Further Geometric Properties of a Subclass of Univalent Functions

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Abstract

This present paper aims to investigate further, certain characterization properties for a subclass of univalent function defined by a generalized differential operator. In particular, necessary and sufficient conditions for the function $f(z)$ to belong to the subclass $\Phi^n_d(\beta, \alpha)$ is established. Additionally, we provide the $\delta$-neighborhood properties for the function $[f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0] \in \Phi^n_d(\beta, \alpha)$ by making use of the necessary and sufficient conditions. The results obtained are new geometric properties for the subclass $\Phi^n_d(\beta, \alpha)$.

Keywords: Analytic Functions; Univalent Functions; Differential Operator; Neighborhood.

1. Introduction

Let $A$ denotes the class of functions $f(z)$ which are analytic in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Also, let the class of all functions in $A$ which are univalent in $U$ be denoted by the symbol $S$ and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

(1)

It is well known that any function $f \in S$ has the Taylor series expansion of the form (1), for details (see Duren [1] and Pommerenke [2]). The form (1) is the normalized form of functions $f(z) \in A$ for which the normalization condition is given by

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Thus,

$$S = \{ f \in A : f(0) = f'(0) - 1 = 0 \}.$$

Some well-known properties of functions in the class $S$ can be found elsewhere (see [1], [3] and [4]), while some special classes of univalent functions have also been investigated by various authors (see [5], [6], [7], [8], [9], [10] and [11]). Furthermore, we denote by $T$ the subclass of $A$ consisting of functions $f(z) \in A$ which are analytic and univalent in $U$ and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$$

(2)

The class $\Phi^n_d(\beta, \alpha)$, a subclass of univalent functions was introduced and studied by Oyekan [10]. For this class, the author established both convolution and inclusion properties for the class. Other subsequent work on the class can be found in Oyekan and Kehinde [12].

**Definition 1**: [10] A function $f(z) \in A$ is in the class $\Phi^n_d(\beta, \alpha)$ of provided $D^n_{\beta, \alpha}[f(z)]' \in p(\alpha)$. That is, if

$$\text{Re}\left[D^n_{\beta, \alpha}(f(z))'\right] > \alpha, z \in U, \text{ for } 0 \leq \alpha < 1, 1 \leq \mu \leq \beta, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We note that $p(\alpha) \in P$ which is the class of the Carathéodory functions.

In the sequel, we shall state and prove our new results for the class $\Phi^n_d(\beta, \alpha)$. These new results presented in section 2, are motivated by the results in Opoola [9].
2. Results and discussion

2.1. Necessary and sufficient conditions

Theorem 2.1: Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A \).

If \( z + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k |z| < 1 - \alpha \),
then \( f(z) \in \mathcal{P}^n(\beta, \alpha) \).

Proof: It suffices to show that

\[
\left| \frac{D_{\mu, \beta} f(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad 0 \leq \alpha < 1
\]

Now,

\[
\left| \left( D_{\mu, \beta} f(z) \right)' - 1 \right| = \left| 1 + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k z^{k-1} - 1 \right|
\]

\[
= \left| \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] z^{k-1} a_k \right|
\]

\[
\leq \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k |z|^{k-1}
\]

Thus, by the condition of the theorem, we have that

\[
\left| \frac{D_{\mu, \beta} f(z)}{f(z)} - 1 \right| < 1 - \alpha.
\]

Hence, the proof is complete.

Theorem 2.2: A function \( f(z) \) of the form given by (2) belongs to the class \( \mathcal{P}^n(\beta, \alpha) \) if and only if

\[
\sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k < 1 - \alpha, \quad 0 \leq \alpha < 1.
\]

Proof: Let \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{P}^n(\beta, \alpha), a_k \geq 0 \).

Then

\[
\text{Re} \left( \frac{D_{\mu, \beta} f(z)}{f(z)} \right)' > \alpha \quad (3)
\]

Which implies

\[
\left| \left( D_{\mu, \beta} f(z) \right)' - 1 \right| < 1 - \alpha \quad (4)
\]

\[
\left| \left( D_{\mu, \beta} f(z) \right)' - 1 \right| = \left| 1 + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k z^{k-1} - 1 \right|
\]

\[
\text{Re} \left( \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k z^{k-1} \right) < 1 - \alpha \quad (5)
\]

Taking values of \( z \) on real axis and letting \( z \to -1 \) through real values we have

From (5) that

\[
\sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k < 1 - \alpha.
\]

Conversely,

\[
\left| \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k
\]

\[
= \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)^n] a_k.
\]

Hence, by the condition of the theorem we have that

\[
\left| \frac{D_{\mu, \beta} f(z)}{f(z)} - 1 \right| < 1 - \alpha.
\]

Consequently
\[ \text{Re} \left( D^n_{\beta, \alpha} f(z) \right)^T > \alpha, \]

And hence
\[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \varphi_n^\beta(\beta, \alpha). \]

### 2.2 Neighborhoods for \( \varphi_n^\beta(\beta, \alpha) \)

Let \( f(z) \in \varphi_n^\beta(\beta, \alpha) \) and \( \delta \geq 0 \), we define the \( \delta - \) neighborhood of \( f(z) \) as
\[ N_\delta(f) := \{ g \in A : g(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \varphi_n^\beta(\beta, \alpha) \text{ and } \sum_{k=2}^{\infty} b_k |a_k - b_k| \leq \delta \} \]

In particular, for the identity function \( e(z) = z \), we immediately have
\[ N_\delta(e) := \{ g \in A : g(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \varphi_n^\beta(\beta, \alpha) \text{ and } \sum_{k=2}^{\infty} b_k |a_k| \leq \delta \}. \]

The concept of neighborhood of analytic functions above was sequel to the works of Goodman [13] and Ruscheweyh [14]. The main goal in this subsection is to investigate the \( \delta - \) neighborhood of \( f(z) \in \varphi_n^\beta(\beta, \alpha) \) with negative coefficients.

**Theorem 2.3:** If
\[ \delta = \frac{2(1 - \beta - \mu)^2}{(21 - \mu - \mu + 2\.\beta\cdot\mu)^2}, \]

then \( \varphi_n^\beta(\beta, \alpha) \subset N_\delta(e) \).

**Proof:** Let \( f(z) \in \varphi_n^\beta(\beta, \alpha) \).

Then from Theorem 2.1, we have that
\[ \sum_{k=2}^{\infty} k(1 + \beta - \mu)^n |a_k| < 1 - \alpha, \]

Which implies that
\[ [2(1 + \beta - \mu)]^n \sum_{k=2}^{\infty} k|a_k| < 1 - \alpha, \]

That is,
\[ \sum_{k=2}^{\infty} k|a_k| < \frac{1 - \alpha}{[2(1 + \beta - \mu)]^n}. \]

Which by (7) gives that \( f(z) \in N_\delta(e) \).

Hence,
\[ \varphi_n^\beta(\beta, \alpha) \subset N_\delta(e). \]

### 3. Conclusion

For the class \( \varphi_n^\beta(\beta, \alpha) \), various results have been obtained and can be found in [10, 12]. Whereas, the results presented in this present work are new geometric properties for the class.

### References

