Vertex and edge Co-PI indices of bridge graphs

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Abstract

The Co-PI index of a graph G, denoted by Co – PI(G), is defined as Co – PI(G) = \( \sum_{uv \in E(G)} |n^G_0(u) - n^G_0(v)| \), where \( n^G_0(u) \) is the number of vertices of G whose distance to the vertex u is less than the distance to the vertex v in G. Similarly, the edge Co-PI index of G is defined as Co – Pl(G) = \( \sum_{uv \in E(G)} |m^G_0(u) - m^G_0(v)| \), where \( m^G_0(u) \) is the number of edges of G whose distance to the vertex u is less than the distance to the vertex v in G. In this paper, the upper bound for the Co-PI and edge Co-PI indices of bridge graphs are obtained.

Keywords: Bridge Graph; Co-PI Index; Edge Co-PI Index.

1. Introduction

All the graphs considered in this paper are connected and simple. A vertex \( x \in V(G) \) is said to be equidistant from the edge \( e = uv \) of G if \( d_G(x,u) = d_G(x,v) \), where \( d_G(x,u) \) denotes the distance between u and x in G. The degree of the vertex u in G is denoted by \( d_G(u) \).

For an edge \( e \in E(G) \), the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G is denoted by \( n^G_0(u) \); analogously, \( n^G_0(v) \) is the number of vertices of G whose distance to the vertex v in G is smaller than the distance to the vertex u; the vertices equidistant from both the ends of the edge \( e \) are not counted. Similarly, \( m^G_0(u) \) denotes the number of edges of G whose distance to the vertex u is less than the distance to the vertex v.

The vertex PI index of G, denoted by PI(G), is defined as

\[
PI(G) = \sum_{e=uv \in E(G)} (n^G_0(u) + n^G_0(v))
\]

and the edge PI index of G, denoted by Pl(G), is defined as

\[
Pl(G) = \sum_{e=uv \in E(G)} (m^G_0(u) + m^G_0(v))
\]

Similarly, the Co-PI index of G, denoted by Co-PI(G), is defined as

\[
Co – PI(G) = \sum_{e=uv \in E(G)} |n^G_0(u) - n^G_0(v)|
\]

and the edge Co-PI index of G, denoted by Co-Pl(G), is defined as

\[
Co – Pl(G) = \sum_{e=uv \in E(G)} |m^G_0(u) - m^G_0(v)|
\]

Khuddar [12] first introduced edge PI index of graphs and they investigated the chemical applications of the PI index. The PI index of the graph G is a topological index related to equidistant vertices. Another topological index of G related to distance of G is the Wiener index of G, first introduced by Wiener; see [18].

Karmakar and Agarwal [12] first introduced edge Padmakanth-Ivan index of graphs and they investigated the chemical applications of the Padmakanth-Ivan index. The mathematical properties of the Pl and its applications in chemistry and nanoscience are well studied by Ashifi and Loghman [1, 3], Ashrafi and Rezai [2], Deng, Chen and Zhang [6], Khullar [10], Khullar, Youssefi-Azani and Ashrafi [11], Klavzar [13] and Youssefi-Azani, Manoochehrad and Ashrafi [17]. The vertex PI indices of the tensor and strong products of graphs are studied in [14, 16]. In [20, 21, 22] the PI indices of bridge graphs and chain graphs are discussed. In this paper, the upper bounds for the Co-PI and edge Co-PI indices of bridge graphs are obtained. Let \( \{G_i\}_{i=1}^s \) be a set of \( s \) finite pairwise vertex disjoint connected graphs with \( v_i \in V(G_i) \). The bridge graph \( B(G_1, G_2, \ldots, G_s) = B(G_1, G_2, \ldots, G_s; v_1, v_2, \ldots, v_s) \) of \( \{G_i\}_{i=1}^s \) with respect to the vertices \( \{v_i\}_{i=1}^s \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_s \) by connecting the vertices \( v_i \) and \( v_{i+1} \) by an edge for all \( i = 1, 2, \ldots, s-1 \).

2. Co-PI Index of Bridge Graph

Let \( G \) be a graph and let \( v \in V(G) \). The set of all edges \( xy \) such that \( d_G(x,v) = d_G(y,v) \) is denoted by \( N_d(x,v) \). Define \( K(G_i) = \{ e = xy \in E(G_i) \mid d_G(x,v) < d_G(y,v) \} \) and \( L(G_i) = \{ e = xy \in E(G_i) \mid d_G(x,v) > d_G(y,v) \} \).

Theorem 2.1

Let \( G = B(G_1, G_2, \ldots, G_s) \) of \( \{G_i\}_{i=1}^s \) with respect to the vertices \( \{v_i\}_{i=1}^s \) and \( |V(G)| = a \). Then \( Co – PI(G) \leq \sum_{i=1}^s (Co – PI(G_i)) + \sum_{i=1}^s (|V(G)| – |V(G_i)|)(k_i + \ell_i) + \sum_{i=1}^s 2a_i – a \), where \( a_i = \sum_{j=1}^s |V(G_j)|, k_i = |E(K(G_i))| \) and \( \ell_i = |E(L(G_i))| \).
Proof. From the definition of $Co - PL(G)$,
\[
Co - PL(G) = \sum_{e \in \text{arc}(G)} \left| n^G_e(e) - n^G_e(e) \right|
\]
\[
= \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| n^G_e(e) - n^G_i(e) \right| + \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| n^G_i(e) - n^G_i(e) \right|
\]
\[
= \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| n^G_i(e) - n^G_i(e) \right|
\]
\[
+ \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| n^G_i(e) - n^G_i(e) \right|
\]
\[
= \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| n^G_i(e) - n^G_i(e) \right|
\]  
\[
= \sum_{i=1}^{s} \left( |V(G)| - |V(G)| \right).
\]
Hence, $Co - PL(G) = \sum_{i=1}^{s} \left( |V(G)| - |V(G)| \right)$.

3. Edge Co-PI Index of Bridge Graph

For a graph $G$ with $v \in V(G)$, let $T_0(G)$ be the set of edges $uv$ of $G$ such that $d_G(u,v) = d_G(v,u)$. For a bridge graph $B(G_1, G_2, \ldots, G_i)$, $i = 1, 2, \ldots, s - 1$, let $K(G)$ be the set of edges $e = uv \in E(G) \setminus T_0(G)$ such that $d_G(u,v) < d_G(v,u)$ and $L(G)$ the set of edges $e = uv \in E(G) \setminus T_0(G)$ such that $d_G(u,v) > d_G(v,u)$.

Theorem 3.1

Let $G = B(G_1, G_2, \ldots, G_i)$ of $(G_i)_i$ with respect to the vertices $\{v_i\}_{i=1}^s$. Then $Co - PL(e) = \sum_{i=1}^{s} \left( |V(G)| - |V(G)| \right)$.

Proof. Let $G = B(G_1, G_2, \ldots, G_i)$. Observe that $E(G_i) = T_0(G_i) \cup K(G_i) \cup L(G_i)$ for $i = 1, 2, \ldots, s$. By the definition of the edge Co-PI index,
\[
Co - PL(e) = \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]
\[
= \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]
\[
+ \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]
\[
+ \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]  
\[
= \sum_{i=1}^{s} \left( |E(G)| - |E(G)| \right).
\]

• For $i = 1, 2, \ldots, s$, if $e = uv \in T_0(G_i)$, then $d_G(u,v) = d_G(v,u)$ for any edge $e_i \in E(G_i \setminus E(G))$, then $d_G(u,v) = d_G(v,u)$. This implies $m^G(e) = m^G(e)$ and $m^G(e) = m^G(e)$. Then
\[
Co - PL(e) = \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]
\[
= \sum_{i=1}^{s} \left( |E(G)| - |E(G)| \right).
\]

• Similarly, if $e = uv \in L(G_i)$, then $d_G(u,v) > d_G(v,u)$, thus
\[
Co - PL(e) = \sum_{i=1}^{s} \sum_{e \in \text{arc}(G)} \left| m^G(e) - m^G(e) \right|
\]
\[
= \sum_{i=1}^{s} \left( |E(G)| - |E(G)| \right).
\]  
From (5) and (6)
\[ \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_i}_e(e)| \]
\[ = \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_i}_e(e)| + \sum_{e \in \text{e}(G_i)} |m^{G_i}_e(e) - m^{G_i}_e(e)| \]
\[ = \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_j}_e(e) + (|E(G)| - |E(G_j)|)| + \sum_{e \in \text{e}(G_i)} |m^{G_i}_e(e) - m^{G_i}_e(e)| - (|E(G)| - |E(G_i)|) \]
\[ \leq \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_j}_e(e)| + \sum_{e \in \text{e}(G_i)} (|E(G)| - |E(G_i)|) \]
\[ \leq \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_j}_e(e)| + \sum_{e \in \text{e}(G_i)} (|E(G)| - |E(G_i)|). \]

* For an edge \( e = v_i v_{i+1} \), \( i = 1, 2, \ldots, s - 1 \), one can easily observe that \( m^{G_j}_e(e) \) and \( m^{G_j}_e(e) \) are:

\[ \left( \sum_{j=i+1}^{s} |E(G_j)| + s - (i + 1) \right) \]
\[ = \sum_{i=1}^{s-1} \left( \sum_{j=i+1}^{s} |E(G_j)| + 1 \right) - \sum_{i=1}^{s-1} |E(G_j) - 1| \]
\[ = \sum_{i=1}^{s-1} \left( 2a_i - |E(G)| + 1 \right). \]

Hence the edge Co-PI index of the bridge graph is given by,

\[ \text{Co - PLE}(G) = \sum_{i=1}^{s} \sum_{e \in \text{e}(G_j)} |m^{G_j}_e(e) - m^{G_j}_e(e)| \]
\[ + \sum_{i=1}^{s} \sum_{e \in \text{e}(G_i)} |m^{G_i}_e(e) - m^{G_i}_e(e)| \]
\[ + \sum_{i=1}^{s} \sum_{e \in \text{e}(G_i)} (|E(G)| - |E(G_i)|) \]
\[ + \sum_{i=1}^{s} \sum_{e \in \text{e}(G_j)} (|E(G)| - |E(G_j)|) + \sum_{i=1}^{s-1} \left( 2a_i - |E(G)| + 1 \right) \]
\[ = \sum_{i=1}^{s} \text{Co - PLE}(G) + \sum_{i=1}^{s} \sum_{e \in \text{e}(G_i)} (|E(G)| - |E(G_i)|)(k_i + \ell_i) \]
\[ + \sum_{i=1}^{s-1} \left( 2a_i - |E(G)| + 1 \right). \]

References
