Parameter estimation for multiple weibull populations under joint type-II censoring

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Abstract

In this paper, we introduce the maximum likelihood estimation for k Weibull populations under joint type II censored scheme and different special cases have been obtained. The asymptotic variance covariance matrix and approximate confidence region based on the asymptotic normality of the maximum likelihood estimators have been obtained. A numerical example is considered to illustrate the proposed estimators.

Keywords: Approximate Inference; Coverage Probabilities; Joint Type II Censored Scheme; Maximum Likelihood Estimation; Weibull Distribution.

1. Introduction

Censoring schemes are used to reduce the costs of experiments and to accelerate the performing of the design. There are various types of censored data to be dealt with in the analysis of lifetime experiments see [Lawless [6]]. Almost all of these types of data are concerned with the one-sample problems. But, there are situations in which the experimenter plans to compare different populations. In such problems, the joint censoring scheme has been suggested in the literature. As mentioned by Balakrishnan and Rasouli [3] and Rasouli and Balakrishnan [8], a joint censoring scheme is quite useful in conducting comparative lifetime test of products coming from different units within the same facility. The joint censoring scheme is of practical significance in conducting comparative life tests of products from different lines within the same facility. Suppose products are being manufactured by different lines within the same facility, and that k independent samples of sizes n1, 1 ≤ h ≤ k are selected from these k lines and placed simultaneously on a life-testing experiment. In order to reduce the cost of the experiment as well as the experimental time, the experimenter may choose to terminate the experiment after a certain number (say, r) of failures have been observed altogether. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines.

Let us suppose that (X1,...,XN) are N jointly distributed random variables, with \{X1,...,X_N\} = \{X_{11},...,X_{1n_1};X_{21},...,X_{2n_2};X_{k1},...,X_{k_{n_k}}\}, with N = \sum_{k=1}^{K} n_k. Suppose X_{11}, X_{12},...,X_{n_1} are the lifetimes of n_1 specimens from production line A_1, and are independent and identically distributed (iid) variables from a population with cdf \(F_1(x)\) and pdf \(f_1(x)\). Similarly, \(X_{21},X_{22},...,X_{2n_2}\) are the lifetimes of \(n_2\) specimens from production line \(A_2\), and are assumed to be a sample from pdf \(f_2(x)\) and cdf \(F_2(x)\), and so on, with \(X_{k1},X_{k2},...,X_{kn_k}\) denoting the lifetimes of \(n_k\) specimens from production line \(A_k\) being iid variables from pdf \(f_k(x)\) and cdf \(F_k(x)\). Denote the order statistics of these \(k\) random samples by \(W_1 \leq W_2 \leq ... \leq W_N\), where \(N\) is the total sample size.

Let \(r\) denote a pre-fixed total number of failures to be observed. Then, under the joint type-II censoring scheme for the \(k\)-samples, the observable data consist of \((z,w)\), where \(w = (w_1,w_2,...,w_r); w_r \in \{X_{h1},X_{h2},...,X_{hn}\}\) for \(1 \leq h_1,h_2,...,h_r \leq k\).
Let $M_j(h)=\sum_{i=1}^{n} z_i(h)$ denote the number of $X_i$ failures in $W$ for $1 \leq h \leq k$ and $r=\sum_{i=1}^{n} M_i(h)$ . Then the likelihood of $(Z, W)$ given by Balakrishnan and Feng [4] as

$$L=c_{r}(\prod_{i=1}^{n} (f_{w}(w_i))^{c_{i}}/\prod_{k=1}^{r} (F_{w}(w_i))^{c_{i}-M_{i}(h)})^{1/(n-r-M_{i}(h))}$$

where $F_{w}(w_i)=1-F_{w}(w_i)$ and $c_r=\prod_{i=1}^{n} n_{i}!/\prod_{i=1}^{n} (n_{i}-M_{i}(h))!$.

In the literature, Balakrishnan and Rasouli [3] developed likelihood inference for the parameters of two exponential populations under joint Type-II censoring. They developed inferential methods based on maximum likelihood estimates (MLE) and compared their performance with those based on some other approaches such as Bootstrap. Shafay et al. [10] derived the Bayesian inference for the unknown parameters of two exponential populations under joint Type-II censoring they developed with the use of squared-error, linear-exponential and general entropy loss functions. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint Type-II censored data is obtained; see also Rasouli and Balakrishnan [8] for a generalization of their results to progressive type-II censoring for the parameters of two exponential populations. Balakrishnan and Feng [4] generalized Balakrishnan and Rasouli [3], Rasouli and Balakrishnan [8] and Shafay et al. [10] works by considered a jointly Type-II censored sample arising from $k$ independent exponential populations. Ashour and Abo-Kasem [1] derived Bayesian and non-Bayesian estimators for two generalized exponential populations under joint type II censored scheme. Finally Ashour and Abo-Kasem [2] obtained MLEs for two Weibull populations under joint type II censored scheme.

In this paper, we discuss the maximum likelihood estimation for $k$ Weibull populations under joint type II censored scheme in section 2. The asymptotic variance covariance matrix and approximate confidence region based on the asymptotic normality of the maximum likelihood estimators have been obtained in section 3. The performance analysis of the obtained estimators is carried out by conducting a simulation study in section 4. Finally, in section 5, we use a numerical example to illustrate all the methods of inference developed here.

## 2. Maximum likelihood estimators

Suppose that the $k$ populations are Weibull with density and distribution functions as

$$f_{w}(x)=\frac{1}{\theta_{i}}\left(\frac{x}{\theta_{i}}\right)^{\alpha_{i}-1}e^{-\left(\frac{x}{\theta_{i}}\right)^{\alpha_{i}}} \quad \text{and} \quad F_{w}(x)=1-e^{-\left(\frac{x}{\theta_{i}}\right)^{\alpha_{i}}}, \quad \text{where} \quad \alpha_{i}, \theta_{i}>0, x>0, \text{for } 1 \leq h \leq k$$

In this case, the likelihood function in (1) becomes

$$L=C_{r}(\prod_{i=1}^{n} \left(\frac{\alpha_{i}}{\theta_{i}}\right)^{1/(n-r-M_{i}(h))} \prod_{i=1}^{n} \left[\left(\frac{w_{i}}{\theta_{i}}\right)^{\alpha_{i}} \exp\left(-\frac{w_{i}}{\theta_{i}}\right)\right]^{c_{i}} \prod_{i=1}^{n} \left[\left(\frac{w_{i}}{\theta_{i}}\right)^{\alpha_{i}} \exp\left(-\frac{w_{i}}{\theta_{i}}\right)\right]^{1-(c_{i})}$$

Therefore, to obtain the MLE’s of $\hat{\alpha}_{i}$ and $\hat{\theta}_{i}$ we find the first derivatives of the natural logarithm of the likelihood function (2) with respect to $\alpha_{i}$ and $\theta_{i}$, we get the following equations

$$\frac{\partial \ln L}{\partial \alpha_{i}}=\left[m_{i}(h)+\sum_{i=1}^{n} z_{i}(h) \ln\left(\frac{w_{i}}{\theta_{i}}\right)-\sum_{i=1}^{n} z_{i}(h) \frac{w_{i}}{\theta_{i}} \ln\left(\frac{w_{i}}{\theta_{i}}\right)-\frac{n_{i}-m_{i}(h)}{\theta_{i}} \ln\left(\frac{w_{i}}{\theta_{i}}\right)\right]$$

$$\frac{\partial \ln L}{\partial \theta_{i}}=\left[-m_{i}(h)+\sum_{i=1}^{n} z_{i}(h) \frac{w_{i}}{\theta_{i}}+(n_{i}-m_{i}(h)) \frac{w_{i}}{\theta_{i}}\right]$$

By equating (3) to zero, we get the following MLEs of $\hat{\alpha}_{i}$ and $\hat{\theta}_{i}$ for $1 \leq h \leq k$ as

$$\frac{(n_{i}-m_{i}(h))(\theta_{i})^{\alpha_{i}} \ln(\theta_{i})+\sum_{i=1}^{n} z_{i}(h)(\theta_{i})^{\alpha_{i}} \ln(\theta_{i})}{(n_{i}-m_{i}(h))(\theta_{i})^{\alpha_{i}}+\sum_{i=1}^{n} z_{i}(h)(\theta_{i})^{\alpha_{i}}} \frac{1}{\hat{\alpha}_{i}}=\frac{1}{m_{i}(h)} \sum_{i=1}^{n} z_{i}(h) \ln(\theta_{i})$$

which can be solved by using an iterative numerical method, and
\[ \hat{\theta}_a = \left( \frac{\sum_{i=1}^{n} z_i (h) (w_i)^{\alpha} + \sum_{i=1}^{n} z_i (h) (w_i)^{\alpha}}{m(h)} \right)^{\frac{1}{\alpha}} \]  

(4)

**Special cases**

From equation (2), different special cases can be obtained such as:

1. For \( h = 2 \), we obtain the two Weibull populations under joint type II censored introduced by Ashour and Abo-Kasem [2] with MLEs as

\[ \hat{\alpha}_1 = \frac{1}{\alpha_1} \sum_{i=1}^{r} \ln(w_i), \]

\[ \hat{\alpha}_2 = \frac{1}{\alpha_2} \sum_{i=1}^{r} \ln(1 - w_i), \]

\[ \hat{\theta}_a = \left( \frac{\sum_{i=1}^{n} z_i (h) (w_i)^{\alpha} + \sum_{i=1}^{n} z_i (h) (w_i)^{\alpha}}{m(h)} \right)^{\frac{1}{\alpha}}, \]

and

\[ \hat{\theta}_z = \left( \frac{\sum_{i=1}^{n} z_i (h) (w_i)^{\alpha} + \sum_{i=1}^{n} z_i (h) (w_i)^{\alpha}}{m(h)} \right)^{\frac{1}{\alpha}}. \]

2. For \( 1 \leq h \leq k \) and \( \alpha_h = 1 \), we obtain multiple exponential populations under joint type-II censoring introduced by Balakrishnan and Feng [4].

3. For \( h = 2 \) and \( \alpha_2 = 1 \), we obtain two exponential populations under joint type-II censoring introduced by Balakrishnan and Rasouli [3].

**Remark:** From the MLEs in (4), it is evident that when \( M(h) = \sum_{i=1}^{r} z_i (h) = 0 \) or \( \hat{\alpha}_a \) or \( \hat{\theta}_a \) do not exist, respectively. Hence, the MLEs in (4) are only conditional MLEs, conditioned on \( 1 \leq M(h) \leq r - 1 \).

### 3. Approximate inference

The approximate asymptotic variance-covariance matrix for \( \alpha_a \) and \( \theta_a \) can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen [5] concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLE's. To obtain elements for information matrix, let

\[ I(\alpha_i, \theta_j) = I_{i,j}(\alpha_i, \theta_j), i, j = 1, 2, \ldots, 2k, \]

and

\[ I(\hat{\alpha}_a, \hat{\theta}_a, \ldots, \hat{\alpha}_a, \hat{\theta}_a) \]

where

\[ I_{i,j}(\alpha_1, \alpha_2, \ldots, \alpha_k, \theta_1, \theta_2, \ldots, \theta_k) = -E \left( \frac{\partial^2 \ln L}{\partial \alpha_i \partial \theta_j} \right) \]

and

\[ I(\hat{\alpha}_a, \hat{\theta}_a, \ldots, \hat{\alpha}_a, \hat{\theta}_a) \]

where

\[ \frac{\partial^2 \ln L}{\partial \alpha_i} = \frac{m(h)}{\alpha_i^2} \sum_{i=1}^{n} z_i (h) \left( w_i - \frac{w_i}{\alpha_i} \right)^{\alpha_i} \ln \left( \frac{w_i}{\alpha_i} \right)^2 + (n_i - m(h)) \left( \frac{w_i}{\alpha_i} \right)^{\alpha_i} \ln \left( \frac{w_i}{\alpha_i} \right)^2, \]

and

\[ \frac{\partial^2 \ln L}{\partial \theta_a} = -m(h) \left( \frac{\alpha_a}{\theta_a} \right) + \left( \frac{\alpha_a + 1}{\theta_a} \right) \sum_{i=1}^{n} z_i (h) \left( w_i - \frac{w_i}{\theta_a} \right) + (n_i - m(h)) \left( \frac{w_i}{\theta_a} \right), \]

and if \( ik = jk \), we obtain
Using the asymptotic normality of the MLEs, we can express the approximate \(100(1-\alpha)\%\) confidence intervals for \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) and \((\theta_1, \theta_2, \ldots, \theta_k)\).

Suppose that \(\hat{\delta}\) is the MLE of the parameter vector \(\delta=(\alpha_1, \alpha_2, \ldots, \alpha_k; \theta_1, \theta_2, \ldots, \theta_k)\). Denote the Fisher information matrix corresponding to \(\delta\) by \(I_{\delta}\) and \(\phi=\lim_{n\to\infty}nI_{\delta}^{-1}\). Then, \(\hat{\delta}\) is asymptotically normal distributed (see Serfling [9]), i.e., \(\sqrt{n}(\hat{\delta}-\delta) \sim N(0, \phi)\). In particular, let \((\hat{S}_{hi}) = \hat{\phi}_{hi}/n, i=1,2,\ldots,k\) are the \((i,i)\) elements in the matrix \(\hat{\phi}=nI_{\delta}^{-1}\) and \(\hat{I}_{\delta}\) is the estimator of \(I_{\delta}\). Therefore, asymptotic normality confidence intervals of \(\delta_{hi}, h=1,2,\ldots,k\) with confidence level \(100(1-\alpha)\%\) are given by

\[
\hat{\delta}_{hi} \pm Z_{\alpha/2} \hat{S}_{hi}
\]

where \(Z_{1-\alpha/2}\) denotes the upper \((1-\alpha)/2\) percentage point of the standard normal distribution.

### 4. Simulation results and discussion

A simulation study was carried out to evaluate the performance of the MLEs and also the 95\% approximate confidence intervals discussed in the preceding sections. We considered different sample sizes for three populations \((i,e,h=3)\) as \(n_1=20,30,60,120\), \(n_2=20,35,75\) and \(n_3=20,35,75\), and different choices of \(r=30,40,50,60,80,100,120,160,200\).

We also chose the parameters \((\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)\) to be \((1,1.5,2,0.5,0.7,0.9)\). For these cases, we computed the MLEs for the parameters \((\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)\), root mean squared errors \(\sqrt{MSE}\), the 95\% approximate confidence intervals, the average widths and the corresponding coverage probabilities. We repeated this process 5000 times and computed the average values of all the estimates. The average value of the MLEs and \(\sqrt{MSE}\) summarized in tables 1. From these values, it is clear that the MLEs have a moderate bias when the essential sample size \(r\) is small even when the sample sizes \((n_1,n_2,n_3)\) are not small. This bias also seems to affect the approximate confidence intervals based on normality as they are not centered properly in this case. However, the biases of the MLEs become negligible when \(r\) increases, and MSE of all the estimates decrease with increasing \(r\) even when the sample sizes \((n_1,n_2,n_3)\) are small, as is evident from table 1.

In table 2, the coverage probabilities of 95\% approximate confidence intervals and the average widths of \((\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)\) for different sample sizes for three populations and different choices of \(r\). From these values, it is clear that the approximate confidence intervals have its coverage probability to be very nearly 95\%.

#### Table 1: The Average Value of the MLEs \((\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)\) and \((\sqrt{MSE})\) For Small, Moderate and Large Values of \((n_1,n_2,n_3)\) and Different Choices of \(r\)

<table>
<thead>
<tr>
<th>((n_1,n_2,n_3))</th>
<th>(r)</th>
<th>(\alpha_1=1)</th>
<th>(\alpha_2=1.5)</th>
<th>(\alpha_3=2)</th>
<th>(\theta_1=0.5)</th>
<th>(\theta_1=0.7)</th>
<th>(\theta_1=0.9)</th>
<th>(\sqrt{MSE})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,20,20)</td>
<td>30</td>
<td>1.097</td>
<td>0.312</td>
<td>1.671</td>
<td>0.529</td>
<td>2.501</td>
<td>0.6</td>
<td>0.518</td>
</tr>
<tr>
<td>40</td>
<td>1.078</td>
<td>0.262</td>
<td>1.577</td>
<td>0.378</td>
<td>2.176</td>
<td>0.562</td>
<td>0.513</td>
<td>0.137</td>
</tr>
<tr>
<td>50</td>
<td>1.071</td>
<td>0.237</td>
<td>1.532</td>
<td>0.305</td>
<td>2.083</td>
<td>0.557</td>
<td>0.51</td>
<td>0.125</td>
</tr>
<tr>
<td>60</td>
<td>1.07</td>
<td>0.253</td>
<td>1.419</td>
<td>0.328</td>
<td>2.385</td>
<td>0.662</td>
<td>0.515</td>
<td>0.145</td>
</tr>
<tr>
<td>70</td>
<td>1.052</td>
<td>0.208</td>
<td>1.369</td>
<td>0.256</td>
<td>2.136</td>
<td>0.668</td>
<td>0.511</td>
<td>0.112</td>
</tr>
<tr>
<td>80</td>
<td>1.045</td>
<td>0.183</td>
<td>1.425</td>
<td>0.221</td>
<td>1.994</td>
<td>0.612</td>
<td>0.508</td>
<td>0.102</td>
</tr>
<tr>
<td>90</td>
<td>1.038</td>
<td>0.168</td>
<td>1.794</td>
<td>0.501</td>
<td>2.603</td>
<td>0.456</td>
<td>0.506</td>
<td>0.089</td>
</tr>
<tr>
<td>100</td>
<td>1.029</td>
<td>0.144</td>
<td>1.644</td>
<td>0.294</td>
<td>2.273</td>
<td>0.443</td>
<td>0.506</td>
<td>0.078</td>
</tr>
<tr>
<td>110</td>
<td>1.026</td>
<td>0.129</td>
<td>1.611</td>
<td>0.248</td>
<td>2.043</td>
<td>0.448</td>
<td>0.505</td>
<td>0.072</td>
</tr>
<tr>
<td>120</td>
<td>1.026</td>
<td>0.141</td>
<td>1.31</td>
<td>0.238</td>
<td>2.071</td>
<td>0.705</td>
<td>0.505</td>
<td>0.077</td>
</tr>
<tr>
<td>130</td>
<td>1.022</td>
<td>0.126</td>
<td>1.375</td>
<td>0.185</td>
<td>1.953</td>
<td>0.64</td>
<td>0.504</td>
<td>0.071</td>
</tr>
<tr>
<td>140</td>
<td>1.02</td>
<td>1.087</td>
<td>1.786</td>
<td>0.429</td>
<td>2.543</td>
<td>0.385</td>
<td>0.504</td>
<td>0.072</td>
</tr>
<tr>
<td>150</td>
<td>1.014</td>
<td>0.696</td>
<td>1.58</td>
<td>0.189</td>
<td>2.2</td>
<td>0.454</td>
<td>0.502</td>
<td>0.055</td>
</tr>
<tr>
<td>160</td>
<td>1.012</td>
<td>0.888</td>
<td>1.559</td>
<td>0.160</td>
<td>2.012</td>
<td>0.462</td>
<td>0.502</td>
<td>0.051</td>
</tr>
<tr>
<td>170</td>
<td>1.027</td>
<td>0.146</td>
<td>1.422</td>
<td>0.223</td>
<td>1.916</td>
<td>0.617</td>
<td>0.505</td>
<td>0.083</td>
</tr>
<tr>
<td>180</td>
<td>1.02</td>
<td>0.123</td>
<td>1.373</td>
<td>0.191</td>
<td>1.817</td>
<td>0.641</td>
<td>0.504</td>
<td>0.068</td>
</tr>
<tr>
<td>190</td>
<td>1.018</td>
<td>0.113</td>
<td>1.419</td>
<td>0.16</td>
<td>1.79</td>
<td>0.597</td>
<td>0.503</td>
<td>0.064</td>
</tr>
</tbody>
</table>
5. Illustrative example

Nelson [7], (Ch. 10, Table 4.1) has given times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The failure times were observed in the form of groups with each group reporting data on 10 insulating fluids. For the purpose of illustrating the methods of inference detailed in the preceding sections, let us consider the following three groups of samples of failure time data presented in table 3.

<table>
<thead>
<tr>
<th>Group</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.31</td>
</tr>
<tr>
<td>2</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>0.49</td>
</tr>
</tbody>
</table>

Suppose the samples of sizes \( n_1=10, n_2=10 \) and \( n_3=10 \) in table 3 are from three Weibull populations with \( (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3) \). Suppose joint type-II censoring with \( r \) as 12, 13 and 15 had been enforced on these data. For example, table 4 presents the jointly type-II censored data that would have been obtained from the data in table 3 with \( r = 15 \).

<table>
<thead>
<tr>
<th>( w )</th>
<th>0.00</th>
<th>0.18</th>
<th>0.31</th>
<th>0.49</th>
<th>0.55</th>
<th>0.64</th>
<th>0.66</th>
<th>0.67</th>
<th>0.82</th>
<th>0.93</th>
<th>1.08</th>
<th>1.30</th>
<th>1.54</th>
<th>1.63</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z(1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z(2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Z(3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We then computed the MLEs of \( (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3) \) and the estimates of their standard deviations for the choices of \( r = 12; 13; 15 \) and these are presented in table 5.

<table>
<thead>
<tr>
<th>( r )</th>
<th>MLEs ( (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3) )</th>
<th>SD ( (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>(1.236,0.552,3.337,3.568,2.328,1.207)</td>
<td>(0.856,0.236,3.723,3.716,2.146,0.18)</td>
</tr>
<tr>
<td>13</td>
<td>(1.022,0.624,3.085,5.002,1.752,1.481)</td>
<td>(0.696,0.241,0.935,5.852,2.121,0.319)</td>
</tr>
<tr>
<td>15</td>
<td>(1.239,0.678,1.653,3.736,1.521,1.923)</td>
<td>(0.683,0.239,0.663,2.535,0.859,0.576)</td>
</tr>
</tbody>
</table>

We have also computed the estimates of the covariance matrix of \( (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3) \) and these are presented in table 6.

From the results in tables 5 and 6, we find the estimates to be quite stable excepting \( \alpha_1 \).
Table 6: Estimates of the Covariance Matrix of the MLEs Based on Jointly Type-II Censored Data from Table 4

<table>
<thead>
<tr>
<th>r</th>
<th>Covariance matrix ( \rho(\alpha, \theta) )_{i,j}</th>
</tr>
</thead>
</table>
|    | \[
| 12 | \begin{pmatrix}
|    | 0.733 & 0 & 0 & -2.658 & 0 & 0 \\
|    | 0 & 0.056 & 0 & 0 & -0.241 & 0 \\
|    | 0 & 0 & 1.882 & 0 & 0 & -0.109 \\
|    | -2.658 & 0 & 0 & 13.811 & 0 & 0 \\
|    | 0 & -0.241 & 0 & 0 & 4.606 & 0 \\
|    | 0 & 0 & -0.109 & 0 & 0 & 0.032 \\
|    | \end{pmatrix} \] |
| 13 | \begin{pmatrix}
|    | 0.485 & 0 & 0 & -3.966 & 0 & 0 \\
|    | 0 & 0.058 & 0 & 0 & -0.094 & 0 \\
|    | 0 & 0 & 0.874 & 0 & 0 & -0.129 \\
|    | -3.966 & 0 & 0 & 46.956 & 0 & 0 \\
|    | 0 & -0.094 & 0 & 0 & 1.466 & 0 \\
|    | 0 & 0 & -0.129 & 0 & 0 & 0.101 \\
|    | \end{pmatrix} \] |
| 15 | \begin{pmatrix}
|    | 0.466 & 0 & 0 & -1.258 & 0 & 0 \\
|    | 0 & 0.057 & 0 & 0 & -0.034 & 0 \\
|    | 0 & 0 & 0.439 & 0 & 0 & -0.163 \\
|    | -1.258 & 0 & 0 & 6.425 & 0 & 0 \\
|    | 0 & -0.034 & 0 & 0 & 0.738 & 0 \\
|    | 0 & 0 & -0.163 & 0 & 0 & 0.332 \\
|    | \end{pmatrix} \] |

Table 7 presents the 95% confidence intervals for \( (\alpha_r, \alpha_s, \theta_1, \theta_2, \theta_3) \) based on the approximate method corresponding to the cases \( r = 12, 13 \) and \( r = 15 \).

Table 7: The 95% Approximate Confidence Intervals for \( (\alpha_r, \alpha_s, \theta_1, \theta_2, \theta_3) \) Based on Jointly Type-II Censored Data from Table 4

<table>
<thead>
<tr>
<th>r</th>
<th>CI for ( \alpha_r )</th>
<th>CI for ( \alpha_s )</th>
<th>CI for ( \alpha_r )</th>
<th>CI for ( \theta_1 )</th>
<th>CI for ( \theta_2 )</th>
<th>CI for ( \theta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>(0.2, 0.9)</td>
<td>(0.08, 1.0)</td>
<td>(0.65, 6.0)</td>
<td>(0.10, 8.5)</td>
<td>(0.6, 5.3)</td>
<td>(0.8, 5.1)</td>
</tr>
<tr>
<td>13</td>
<td>(0.2, 2.3)</td>
<td>(0.07, 1.0)</td>
<td>(2.1, 6.4)</td>
<td>(0.18, 9.3)</td>
<td>(0.4, 1.2)</td>
<td>(0.8, 5.2)</td>
</tr>
<tr>
<td>15</td>
<td>(0.2, 0.5)</td>
<td>(0.08, 1.1)</td>
<td>(2.0, 4.1)</td>
<td>(0.8, 7.0)</td>
<td>(0.3, 2.0)</td>
<td>(0.7, 9.3)</td>
</tr>
</tbody>
</table>

References