

Estimating the Parameters of the Exponential-Geometric distribution based on progressively type-II censored data

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Abstract

This paper deals with the problem of parameters estimation of the Exponential-Geometric (EG) distribution based on progressive type-II censored data. It turns out that the maximum likelihood estimators for the distribution parameters have no closed forms, therefore the EM algorithm are alternatively used. The asymptotic variance of the MLEs of the targeted parameters under progressive type-II censoring is computed along with the asymptotic confidence intervals. Finally, a simple numerical example is given to illustrate the obtained results.

Keywords: EM algorithm ; Exponential-Geometric distribution ; Maximum Likelihood Estimator ; Progressive type-II Censoring

1. Introduction

With no doubt, the analysis of lifetime data has become a very important subject in different scientific disciplines. A numerous lifetime distributions have shown how they are so important to model perfectly practical situations. In the literature, there are a huge number of papers dealing with distributions that have been developed by compounding some other life distributions. This is actually necessary to fulfill the need of modeling so many complicated real problems. For example, Adamidis and Loukas 1998, (see reference [1]), introduced a two-parameter exponential-geometric distribution with decreasing failure rate by compounding an exponential with a geometric distributions. The new proposed distribution, in fact, has some desirable properties and nice physical interpretations. Another distribution that can be considered as a counterpart of the EG distribution is established by [2], and is called the complementary exponential-geometric (CEG) distribution.

In real life, especially in scientific experiments, it is desirable to study a censored data rather than a complete data set. Censoring is a common technique in survival analysis and is used effectively in various fields such as statistics, engineering and medical disciplines. It appears when exact lifetimes of entities or components are only partially known. There are different types of censoring that can be considered; type-I, type-II, progressively type-II and hybrid censoring schemes. The type-I and type-II censoring schemes are known to be the most extensively used. A short summary of censoring types is given here: In type-I censoring, a life test is implemented for a fixed-time period, while in type-II censoring, an experiment stops when a pre-specified number of units fail. It is noted that both type-I and type-II do not allow for removal of elements at points rather than termination point of an experiment. Therefore, the progressive type-II has been introduced to overcome the problem of units' removals. For comprehensive overview on this topic, reader might be referred to [3].

The parameter estimation is a challenging task especially in the case of compounded distributions. For this reason, computational methods are often sought, see for example, [3] and references therein for a comprehensive review of results in this direction. Inference procedures based on the maximum likelihood estimators have been considered when the underlying distribution is Gaussian, Gumpertz, bivariate normal, generalized logistic and the lifetime of a k -unit parallel system distributions, see for more details, [4], [5], [6], [7] and [8]. In some cases, MLEs cannot be obtained in a closed form; therefore, numerical methods can be employed. For example, [9], [10] studied parameter estimation under progressive type-II censored data using the Expectation-Maximization (EM) algorithm. It is an alternative procedure to compute MLEs in cases where the observed (complete) data could be incomplete. For more details on how to apply the EM algorithm, see [11], [12], [13], [14] and [15].

In this paper, we mainly study the properties of the EG distribution and estimate the unknown parameters based on progressive type-II censored data. A continuous random variable X is said to have EG distribution with parameters β and p if its probability density function (pdf) is given by

$$f(x; \theta) = \frac{\beta(1-p)e^{-\beta x}}{(1-pe^{-\beta x})^2}; x > 0, \beta > 0, 0 < p < 1, \quad (1)$$

where $\theta = (\beta, p)$; such that β and p are scale and shape parameters, respectively. The corresponding cumulative distribution function (cdf) and the survival function of EG distribution; respectively, are given by

$$F(x; \theta) = \frac{1 - e^{-\beta x}}{1 - pe^{-\beta x}}; x > 0, \beta > 0, 0 < p < 1, \quad (2)$$

$$s(x; \theta) = \frac{e^{-\beta x}(1-p)}{1 - pe^{-\beta x}}; x > 0, \beta > 0, 0 < p < 1, \quad (3)$$

The rest of the paper is structured as follows: In Section 2, the maximum likelihood estimators of the EG distribution (MLEs) based on progressive type-II censored data will be derived, using the EM algorithm. In Section 3, the asymptotic variance of the MLE of the targeted parameter will be computed along with their asymptotic confidence intervals. In Section 4, a numerical example will be introduced to illustrate the obtained results under different censoring schemes. Finally, a conclusion will be stated in Section 5.

2. Parameters estimation

In progressively type-II censored scheme, n units are placed on a test at time zero, when $m < n$ failures to be observed and the censoring scheme $\mathbf{R} = \mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_m$ are fixed before the experiment with $\mathbf{R}_i \geq 0$ and $\sum_{i=1}^m \mathbf{R}_i = n - m$ are specified. At the first failure time $\mathbf{X}_{1:m:n}$, $\mathbf{R}_1$ units, chosen at random, are removed from the $n - 1$ remaining units. At the second failure time $\mathbf{X}_{2:m:n}$, $\mathbf{R}_2$ randomly chosen units from the remaining $n - 2 - \mathbf{R}_1$ units are removed. The experiment continues until the m^{th} failure time $\mathbf{X}_{m:m:n}$. At this time, all remaining units are removed, and $\mathbf{R}_m = n - m - \sum_{i=1}^{m-1} \mathbf{R}_i$. The maximum likelihood function is given in the following subsection.

2.1. Maximum likelihood estimation

Suppose that the failure times of n units on experiment have a pdf and cdf given by (1) and (2) respectively. Then the likelihood function of $X = (X_1, \dots, X_n)$ under progressively type-II censored sample is given by

$$L(X|\theta) = C \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{R_i} \\ = C \prod_{i=1}^m \frac{\beta(1-p)e^{-\beta x_i}}{(1-pe^{-\beta x_i})^2} \left[\frac{(1-p)e^{-\beta x_i}}{(1-pe^{-\beta x_i})} \right]^{R_i}, \quad (4)$$

where $C = n(n - R_1 - 1) \dots (n - \sum_{i=1}^{m-1} R_i - m + 1)$. The log-likelihood function with ignoring constant can be written as follows.

$$\mathcal{L}(X|\theta) = m \ln \beta + \sum_{i=1}^m (1 + R_i) \ln(1 - p) - \beta \sum_{i=1}^m x_i (R_i + 1) \\ = - \sum_{i=1}^m (R_i + 2) \ln(1 - pe^{-\beta x_i}) \quad (5)$$

Consequently, the Fisher's score functions is given by

$$\frac{\partial \log \mathcal{L}}{\partial \beta} = \frac{m}{\beta} - \frac{p \sum_{i=1}^m (R_i + 2) x_i e^{-\beta x_i}}{(1 - pe^{-\beta x_i})} \\ - \sum_{i=1}^m (R_i + 1) x_i. \quad (6)$$

$$\frac{\partial \log \mathcal{L}}{\partial p} = - \frac{\sum_{i=1}^m (R_i + 1)}{(1 - p)} + \frac{\sum_{i=1}^m (R_i + 2) e^{-\beta x_i}}{(1 - pe^{-\beta x_i})}. \quad (7)$$

The maximum likelihood estimators, say $\hat{\beta}$ and $\hat{p}$, of β and p are the solutions of the equations (6) and (7). We note that they cannot be achieved in closed forms. Therefore, we use the EM algorithm to solve these equations.

2.2. Expectation-Maximization algorithm

The EM algorithm is used to compute the estimators based on the maximum likelihood in the presence of missing or incomplete data. Since the progressive type-II censoring can be considered as an

incomplete data set (see [8] and [9]), therefore, the EM algorithm is an alternative method to find the MLEs numerically. The EM algorithm consists of an E-step (Estimation step) followed by an M-step (Maximization step) defined as follows:

E-step: Compute $Q(\theta; \theta^t)$ where $Q(\theta; \theta^t) = E_{\theta^t}[\ell(\theta; x)|Y]$.

M-step: Find θ^{t+1} in θ such that $Q(\theta^{t+1}; \theta^t) \geq Q(\theta; \theta^t)$ for all $\theta \in \theta$.

The E-step and the M-step are repeated interchangeably until the difference $L(\theta^{t+1}) - L(\theta^t)$ is less than δ , where δ is a small quantity. Now, assume that $Y = (Y_1, \dots, Y_m)$ in such $Y_1 < \dots < Y_m$ represents the progressive type-II censored (observed) data from a population with pdf and cdf that given in equations (1) and (2), respectively. Further, let $Z = (Z_1, Z_2, \dots, Z_{n-m})$ be the censored (unobserved) data, which can be considered as missing data. The combination vector of Y and Z , say $W = (X, Z)$, is the complete data. Then, the maximum likelihood estimates of $\theta = (\beta, p)$ based on the distribution of W , ignoring the additive constant, is

$$L_c(W; \theta) = n \ln(\beta) + \sum_{i=1}^m (R_i + 1) \ln(1 - p) \\ - \beta \sum_{i=1}^m y_i (R_i + 1) \\ - \sum_{i=1}^m (R_i + 2) \ln(1 - pe^{-\beta y_i}) \\ - \beta \sum_{i=1}^{n-m} z_i (R_i + 1) \\ - \sum_{i=1}^{n-m} (R_i + 2) \ln(1 - pe^{-\beta z_i}). \quad (8)$$

The E-step of the EM algorithm emerges from $L_c(W; \theta)$ by substituting any function of z_i say $g(z_i)$ with $L_s(W; \theta) = E[g(z_i) | z_i > y_m]$. Therefore, the log-likelihood function becomes

$$L_s(W; \theta) = n \ln(\beta) + \sum_{i=1}^m (R_i + 1) \ln(1 - p) \\ - \beta \sum_{i=1}^m y_i (R_i + 1) \\ - \sum_{i=1}^m (R_i + 2) \ln(1 - pe^{-\beta y_i}) \\ - (n - m)[\beta A(y_m, \theta) + B(y_m, \theta)], \quad (9)$$

Here, $A(y_m, \theta)$ and $B(y_m, \theta)$ are as follows

$$A(y_m, \theta) = E[(R_i + 1) z_i | z_i > y_m] \\ B(y_m, \theta) = E[(R_i + 2) \log(1 - pe^{-\beta z_i}) | z_i > y_m] \quad (10)$$

To solve the above two equations in Equation (9), the following theorem should be stated first:

Theorem 1 Given a random variable $Y = (Y_1, Y_2, \dots, Y_m)$ distributed as EG distribution, $Y \sim EG(\beta, p)$, the conditional distribution of $Z_i; i = 1, 2, 3, \dots, n - m$, can be written as

$$f_{Z|Y}(Z_i | Y_{m:n} = y_{m:n}) = \begin{cases} \frac{f(Z_i|\theta)}{[1 - F(Y_{m:n}|\theta)]} & ; z_i > y_{m:n} \\ 0 & ; \text{otherwise} \end{cases} \\ = \begin{cases} \beta D \frac{e^{-\beta z_i}}{(1 - pe^{-\beta z_i})^2} & ; z_i > y_{m:n} \\ 0 & ; \text{otherwise}, \end{cases} \quad (11)$$

Here D is a function of y and is given by

$$D = \frac{(1 - pe^{-\beta y_m})}{e^{-\beta y_m}}.$$

A proof of the theorem can be found in details in ref. [9]. Therefore, the expectations in (10) can be written as follows:

$$A(y_m, \theta) = -\frac{(R+1)D}{p\beta} [\ln(1-p)],$$

$$B(y_m, \theta) = -(R+2) \frac{1}{pe^{-\beta y_m}} [\ln(1-pe^{-\beta y_m}) + pe^{-\beta y_m}].$$

In M-step of the EM algorithm which involves the maximization of equation (9) with respect to θ . If at the k th stage the estimate of θ is $\hat{\theta}^{(t)}$ then $\hat{\theta}^{(t+1)}$ is computed by maximizing the following equation:

$$g(\theta) = n \ln(\beta) + \sum_{i=1}^m (R_i + 1) \ln(1-p) - \beta \sum_{i=1}^m y_i (R_i + 1) - \sum_{i=1}^m (R_i + 2) \ln(1-pe^{-\beta y_i}) - (n-m)[\beta A(y_m, \hat{\theta}^{(t)}) + B(y_m, \hat{\theta}^{(t)})]. \quad (12)$$

Differentiating (12) with respect to $\theta = (\beta, p)$ and setting the results equal to 0, yields the two following equations:

$$\frac{\partial g}{\partial \beta} = \frac{n}{\beta^{(t+1)}} - \frac{p \sum_{i=1}^m (R_i + 2) y_i e^{-\beta^{(t+1)} y_i}}{(1 - pe^{-\beta^{(t+1)} y_i})} - \sum_{i=1}^m (R_i + 1) y_i - (n-m) A(y_m, \hat{\theta}^{(t)}) = 0$$

$$\frac{\partial g}{\partial p} = -\frac{\sum_{i=1}^m (R_i + 1)}{(1 - p^{(t+1)})} + \frac{\sum_{i=1}^m (R_i + 2) e^{-\beta y_i}}{(1 - p^{(t+1)} e^{-\beta y_i})} = 0$$

Next, we find $\hat{\theta}^{(t)} = (\hat{\beta}^{(t+1)}, \hat{p}^{(t+1)})$ by solving fixed-point type equations, $h(\theta) = \theta$, where

$$h(\beta) = n \left[\frac{p \sum_{i=1}^m (R_i + 2) y_i e^{-\beta^{(t+1)} y_i}}{(1 - pe^{-\beta^{(t+1)} y_i})} + \sum_{i=1}^m (R_i + 1) y_i + (n-m) A(y_m, \hat{\theta}^{(t)}) \right]^{-1},$$

and

$$h(p) = 1 - 1 - \sum_{i=1}^m (R_i + 1) \left[\frac{\sum_{i=1}^m (R_i + 2) e^{-\beta y_i}}{(1 - p^{(t+1)} e^{-\beta y_i})} \right]^{-1}.$$

Subsequently, the E-step and the M-step are then repeated until convergence is achieved to the desired level of accuracy.

3. Asymptotic variance

In this section, we compute the asymptotic variance of MLEs using the idea of missing information principle see [9]. In addition, the asymptotic variance of MLEs can be used to construct the asymptotic confidence intervals through the observed information. The idea of missing information principle can be expressed through the following:

$$I_Y(\theta) = I_W(\theta) - I_{Z|Y}(\theta),$$

where $I_Y(\theta)$ is the observed information, $I_W(\theta)$ is the complete information and $I_{Z|Y}(\theta)$ is the (unobserved) missing information. The complete information matrix is

$$I_W(\theta) = -E \left[\frac{\partial^2 l_c(W, \theta)}{\partial \theta^2} \right], \quad (13)$$

where $l_c(W, \theta)$ is the log-likelihood function of the complete data. The missing information matrix can be written as

$$I_{Z|Y}(\theta) = -(n-m)E \left[\frac{\partial^2 \ln f_{Z|Y}(Z|y)}{\partial \theta^2} \right]. \quad (14)$$

Now, the complete information matrix of $Y = (Y_1, \dots, Y_n)$ where Y is a progressive type-II censored data selected from a population with pdf and cdf given in equations (1) and (2), respectively, is given as follows:

$$I_W(\beta) = -E \left[\frac{\partial^2 l_c(W, \theta)}{\partial \beta^2} \right] = \frac{n}{\beta^2} - p \sum_{i=1}^m (R_i + 2) E \left[\frac{y_i^2 e^{-\beta y_i}}{(1 - pe^{-\beta y_i})^2} \right].$$

$$I_W(p) = -E \left[\frac{\partial^2 l_c(W, \theta)}{\partial p^2} \right] = \frac{\sum_{i=1}^m (R_i + 1)}{(1-p)^2} - \sum_{i=1}^m (R_i + 2) E \left[\frac{e^{-2\beta y_i}}{(1 - pe^{-\beta y_i})^2} \right].$$

The logarithm of the conditional distribution is given by

$$\ln f_{Z|Y}(Z|y) = \ln(\beta) + \ln(1 - pe^{-\beta y_m}) + \beta y_m - \beta z - 2 \ln(1 - pe^{-\beta z}). \quad (15)$$

Next, the expectations of the second order derivative of (15) with respect to $\theta = (\beta, p)$ are:

$$E \left[\frac{\partial^2 \ln f_{Z|Y}(Z|y)}{\partial \beta^2} \right] = -\frac{1}{\beta^2} - \frac{py_m^2 e^{-\beta y_m}}{(1 - pe^{-\beta y_m})^2} + E \left[\frac{2pz^2 e^{-\beta z}}{(1 - pe^{-\beta z})^2} \right].$$

$$E \left[\frac{\partial^2 \ln f_{Z|Y}(Z|y)}{\partial p^2} \right] = \frac{-e^{-2\beta y_m}}{(1 - pe^{-\beta y_m})^2} + E \left[\frac{2e^{-2\beta z}}{(1 - pe^{-\beta z})^2} \right].$$

Therefore, the matrix $I_{Z|Y}(\theta)$, where $\theta = (\beta, p)$, can be written as follows:

$$I_{Z|Y}(\beta) = (n-m) \left\{ \frac{1}{\beta^2} + \frac{py_m^2 e^{-\beta y_m}}{(1 - pe^{-\beta y_m})^2} - 2E \left[\frac{pz^2 e^{-\beta z}}{(1 - pe^{-\beta z})^2} \right] \right\}.$$

$$I_{Z|Y}(p) = (n-m) \left\{ \frac{e^{-2\beta y_m}}{(1 - pe^{-\beta y_m})^2} - 2E \left[\frac{e^{-2\beta z}}{(1 - pe^{-\beta z})^2} \right] \right\}.$$

Subsequently, the asymptotic variance of $\hat{\theta} = (\hat{\beta}, \hat{p})$ can be obtained by taking the inverse of the matrix $I_Y(\theta)$. Applying the normality properties of MLEs, we can use 100%(1 - γ) approximate confidence intervals of parameters $\theta = (\beta, p)$ as follow.

$$\hat{\beta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\beta})}, \quad \hat{p} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{p})},$$

where $\text{var}(\hat{\beta})$ and $\text{var}(\hat{p})$ are asymptotic variance of $\hat{\theta} = (\hat{\beta}, \hat{p})$ and $z_{\alpha/2}$ is percentile of standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

4. Result and Discussions

To assess the mathematical derivations and the behavior of the estimated parameters, a simulation study has been conducted using R software. The program has been designed to obtain the MLEs, $\hat{\theta} = (\hat{\beta}, \hat{p})$, and other important measurements such as mean squared error (MSE), bias and a 95% confidence interval with lower and upper limits (LL, UL) along with its length. Samples of different sizes, $n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$, have been generated from EG distribution. The process has been run based on 2000 iterations by setting $\theta = (\beta = 1, p = 0.9)$. Also, different censoring schemes have been considered at different number of failures m to each sample. The obtained results are shown in two tables.

Table 1: The length of confidence intervals of MLEs of EG distribution under progressive type-II censored data for $\beta = 1$ and $p = 0.9$ with different sample size and fixed number of failures observed.

m	n	Length $\hat{\beta}, \hat{p}$	m	n	Length $\hat{\beta}, \hat{p}$
15	20	0.81735	50	60	0.45000
		0.13793			0.05742
	30	0.57272		70	0.38249
18	20	0.05430	80		0.03407
		0.85069			0.32516
	30	0.14561		90	0.02509
25	30	0.52005	100		0.29762
		0.04203			0.02552
	40	0.74598			0.27042
	40	0.15101	60	70	0.02335
		0.61806			0.43663
	50	0.09461		80	0.05479
30	40	0.50023	90		0.40866
		0.07328			0.05294
	50	0.60046		100	0.32432
40	50	0.09558	70	80	0.02839
		0.54362			0.24488
	60	0.07391		90	0.01897
	60	0.53822	90		0.40483
		0.07848			0.05364
	100	0.47204			0.30732
		0.05489			0.02423

Table 1 shows the length of the 95% confidence intervals of the MLEs for the parameters of EG distribution under progressive type-II censored sample at starting point $\theta = (\beta = 1, p = 0.9)$ using different sample size and fixed number of failures observed. It has been noted that, for fixed number of failures and as the sample size increased, the lengths of the 95% confidence intervals tend to be narrower.

Table 2 presents the simulated values of MLEs, MSE, bias and (LL, UL) of 95% confidence interval along with its length, under the consideration of the censoring schemes for different samples sizes $n = 10, 20, 30, \dots, 100$ and different numbers of failures m for parameters $\theta = (\beta = 1, p = 0.9)$. From this table we could conclude that regardless of the censoring schemes and at which point the censored units are removed from the sample, for increasing sample size; the estimated value of the parameters converges to the true value, and the MSE of the MLEs decreases. It is also noted that for $n = 100$, complete case, as m increases (i.e. from 50 to 100) the MSE values decrease too. However, this may not satisfy, in some cases, as the simulated samples have been generated randomly. Additionally, different censoring schemes have been used for different samples sizes.

5. Conclusion

The problem of estimating the unknown parameters of the EG distribution under progressive type-II censored was discussed. The EM algorithm was used to find the maximum likelihood estimators of the scale and shape parameters of the model. Furthermore, the asymptotic variance was derived along with the asymptotic confidence intervals. The simulation study was conducted using different censoring schemes and different samples sizes to figure out the MLEs numerically. In addition, the MSE, Bias and their confidence intervals were presented under various aspects. The results show that as the sample size increased the estimated values of the unknown parameters converge to the true values.

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Table 2: The MLEs, MSE, Bias, and 95% Confidence limits and Length of the MLEs of the EG distribution based on progressive Type-II censored data for $\beta = 1$ and $p = 0.9$ with different sample size and different number of failures observed.
for $\beta = 1$ and $p = 0.9$

n	m	Censoring Scheme	MLE $\hat{\theta} = (\hat{\beta}, \hat{p})$	MSE	Bias	(LCI, UCI)	Length
10	5	(0,0,3,1,1)	1.66388	0.82656	0.66388	(1.06736, 2.26040)	1.19303
			0.90400	0.00138	0.00400	(0.83533, 0.97267)	0.13734
	7	(3,0,0,0,0,0,0)	1.09252	0.20420	0.09252	(0.51108, 1.67396)	1.16289
			0.89496	0.00145	-0.00504	(0.83066, 0.95926)	0.12861
	9	(0*8,1)	0.90648	0.15708	-0.09352	(0.33203, 1.48093)	1.14890
			0.89648	0.00385	-0.00352	(0.78181, 1.01115)	0.22935
20	10	(0*6,4,3,2,1)	1.36896	0.26403	0.36896	(0.95547, 1.78245)	0.82697
			0.89996	0.00078	-0.00004	(0.84849, 0.95143)	0.10295
	15	(0*13,4,1)	0.98160	0.13509	0.13509	(0.57293, 1.39027)	0.81735
			0.89336	0.00160	-0.00664	(0.82439, 0.96233)	0.13793
	18	(0*17,2)	0.98554	0.08401	-0.01446	(0.56020, 1.41088)	0.85069
			0.89926	0.00171	-0.00074	(0.82645, 0.97207)	0.14561
30	15	(0*12,5,5,5)	1.34608	0.23543	0.34608	(1.05972, 1.63244)	0.57272
			0.89810	0.00034	-0.00190	(0.87095, 0.92525)	0.05430
	18	(0*17,12)	1.10724	0.05195	0.10724	(0.84722, 1.36726)	0.52005
			0.89552	0.00028	-0.00448	(0.87450, 0.91654)	0.04203
	25	(0*25,5)	1.08082	0.05171	0.08082	(0.70783, 1.45381)	0.74598
			0.91292	0.00168	0.01292	(0.83742, 0.98842)	0.15101
40	20	(0*16,5,5,5,5)	1.56552	0.43857	0.56552	(1.32194, 1.80910)	0.48716
			0.90570	0.00025	0.00570	(0.88271, 0.92869)	0.04598
	25	(0*25,5,5,3,2)	1.46574	0.26964	0.46574	(1.15671, 1.77477)	0.61806
			0.92386	0.00116	0.02386	(0.87656, 0.97116)	0.09461
	30	(0*25,3,3,2,1,1)	1.02084	0.06607	0.02084	(0.72061, 1.32107)	0.60046
			0.90240	0.00067	0.00240	(0.85461, 0.95019)	0.09558
50	25	(0*25,10,10,3,2)	1.81910	0.71397	0.81910	(1.56899, 2.06921)	0.50023
			0.92960	0.00124	0.02960	(0.89296, 0.96624)	0.07328
	30	(0*26,15,3,1,1)	1.23950	0.09143	0.23950	(0.96769, 1.51131)	0.54362
			0.90228	0.00039	0.00228	(0.86533, 0.93923)	0.07391
	40	(0*36,3,3,2,2)	1.02142	0.02803	0.02142	(0.75231, 1.29053)	0.53822
			0.90344	0.00046	0.00344	(0.86420, 0.94268)	0.07848
60	30	(0*24,5,5,5,5,5,5)	1.42616	0.22214	0.42616	(1.22564, 1.62668)	0.40104
			0.90298	0.00014	0.00298	(0.88369, 0.92227)	0.03858
	40	(0*35,5,5,4,3,3)	1.15124	0.06306	0.15124	(0.91522, 1.38726)	0.47204
			0.90144	0.00024	0.00144	(0.87400, 0.92888)	0.05489
	50	(0*47,5,0,5)	0.91110	0.02962	-0.08890	(0.68610, 1.13610)	0.45000
			0.89788	0.00023	-0.00212	(0.86917, 0.92659)	0.05742
70	45	(0*40,5,5,5,5,5)	1.19830	0.05652	0.19830	(0.99273, 1.40387)	0.41115
			0.90298	0.00031	0.00298	(0.88225, 0.92371)	0.04147
	50	(0*48,10,10)	1.02394	0.01300	0.02394	(0.83270, 1.21518)	0.38249
			0.89950	0.00009	-0.00050	(0.88247, 0.91653)	0.03407
	60	(0*58,5,5)	1.00164	0.04089	0.00164	(0.78332, 1.21996)	0.43663
			0.90282	0.00035	0.00282	(0.87543, 0.93021)	0.05479
80	50	(0*49,10,10,10)	1.47022	0.23056	0.47022	(1.30764, 1.63280)	0.32516
			0.91946	0.00042	0.01946	(0.90691, 0.93201)	0.02509
	60	(0*56,6,5,6,3)	1.02088	0.03108	0.02088	(0.81655, 1.22521)	0.40866
			0.89970	0.00024	-0.00030	(0.87323, 0.92617)	0.05294
	70	(0*68,5,5)	0.94950	0.01463	-0.05050	(0.74709, 1.15191)	0.40483
			0.90178	0.00021	0.00178	(0.87496, 0.92860)	0.05364
90	50	(0*46,10,10,10,10)	1.30784	0.11901	0.30784	(1.15903, 1.45665)	0.29762
			0.90276	0.00008	0.00276	(0.89000, 0.91552)	0.02552
	60	(0*57,10,10,10)	1.14750	0.07093	0.14750	(0.98534, 1.30966)	0.32432
			0.90128	0.00019	0.00019	(0.88709, 0.91547)	0.02839
	70	(0*69,20)	0.97186	0.02031	-0.02814	(0.81820, 1.12552)	0.30732
			0.89860	0.00009	-0.00140	(0.88648, 0.91072)	0.02423
100	50	(0*45,10,10,10,10,10)	1.29598	0.11145	0.29598	(1.16077, 1.43119)	0.27042
			0.89860	0.00007	-0.00140	(0.88693, 0.91027)	0.02335
	60	(0*58,20,20)	1.21526	0.05549	0.21526	(1.09282, 1.33770)	0.24488
			0.90118	0.00004	0.00118	(0.89169, 0.91067)	0.01897
	100	(0*100)	1.00134	0.01556	0.00134	(0.80413, 1.19855)	0.39442
			0.90028	0.03078	0.00028	(0.55950, 1.24106)	0.68157