-0.7937	0.537	0.7328
	$\hat{\beta}$	0.2039	0.0039	1.9283	0.0008	0.0277
	$\hat{\sigma}$	2.0814	-0.0186	-0.8852	0.1912	0.4373
$n = 200$	$\hat{\alpha}$	1.4728	-0.0272	-1.8126	0.3072	0.5543
	$\hat{\beta}$	0.2013	0.0013	0.6305	0.0003	0.0187
	$\hat{\sigma}$	2.0704	-0.0296	-1.409	0.1086	0.3295
$n = 500$	$\hat{\alpha}$	1.5456	0.0456	3.0373	0.1338	0.3658
	$\hat{\beta}$	0.2028	0.0028	1.4173	0.0002	0.0128
	$\hat{\sigma}$	2.0938	-0.0062	-0.295	0.0437	0.209

Table 3: Simulation at $\alpha = 1.58, \beta = 0.8, \sigma = 0.7$

Sample Size	Parameter	Mean	Bias	Relative Bias	MSE	RMSE
$n = 30$	$\hat{\alpha}$	1.3653	-0.2147	-13.5886	2.0112	1.4182
	$\hat{\beta}$	0.8211	0.0211	2.6358	0.0379	0.1948
	$\hat{\sigma}$	0.6746	-0.0254	-3.6356	0.0312	0.1765
$n = 50$	$\hat{\alpha}$	1.6428	0.0628	3.9768	2.3941	1.5473
	$\hat{\beta}$	0.8065	0.0065	0.8114	0.0213	0.1461
	$\hat{\sigma}$	0.7163	0.0163	2.3285	0.0411	0.2026
$n = 100$	$\hat{\alpha}$	1.6557	0.0757	4.7929	2.1539	1.4676
	$\hat{\beta}$	0.7958	-0.0042	-0.5231	0.0113	0.1062
	$\hat{\sigma}$	0.7248	0.0248	3.5385	0.0378	0.1944
$n = 200$	$\hat{\alpha}$	1.6005	0.0205	1.2951	1.4702	1.2125
	$\hat{\beta}$	0.7881	-0.0119	-1.4835	0.0062	0.0789
	$\hat{\sigma}$	0.7154	0.0154	2.2065	0.023	0.1517
$n = 500$	$\hat{\alpha}$	1.7271	0.1471	9.3133	1.1723	1.0827
	$\hat{\beta}$	0.7973	-0.0027	-0.3336	0.0026	0.0508
	$\hat{\sigma}$	0.7226	0.0226	3.231	0.0196	0.1401

6. Application

This section presents the application of the proposed KNG-R distribution to a real-life dataset. Three different datasets (survival times for Chemotherapy Patients, failure times of 50 components, and Breaking Stress of Carbon Fiber) were used together with four competing models as shown below;

1) Weibull–Rayleigh [12]

$$G(x) = 1 - e^{-\frac{1}{2\beta^2}\left(\frac{x^{1-\lambda}}{\theta\lambda}\right)^{2\alpha}}, x > 0 \tag{54}$$

2) Type II exponentiated half-logistic-PLo(TIIEHL-PLo) [16]

$$G(x)=1-\left(\frac{1-\left[1-\left(1+x^{\theta}\right)^{-\eta}\right]^{\gamma}}{1+\left[1-\left(1+x^{\theta}\right)^{-\eta}\right]^{\gamma}}\right)^{\delta},x,\gamma,\delta,\theta,\eta>0 \tag{55}$$

3) Scale mixture of Rayleigh distribution (SMR) [17]

$$G(x)=1-\frac{1}{\left(\frac{x^{2}}{2\sigma}+1\right)^{\frac{q}{2}}},x,\sigma,q>0. \tag{56}$$

4) Baseline Rayleigh

$$G(x)=1-e^{-\frac{x^{2}}{2\sigma^{2}}},x,\sigma>0 \tag{57}$$

The first dataset considered in this study was sourced from [13] which provides survival times (in years) for 46 patients undergoing chemotherapy: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.
The second dataset, was taken from [14] and represents the failure times of 50 components (in 1000 h): 0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.
The third dataset was obtained from [15] and contains 100 observations on Breaking Stress of Carbon Fibers (in Gba): 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2. 59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42,4.70. 4.90, 4.91, 5.08, 5.56
Furthermore, the parameters of the KNG-R model together with the other competing models were estimated using the maximum likelihood estimation approach, as shown in Tables 4-6. Additionally, several statistics, including the Log-Likelihood (Log Lik), Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), and Akaike's Information Criterion corrected (AICc), were employed to assess the superiority of each model in terms of fitting each dataset. Tables (7-9) display the goodness of fit results for the KNG-R together with the other competitive models for the three datasets. The KNG-R outperformed the other models by recording the highest Log Lik values with smaller AIC, BIC, and AICc values for all three datasets. Also, the goodness of fit plots (figs 5-7) show that the KNG-R distribution mimics all three data sets well as compared to the other competitive models.

Table 4: Parameter Estimation Survival Times Data (in years)

Model	Parameter	Estimate	Std. Error	Z. value	P. value
KNG-R	α	1.6361	1.2984	1.2601	0.2076
	β	0.4987	0.0900	5.5407	0.0000
	σ	2.1266	0.5362	3.966	0.0001
WR	α	0.8428	0.1128	7.4718	0.0000
	β	1.2857	12.4767	0.1031	0.9179
	θ	1.2872	22.8134	0.0564	0.9550
	λ	0.3736	0.0250	14.9716	0.0000
TIIEHL-PLo	γ	1.1043	0.14382	7.6782	0.0000
	δ	34.9249	43.9019	0.7955	0.4263
	θ	0.0143	0.0158	0.9023	0.3669
	η	2.1932	3.3069	0.6632	0.5072
SMR	σ	0.1414	0.0852	1.6593	0.0971
	q	1.2471	0.3423	3.6438	0.0003
Rayleigh	σ	1.2755	0.0940	13.5647	0.0000

Table 5: Parameter Estimates Failure Times Data (in 1000 h)

Model	Parameter	Estimate	Std. Error	Z. value	P. value
KNG-R	α	1.5383	1.1250	1.3673	0.1715
	β	0.2851	0.0513	5.5594	0.0000
	σ	8.0806	2.2368	3.6125	0.0003
WR	α	0.4986	64.3333	0.0078	0.9938
	β	1.1848	27.4136	0.0432	0.9655
	θ	1.1891	27.1436	0.0438	0.9651
	λ	0.3369	85.5612	0.0039	0.9969
TIIEHL-PLo	γ	0.6807	0.0933	7.2930	0.0000
	δ	30.19749	109.3514	0.2762	0.7824
	θ	0.00628	0.0064	0.9803	0.3269
	η	0.44611	2.85599	0.1562	0.8759
SMR	σ	0.00807	0.00486	1.6618	0.0966
	q	0.43884	0.07832	5.603	0.0000
Rayleigh	σ	3.76234	0.26604	14.1421	0.0000

Table 6: Parameter Estimates Breaking Stress of Carbon Fiber Data (in Gba)

Model	Parameter	Estimate	Std. Error	Z. value	P. value
KNG-R	α	0.0010	0.0000	50000	0.0000
	β	1.8293	0.2664	6.8678	0.0000
	σ	1.6627	0.0934	17.8038	0.0000
WR	α	2.0340	2.9777	0.6831	0.4946
	β	3.13453	46.6789	0.0672	0.9465
	θ	3.21948	23.5076	0.137	0.8911
	λ	0.31346	1.0020	0.3128	0.7544
TIIEHL-PLo	γ	3.83148	0.6381	6.0042	0.0000
	δ	29.6877	23.1764	1.2809	0.2002
	θ	0.0179	0.0114	1.5684	0.1168
	η	10.6684	8.1915	1.3024	0.1928
SMR	σ	10.0000	2.7759	3.6024	0.0003
	q	6.3865	1.5838	4.0324	0.0001
Rayleigh	σ	1.98613	0.09931	20.0000	0.0000

Table 7: GOF Survival Times Data (in Years)

Model	Log Lik	AIC	BIC	AICc
KNG-R	-58.6034	123.2068	128.6927	123.7782
WR	-58.8262	125.6524	132.9669	126.6280
TIIEHL-PLo	-58.8852	125.7703	133.0849	126.7459
SMR	-62.1893	128.3786	132.0359	128.6577
Rayleigh	-79.0742	160.1484	161.977	160.2393

Table 8: GOF Failure Times Data (in 1000 h)

Model	LogLik	AIC	BIC	AICc
KNG-R	-101.4280	208.8563	214.5924	209.3781
WR	-102.3640	212.7286	220.3767	213.6175
TIIEHL-PLo	-102.7410	213.4826	221.1307	214.3715
SMR	-109.6590	223.3178	227.1419	223.5732
Rayleigh	-179.5990	361.1982	363.1102	361.2815

Table 9: GOF Breaking Stress of Carbon Fiber Data (in Gba)

Model	LogLik	AIC	BIC	AICc
KNG-R	-141.594	289.1878	297.0033	289.4378
WR	-141.529	291.0586	301.4793	291.4797
TIIEHL-PLo	-141.559	291.1185	301.5392	291.5395
Rayleigh	-149.501	301.0018	303.6070	301.0426
SMR	-149.502	303.0030	308.2133	303.1267
EOWIR	-266.446	538.8929	546.7084	539.1429

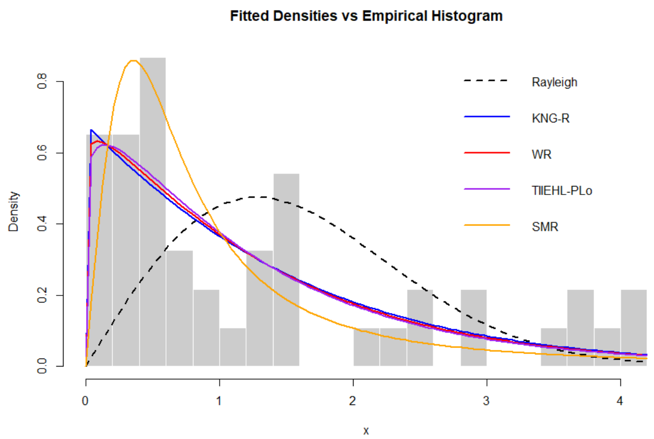

Fig 5: Survival Times Data (in Years).

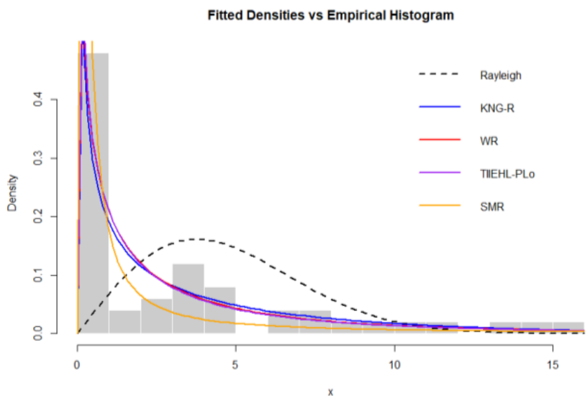

Fig. 6: Failure Times Data (in 1000 h).

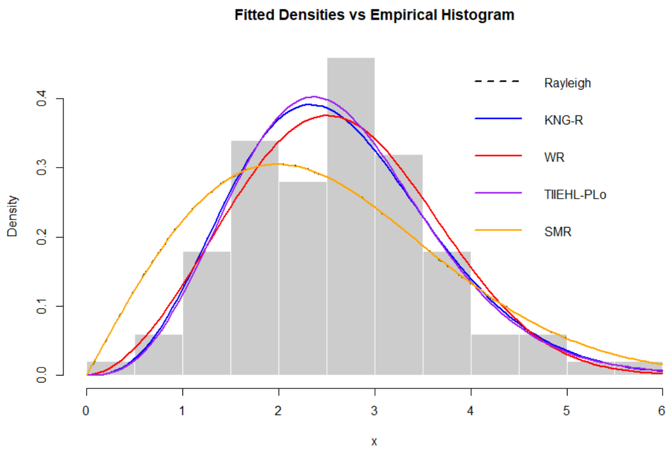

Fig. 7: Breaking Stress of Carbon Fiber Data (in Gba).

7. Conclusion

This study proposed the Khalil New Generalized Rayleigh (KNG-R) distribution, a flexible extension of the Rayleigh model within the Khalil New Generalized Family of probability distributions. The model's statistical properties were derived, and parameter estimation using the maximum likelihood showed consistent and efficient performance through simulation. Applications of the KNG-R to three real datasets demonstrated that the KNG-R provides a superior fit compared to existing models, including WR, TIIEHL-PLo, SMR, and Rayleigh, based on higher log-likelihood and lower AIC, BIC, and AICc values. Generally, the KNG-R distribution offers a powerful and versatile tool for modeling lifetime and reliability data. Future research may explore Bayesian estimation procedures for the KNG-R parameters, regression frameworks based on the KNG-R model, and extensions to multivariate forms for complex lifetime data.

Conflicts of Interest

The authors declare no conflicts of interest.

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