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Research Paper

Weibull-Bayesian analysis based on ranked set sampling

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Abstract

Most of estimation methods reported in the literature are based on simple random sampling (SRS), which to certain extent is considerably less effective in estimating the parameters as compared to a new sampling technique, ranked set sampling (RSS) and its modifications.
In this Paper we address the problem of Bayesian estimation of the parameters for Weibull distribution, based on ranked set sampling. Two loss functions have been studied: (i) the squared-error loss function as symmetric loss function, (ii) the linex loss function as asymmetric loss function. Different estimates are compared using simulations for illustrative purposes.

Keywords: *Bayes, Estimation, Loss function, priors, Ranked set sampling.*

1. Introduction

In certain practical problems, actual measurements of a variable of interest are costly or time-consuming, but the ranking of items according to the variable is relatively easy without actual measurement. Under such circumstances a sampling scheme called ranked set sampling (RSS) can be employed to gain more information than simple random sampling (SRS) while keeping the cost of, or the time constraint on, the sampling about the same. McIntyre[1] proposed a method of sampling that synthesizes the convenience of purposive sampling with the control of simple random sampling (SRS) and referred to it as a method of unbiased selective sampling using ranked set. But the rigorous mathematical support to this technique, without referring to the McIntyre's work, was given by Takahasi and Wakimoto [2] and, independently, by Dell [3] while investigating the McIntyre's contribution. In fact, they coined the name ranked set sampling (RSS) for this technique and found empirically that it was more efficient than SRS. As errors may get involved while ordering due to dependence on the ranker's judgment, Dell and Clutter [4] showed that the RSS estimator of a population mean remains unbiased and is at least as efficient as the SRS estimator with the same number of quantifications. Patil et al. [5] provided a bibliography for RSS; Patil et al. [6] showed its use for estimating the level of concentration of Poly Chlorinated Biphenyls (PCB) in soil, while Johnson et al. [7] described its applications in vegetation research and Mode et al. [8] discussed its relevance for ecological research. Under the nonparametric set up Bohn and Wolfe [9][10], and Hettmansperger [11], and Koti and Babu [12] used the sampling method for developing MRSS based nonparametric methods. Bohn [13] presented a review of nonparametric RSS methodology. Muttlak and Al-Saleh [14] discussed some recent developments in RSS. Jemain

et al. [15] have studied the multistage median ranked set sampling for estimating the population median. Samawi et al. [16] have proved that the sign test using bivariate ranked set sampling is asymptotically more efficient than sign test based on a bivariate simple random sampling. Al-Odat [17] suggested a modification of estimating a ratio in rank set sampling. He found that the proposed estimator more efficient than traditional one. Al-Hadhrami et al. [18] studied a Maximum likelihood estimator of the standard deviation of the normal distribution using MERSS. It was found that the suggested estimator is more efficient than the corresponding one based on SRS.
The paper is organized as follows: In Section 2, we present the general setup and basic concepts. Bayesian estimation for the parameters of Weibull distribution using simple random sample (SRS) is considered in Section 3. In Section 4, we develop Bayesian estimation for the proposed parameters using ranked set sampling. Two loss functions have been studied: (i) the squared-error loss function as symmetric loss function, (ii) the LINEX loss function as asymmetric loss function. Finally, simulation results and conclusions are presented in Section 5.

2. Preliminaries

Let $X_1, X_2, \cdots, X_n$ be iid random variables from a Weibull distribution with two parameters, has a probability density function (pdf)

$$f(x) = \alpha\beta x^{(\beta-1)} \exp[-\alpha x^{\beta}]; \quad \alpha, \beta > 0; x \geq 0, \tag{1}$$

and its cumulative distribution function (cdf)

$$F(x; \alpha, \beta) = 1 - \exp[-\alpha x^{\beta}]; \alpha, \beta > 0; x \geq 0. \tag{2}$$

Assuming that the shape parameter β is knowing.

The choice of loss function is an important part of Bayesian approach. It is well known that the most studies use the squared error loss function (SEL) as the basis of measuring estimators performance (See, Box and Tiao [19]; Berger [20]). That is due to its simplicity and relevance with classical procedures, but the squared error loss function is justified loss function only where losses are symmetric in nature. In the estimation of the survival function or reliability function , using of symmetric loss function may be inappropriate.
A number of asymmetric loss functions may be found in the literature (See, Zellner [21]) but among these the linear - exponential(LINEX) loss function, introduced by Varian [22], is dominantly and widely used because it is a natural extension of the squared error loss function (SEL).

The LINEX loss function for the parameter θ can be expressed as

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1; c \neq 0, \tag{3}$$

where $\Delta = (\theta^* - \theta)$; θ^* is an estimate of θ.

The sign and magnitude of the shape parameter c represent the direction and degree of symmetry, respectively. When value of c is less than zero, the LINEX loss function gives more weight to under-estimation against over-estimation and the situation is reverse when value of c is greater than zero.

For c close to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The Bayes estimator of θ under the LINEX loss function, denoted by $\tilde{\theta}_{Lnx}$, is the value which minimize $E[(L(\theta^* - \theta)]$, it is

$$\tilde{\theta}_{Lnx} = -\frac{1}{c} \ln[E(-c\theta)]. \tag{4}$$

In this study, the Bayes estimates based on SRS and RSS are derived for the unknown parameter α . The Bayes estimates are obtained based on a conjugate prior and a non-informative prior for the scale parameter of this model. This is done with respect to both symmetric loss function (squared error loss) , and asymmetric loss function (LINEX).

3. Bayes estimator based on SRS

3.1. Conjugate prior

Let $X_1, X_2, \cdots, X_n$ be iid random variables has Weibull distribution with parameters α, β but under the assumption that the shape parameter β is knowing and $\pi(\alpha)$ is the conjugate prior of the Weibull density, i.e., $\alpha \sim Gamma(a, b)$ whose density function is given by $\pi(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{(a-1)} \exp[-b\alpha], \alpha > 0; a, b > 0$.

The posterior density of α for this SRS can be written as

$$
\begin{aligned}
g(\alpha \mid \underline{x}) &= \frac{\pi(\alpha) \prod_{i=1}^{n} f(x_i)}{\int_\alpha \pi(\alpha) \prod_{i=1}^{n} f(x_i) d\alpha} \\
&= \frac{[\frac{b^a}{\Gamma(a)}]\alpha^{(n+a-1)}\beta^n(\prod_{i=1}^{n} x_i^{(\beta-1)}) \exp[-\alpha(b+\sum_{i=1}^{n} x_i^\beta)]}{\int_0^\infty [\frac{b^a}{\Gamma(a)}]\alpha^{(n+a-1)}\beta^n(\prod_{i=1}^{n} x_i^{(\beta-1)}) \exp[-\alpha(b+\sum_{i=1}^{n} x_i^\beta)] d\alpha},
\end{aligned}
$$

then,

$$
g(\alpha \mid \underline{x}) = \frac{\alpha^{(n+a-1)}(b+\sum_{i=1}^{n} x_i^\beta)^{(a+n)} \exp[-\alpha(b+\sum_{i=1}^{n} x_i^\beta)]}{\Gamma(a+n)}.
$$

The Bayesian estimation of α based on squared error loss function(SEL) is

$$
\begin{aligned}
\alpha^*_{Se} &= E(\alpha \mid X) = \int_0^\infty \alpha g(\alpha \mid \underline{x}) d\alpha \\
&= \frac{(b+\sum_{i=1}^{n} x_i^\beta)^{(a+n)}}{\Gamma(a+n)} \int_0^\infty \alpha^{(a+n)} \exp[-\alpha(b+\sum_{i=1}^{n} x_i^\alpha)] d\alpha \\
&= \frac{(b+\sum_{i=1}^{n} x_i^\beta)^{(a+n)}\Gamma(a+n+1)}{\Gamma(a+n)(b+\sum_{i=1}^{n} x_i^\beta)^{(a+n+1)}},
\end{aligned}
$$

therefore,

$$
\alpha^*_{Se} = \frac{(a+n)}{(b+\sum_{i=1}^{n} x_i^\beta)}. \tag{5}
$$

The Bayesian estimation of α based on LINEX loss function is

$$
\begin{aligned}
\alpha^*_{Lnx} &= -\frac{1}{c} \ln E(e^{-c\alpha}) \\
&= -\frac{1}{c} \ln \int_0^\infty \frac{e^{-c\alpha}\alpha^{a+n-1}(b+\sum_{i=1}^{n} x_i^\beta)^{a+n} e^{-\alpha(b+\sum_{i=1}^{n} x_i^\beta)} d\alpha}{\Gamma(a+n)} \\
&= -\frac{1}{c} \ln \Big[\frac{(b+\sum_{i=1}^{n} x_i^\beta)^{a+n}}{\Gamma(a+n)} \int_0^\infty \alpha^{a+n-1} e^{-\alpha(b+\sum_{i=1}^{n} x_i^\beta + c)} d\alpha \Big] \\
&= -\frac{1}{c} \ln \Big[\frac{(b+\sum_{i=1}^{n} x_i^\beta)^{a+n}}{(b+\sum_{i=1}^{n} x_i^\beta + c)^{a+n}} \Big],
\end{aligned}
$$

then,

$$
\alpha^*_{Lnx} = -\frac{a+n}{c} \ln \Big[\frac{(b+\sum_{i=1}^{n} x_i^\beta)}{(b+\sum_{i=1}^{n} x_i^\beta + c)} \Big]. \tag{6}
$$

3.2. Non–Informative prior

If prior information about α is scanty , it may be appropriate to resort to the use of a diffuse prior distribution. Improper prior density for α , which can reasonably be accepted , is the Jeffrey*ś* prior distribution. The non–informative prior distribution of the parameter α under complete sampling is given by $g(\alpha) \propto \sqrt{I(\alpha)}$. Therefore, $g(\alpha) \propto \frac{1}{\alpha}$; $\alpha > 0$ where $I(\alpha)$ is the Fisher information for complete sampling , using the likelihood function and the prior distribution of α then the posterior distribution of α is

$$\begin{aligned} g(\alpha \mid \underline{x}) &= \frac{(\frac{1}{\alpha}) \prod\limits_{i=1}^{n} f(x_i \mid \alpha)}{\int_0^\infty (\frac{1}{\alpha}) \prod\limits_{i=1}^{n} f(x_i \mid \alpha) d\alpha} \\ &= \frac{(\frac{1}{\alpha})(\alpha\beta)^n (\prod\limits_{i=1}^{n} x_i^{\beta-1}) e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta}}{\int_0^\infty \beta^n \alpha^{(n-1)} (\prod\limits_{i=1}^{n} x_i^{\beta-1}) e^{-\alpha \sum\limits_{i=1}^{n} x_i{}^\beta} d\alpha} \\ &= \frac{\alpha^{(n-1)} e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta} (\sum\limits_{i=1}^{n} x_i^\beta)^n}{\Gamma(n)} \end{aligned}$$

Then:

$$g(\alpha \mid \underline{x}) = \frac{\alpha^{n-1} e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta} (\sum\limits_{i=1}^{n} x_i^\beta)^n}{\Gamma(n)} \tag{7}$$

we can noted that the posterior distribution of α is gamma distribution with parameter $(n, \sum\limits_{i=1}^{n} x_i^\beta)$. Then, the Bayesian estimation of α based on squared error loss function is :

$$\begin{aligned} \widehat{\alpha}_{Se} = E(\alpha) &= \int_0^\infty \frac{\alpha^n e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta} (\sum\limits_{i=1}^{n} x_i^\beta)^n}{\Gamma(n)} d\alpha \\ &= \frac{(\sum\limits_{i=1}^{n} x_i^\beta)^n}{\Gamma(n)} \int_0^\infty \alpha^n e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta} d\alpha \\ &= \frac{(\sum\limits_{i=1}^{n} x_i^\beta)^n \Gamma(n+1)}{\Gamma(n)(\sum\limits_{i=1}^{n} x_i^\beta)^{n+1}} \\ &= \frac{n}{\sum\limits_{i=1}^{n} x_i^\beta} \end{aligned}$$

The Bayesian estimation of the α based on LINEX loss function is

$$\begin{aligned} \widehat{\alpha}_{Lnx} &= -\frac{1}{c} \ln[E(e^{-c\alpha})] \\ &= -\frac{1}{c} \ln\Big[\int_0^\infty \frac{e^{-c\alpha} \alpha^{n-1} (\sum\limits_{i=1}^{n} x_i^\beta)^n e^{-\alpha \sum\limits_{i=1}^{n} x_i^\beta}}{\Gamma(n)} d\alpha \Big] \\ &= -\frac{1}{c} \ln\Big[\frac{(\sum\limits_{i=1}^{n} x_i^\beta)^n}{(\sum\limits_{i=1}^{n} x_i^\beta + c)^n} \Big] \end{aligned}$$

Then, the Bayesian estimation of the α based on LINEX loss function is

$$\widehat{\alpha}_{Lnx} = -\frac{n}{c}\ln\Big[\frac{\sum\limits_{i=1}^{n} x_i^{\beta}}{\sum\limits_{i=1}^{n} x_i^{\beta}+c}\Big] \tag{8}$$

4. Bayes estimator based on RSS

Assume that the variable of interest has a density function $f(x \mid \alpha)$ and a distribution function $F(x \mid \alpha)$, both of them are known, and α has a prior density function $\pi(\alpha)$. Let $X_1, X_2, \cdots, X_n$ be a SRS and $X_{11}, X_{12}, \cdots, X_{1n}$; $X_{21}, X_{22}, \cdots, X_{2n}$; $\cdots$; $X_{n1}, X_{n2}, \cdots, X_{nn}$ be the visual(judgment) order statistics of n sets of simple random samples each set of size (n).

we will assume throughout the study that there is no error in ranking, i.e. the visual ranking is the same as the actual ranking.

Let $Y_{i1}, Y_{i2}, \cdots, Y_{in}$ be a set of order statistics, where Y_{i1} is taken from the first set, $Y_{i2}$is taken from the second set and Y_{in} is taken from the last set, then $(Y_{11}, Y_{22}, \cdots, Y_{nn})$ is called a balanced RSS (BRSS) and the density of Y_i is

$$\begin{aligned} g(y_i \mid \alpha,\beta) &= \frac{n!}{(i-1)!(n-i)!}[F(y_i \mid \alpha,\beta)]^{i-1}[1-F(y_i \mid \alpha,\beta)]^{n-i} f(y_i \mid \alpha,\beta) \\ &= \sum_{k=0}^{i-1} i\binom{n}{i}\binom{i-1}{k}(-1)^k\,\alpha\beta y_i^{\beta-1} e^{-\alpha y_i^{\beta}(k+n-i+1)} \end{aligned}$$

the joint density is given by:

$$g(\underline{y} \mid \alpha,\beta) = \prod_{i=1}^{n}\sum_{k=0}^{i-1}[i\binom{n}{i}\binom{i-1}{k}(-1)^k]\alpha\beta y_i^{\beta-1} e^{-\alpha y_i^{\beta}(k+n-i+1)}$$

From Takahasi and Wakimoto (1968), we have

$$f(x|\alpha) = \frac{1}{n}\sum_{i=1}^{n} f_i(x|\alpha).$$

Where $f_i(x|\alpha)$ is the density of the ith order statistic in a random sample of size n evaluated at x. Thus,

$$f(\underline{x}|\alpha) = \prod_{i=1}^{n} f(x_i|\alpha) = \frac{1}{n^n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n}\prod_{j=1}^{n} f_{i_j}(x_j|\alpha).$$

Then

$$g(\underline{y} \mid \alpha,\beta) = \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}](\alpha\beta)^n(\prod_{j=1}^{n} y_j)^{\beta-1} e^{-\alpha\sum\limits_{j=1}^{n} y_j^{\beta}(i_j+n-j+1)}.$$

4.1. Conjugate prior for α

Under the assumption that the shape parameter β is know, we assume a gamma with parameters a, b is conjugate prior for $(\alpha.)$ Then the posterior density of α is

$$\begin{aligned} \pi^*(\alpha) &= \frac{\pi(\alpha) g(\underline{y_i} \mid \alpha,\beta)}{\int_0^{\infty}\pi(\alpha) g(\underline{y_i} \mid \alpha,\beta)d\alpha} \\ &= \frac{\sum\limits_{i_1=0}^{0}\sum\limits_{i_2=0}^{1}\cdots\sum\limits_{i_n=0}^{n-1}(\prod\limits_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^{n+a-1} e^{-\alpha[\sum\limits_{j=1}^{n} y_j^{\beta}(i_j+n-j+1)+b]}}{\sum\limits_{i_1=0}^{0}\sum\limits_{i_2=0}^{1}\cdots\sum\limits_{i_n=0}^{n-1}(\prod\limits_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\big[\sum\limits_{j=1}^{n} y_j^{\beta}(i_j+n-j+1)+b\big]^{a+n}}} \end{aligned}$$

Then the Bayesian estimation of α based on squared error loss function; i.e, the posterior mean of α, is

$$\begin{aligned}
\tilde{\alpha}_{Se} &= E(\alpha) \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\int_0^\infty \alpha^{n+a} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b]}d\alpha}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{a+n}}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n+1)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{(a+n+1)}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{a+n}}} \qquad (9)
\end{aligned}$$

To obtain the Bayesian estimation of α based on LINEX loss function, we need to calculate the posterior expectation of $e^{-c\alpha}$,

$$\begin{aligned}
E(e^{-c\alpha}) &= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\int_0^\infty \alpha^{n+a-1} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b+c]}d\alpha}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{a+n}}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b+c\right]^{a+n}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(a+n)}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{a+n}}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{1}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b+c\right]^{a+n}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{1}{\left[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+b\right]^{a+n}}}
\end{aligned}$$

Now the Bayesian estimation of α on LINEX is

$$\tilde{\alpha}_{Lnx} = \frac{-1}{c}\ln[E(e^{-c\alpha})] \qquad (10)$$

4.2. Non–informative prior for α

The non-informative prior distribution of the parameter α is given by $g(\alpha) \propto \frac{1}{\alpha}$, $\alpha > 0$ Then, the posterior density can be written as:

$$\begin{aligned}
\pi^*(\alpha) &= \frac{(\frac{1}{\alpha})g(\underline{y_i}\mid\alpha,\beta)}{\int_0^\infty(\frac{1}{\alpha})g(\underline{y_i}\mid\alpha,\beta)d\alpha} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^{n-1} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]}}{\int_0^\infty\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^{n-1} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]}d\alpha} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^{n-1} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\Gamma(n)[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]^{(-n)}},
\end{aligned}$$

then the Bayesian estimation of α based on squared error loss function; i.e.

$$\begin{aligned}
\check{\alpha}_{Se} &= E(\alpha) \\
&= \frac{\int_0^\infty \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^n e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]}d\alpha}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\Gamma(n)[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]^{(-n)}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(n+1)}{[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]^{(n+1)}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(n)}{[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)]^{n}}}
\end{aligned}$$

To obtain the Bayesian estimation of α based on LINEX loss function, we need to calculate the posterior expectation of $e^{-c\alpha}$

$$\begin{aligned}
E(e^{-c\alpha}) &= \frac{\int_0^\infty \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\alpha^{n-1} e^{-\alpha[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+c]}d\alpha}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\Gamma(n)\big[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)\big]^{(-n)}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(n)}{\big[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+c\big]^{n}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{\Gamma(n)}{\big[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)\big]^{n}}} \\
&= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{1}{\big[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)+c\big]^{n}}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}(\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j})\frac{1}{\big[\sum_{j=1}^{n} y_j^\beta(i_j+n-j+1)\big]^{n}}}
\end{aligned}$$

Then,

$$\check{\alpha}_{Lnx} = \frac{-1}{c}\ln[E(e^{-c\alpha})] \tag{11}$$

5. Simulation study and conclusion

In this section, we carry out Monte Carlo simulations to the Bayes estimators based on ranked set sampling (RSS) and based simple random sampling (SRS) for Weibull distribution with informative prior and non informative prior for the scale parameter (α), and calculate the bias and the mean squared error of the estimators , taking in consideration the shape parameter β is known, for different values of α, [$\alpha = 0.5$ and $\alpha = 1$], Also at different sizes of the sample [n=3, n=4 and n=5], all this to illustrate the performance of the ranked set sampling.

From Table 1, we first of all observe that the Bayesian estimates of α are all biased. Next, we observe that the Bayesian estimates based on RSS are considerably less biased than the corresponding Bayesian estimates based on SRS. From Table 2, we first note that the mean squared error of all estimates decrease when n increases, as one would expect. Next, we observe that the Bayesian estimates based on RSS have a much smaller mean squared error than the corresponding Bayesian estimates based on RSS in all cases considered.

Now calculate the relative efficiency of the estimates based on ranked set sampling with respect to estimates based on simple random sampling as follows

$$eff^J_{(Se)} = \frac{MSE(\widehat{\alpha}_{Se})}{MSE(\check{\alpha}_{Se})}, \quad eff^J_{(Lnx)} = \frac{MSE(\widehat{\alpha}_{Lnx})}{MSE(\check{\alpha}_{Lnx})},$$

$$eff^G_{(Se)} = \frac{MSE(\alpha^*_{Se})}{MSE(\tilde{\alpha}_{Se})}, \quad eff^G_{(Lnx)} = \frac{MSE(\alpha^*_{Lnx})}{MSE(\tilde{\alpha}_{Lnx})}.$$

Table 1: Bias of the Bayesian estimates based on SRS and RSS

For $\alpha = 0.5$, (when $\beta = 0.5,\ a = 1,\ b = 0.5$)									
n	Bias(α_{Se})					Bias (α_{Lnx})			
	Jeffrey prior		Gamma prior		C	Jeffrey prior		Gamma prior	
	SRS $\widehat{\alpha}_{Se}$	RSS $\check{\alpha}_{Se}$	SRS α^*_{Se}	RSS $\tilde{\alpha}_{Se}$		SRS $\widehat{\alpha}_{Lnx}$	RSS $\check{\alpha}_{Lnx}$	SRS α^*_{Lnx}	RSS $\tilde{\alpha}_{Lnx}$
3	-0.9763	-0.7284	-0.3265	-0.1834	1	-0.9418	-0.7113	-0.3151	-0.1790
					-1	-1.0138	-0.7465	-0.3386	-0.1879
4	-0.1648	-0.0623	-0.2408	-0.0987	1	-0.1574	-0.0602	-0.2336	-0.0966
					-1	-0.1724	-0.0644	-0.2482	-0.1008
5	-0.1227	-0.0332	-0.2022	-0.0685	1	-0.1177	-0.0367	-0.1971	-0.0673
					-1	-0.1278	-0.0331	-0.2075	-0.0698
For $\alpha = 1$, (when $\beta = 0.5,\ a = 1,\ b = 0.5$)									
3	-0.4525	-0.2131	-0.5055	-0.2416	1	-0.3944	-0.1963	-0.4702	-0.2275
					-1	-0.5301	-0.2307	-0.5440	-0.2562
4	-0.3027	-0.1222	-0.3683	-0.1848	1	-0.2736	-0.1142	-0.3454	-0.1769
					-1	-0.3350	-0.1303	-0.3925	-0.1930
5	-0.2714	-0.0770	-0.3057	-0.1054	1	-0.2512	-0.0722	-0.2892	-0.1008
					-1	-0.2928	-0.0818	-0.3228	-0.1102

Table 2: MSE of the Bayesian estimates based on SRS and RSS

For $\alpha = 0.5$, (when $\beta = 0.5,\ a = 1,\ b = 0.5$)									
n	MSE(α_{Se})					MSE (α_{Lnx})			
	Jeffrey prior		Gamma prior		C	Jeffrey prior		Gamma prior	
	SRS $\widehat{\alpha}_{Se}$	RSS $\check{\alpha}_{Se}$	SRS α^*_{Se}	RSS $\tilde{\alpha}_{Se}$		SRS $\widehat{\alpha}_{Lnx}$	RSS $\check{\alpha}_{Lnx}$	SRS α^*_{Lnx}	RSS $\tilde{\alpha}_{Lnx}$
3	1.6504	0.9153	0.3655	0.1278	1	1.5145	0.8658	0.3414	0.1236
					-1	1.8096	0.9704	0.3926	0.1322
4	0.1864	0.0506	0.2384	0.0526	1	0.1760	0.0496	0.2263	0.0516
					-1	1978	0.0517	0.2516	0.0537
5	0.1335	0.0249	0.1694	0.0309	1	0.1274	0.0246	0.1630	0.0305
					-1	0.1400	0.0252	0.1764	0.0313
For $\alpha = 1$, (when $\beta = 0.5,\ a = 1,\ b = 0.5$)									
3	1.9958	0.4282	0.9343	0.3505	1	1.5172	0.3974	0.8324	0.3285
					-1	3.1024	0.4633	1.05650	0.3748
4	0.8390	0.1762	0.6196	0.2016	1	0.7266	0.1696	0.5668	0.1939
					-1	0.9946	0.1833	0.6802	0.2097
5	0.5360	0.1064	0.3566	0.1015	1	0.4925	0.1037	0.3785	0.0989
					-1	0.5856	0.1091	0.2967	0.1042

Table 3: Relative efficiency when $\alpha = 0.5$ and $\alpha = 1$

n	$\text{eff}^{J}_{(Se)}$		$\text{eff}^{G}_{(Se)}$		c	$\text{eff}^{J}_{(Lnx)}$		$\text{eff}^{G}_{(Lnx)}$	
	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$		$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
3	1.8031	4.6609	2.8599	2.6656	1	1.7492	3.8178	2.7621	2.5339
					-1	1.8648	6.6963	2.9697	2.8188
4	3.6838	4.7616	4.5323	3.0734	1	3.5483	4.2842	4.3857	2.9232
					-1	3.8259	5.4261	4.6853	3.2437
5	5.3614	5.0376	5.4822	3.9941	1	5.1789	4.7445	5.3443	3.8271
					-1	5.5556	5.3676	5.6294	4.1747

We find that the results of the relative efficiency are greater than one, because the numerator is the mean squared error for Bayes estimators based on simple random sampling (SRS) while the denominator is the mean squared error for Bayes estimators based on ranked set sampling (RSS), so ranked set sampling more effective than simple random sampling.

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