**A Totient Function Inequality**

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**Abstract:**  A new unconditional inequality of the totient function is contributed to the literature. This result is associated with various unsolved problems about the distribution of prime numbers.

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**1 Introduction**

The totient function , which counts the number of relatively prime integers less than *N*, is a sine qua non in number theory. It and its various generalizations appear everywhere in the mathematical literature. The product form representation

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unearths its intrinsic link to the distribution of the prime numbers.

The totient function ϕ(*N*) is an oscillatory function, its value oscillates from its maximum ϕ(*N*) = *N* − 1 at prime integers *N* to its minimum ϕ(*N*) = *N*/*c*0loglog *N*, at the primorial integers , where *pi* is the *k*th prime, *vi* ≥ 1, and *c*0 > 1 is a constant. The new contributions to the literature are the unconditional estimates stated below.

***Theorem***  Let *pi* be the *k*th prime, and let *Nk* = 2⋅3⋅5⋅⋅⋅*pk*, *k* ≥ 1. Then for all sufficiently large primorial integer *Nk*.

This unconditional result is consistent with the Riemann hypothesis, and seems to prove the Nicolas inequality, Theorem 4 below, for all sufficiently large integers. Just a finite number of cases of primorial integers *Nk* ≤ *N*0 remain unresolved as possible counterexamples of the inequality.

***Theorem***  The function for almost every integer *N* ≥ 1, and *c*0 > 0 constant.

Currently the best unconditional estimate of this arithmetical function in the literature is the followings:

***Theorem***  ([13]) Let *N* ∈ ℕ, then with one exception for *N* = 2⋅3 ⋅⋅⋅ 23.

On the other hand, there are several conditional criteria; one of these is listed below.

***Theorem***  ([12]) Let *Nk* = 2⋅3⋅5⋅⋅⋅*pk* be the product of the first *k* primes.

(i) If the Riemann Hypothesis is true then for all *k* ≥ 1.

(ii) If the Riemann Hypothesis is false then both and occur for infinitely many *k* ≥ 1.

Some related and earlier works on this topic include the works of Ramanujan, Erdos, and other on abundant numbers, see [11], [2], and recent related works appeared in [2], [3], [9], [14], and [20].

The next section covers some background materials focusing on some finite sums over the prime numbers and some associated and products. The proofs of Theorems 1 and 2 are given in the last two sections respectively.

**2 Background Materials**

This section provides a survey of supporting materials. An effort was made to have a self contained paper as much as possible, but lengthy proofs available in the literature are omitted.

**2.1 Sums and Products Over the Primes.** The most basic finite sum over the prime numbers is the prime harmonic sum . The refined estimate of this finite sum, stated below, is a synthesis of various results due to various authors. The earliest version is due to Mertens, see [17].

***Theorem***  Let *x* ≥ 2 be a sufficiently large number. Then

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where *B*1 = .2614972128… .

Proof: Use the integral representation of the finite sum

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where *c* > 1 is a small constant. Moreover, the prime counting function has the form

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The unconditional part of the prime counting formula arises from the delaVallee Poussin form of the prime number theorem , see [10, p. 179], the conditional part arises from the Riemann form of the prime number theorem , and the unconditional oscillations part arises from the Littlewood form of the prime number theorem , consult [7, p. 51], [10, p. 479] et cetera. Now replace the logarithm integral , and the appropriate prime counting measure , and simplify the integral. ■

The proof of the unconditional part of this result is widely available in the literature, see [6], [10], [16], et cetera. The omega notation means that both and occur infinitely often as *x* → ∞, where *c*0 > 0 is a constant, see [10, p. 5], [18].

As an application of the last result, there is the following interesting product:

***Theorem***  Let *x* ∈ ℝ be a large real number, then

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Proof: Consider the logarithm of the product

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where the Euler constant is defined by , and the Mertens constant is defined by , see [6, p. 466]. The last equality in (6) stems from the power series expansion , which yields

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Applying Theorem 5 returns

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and reversing the logarithm completes the verification. ■

The third part in (5) above simplifies the proof given in [5] of the following result:

The quantity

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attains arbitrary large positive and negative values as *x* → ∞.

**3 An Estimate of the Totient Function**

The proof of Theorem 3 on the extreme values of the arithmetic function relies on the oscillation theorem of the finite prime product . This technique leads to a concise and simpler proof. A completely elementary proof, but longer, and not based on the oscillation theorem was presented in the earlier version of this paper.

***Theorem* 1.** Let *N* ∈ ℕ be a primorial integer, then holds unconditionally for all sufficiently large *N* = 2⋅3⋅5⋅⋅⋅*pk*.

Proof : Theorem 6 implies that the product

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In particular, it follows that

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and

occur infinitely often as *x* → ∞, where *c*0 > 0, *c*1 > 0, and *c*2 > 0 are constants. It shows that oscillates infinitely often, symmetrically about the line as *x* → ∞.

To rewrite the variable *x* ≥ 1 in terms of the integer *N*, recall that the Chebychev function satisfies , *c*1 > 1, see [15]. The properties of this function lead to

, and . (

So it readily follows that . Moreover, since the maxima of the sum of divisors function

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where the symbol *p*α || *N* denotes the maximal prime power divisor of *N*, occur at the colossally abundant integers, and *v*1 ≥  *v*2 ≥ ⋅⋅⋅ ≥ *vk* ≥ 1, see [2], [3], [9], [11], it follows that the maxima of the inverse totient function *N*/ϕ(*N*) occur at the squarefree primorial integers *Nk* = 2⋅3⋅5⋅⋅⋅*pk*. Therefore, expressions (10) and (11) implies that

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as the primorial integer *Nk* = 2⋅3⋅5⋅⋅⋅*pk* tends to infinity. ∎

**4. Probabilistic Properties**

The natural density function

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is known to be a continuous function of *t* ≥ 0. Some recent works have established the exact asymptotic expression

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as *t* tends to infinity, see [19], [20].

The evaluation of the natural density (15) at as *N* → ∞ suggests that the Nicolas inequality should be . The numerical data are compiled in [8].

Now, note that the evaluation at as *N* → ∞ yields the density function

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Consequently, the subset of integers *N* ≥ 1 such that has zero density with respect to the set of integers ℕ. A simple proof of this result is included here.

***Theorem* 2.** The function for almost every integer *N* ≥ 1, and *c*0 > 0 constant.

Proof: The prime divisors counting function satisfies for almost every integer *N* ≥ 1, this is Ramanujan Theorem. In addition,

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where *x* ≥ 2 is a suitable real number, holds for every integer *N* ≥ 1, this is Mertens Theorem. Furthermore, by the Prime Number Theorem, the *n*th prime . In light of these facts, put

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where *c*1, *c*2, *c*3, *c*4, … are constants. Substituting (19) into the previous relation (18) implies that

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holds for almost every integer *N* ≥ 1. Ergo, the ratio holds for almost every integer *N* ≥ 1 as claimed. ■

***Corollary* 7.** The function for almost every integer *N* ≥ 1, and *c*5 > 0 constant.

Proof: The sigma-phi identity, on the first line below, coupled with Theorem 4 lead to

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where *c*5 is a constant, holds for almost every integer *N* ≥ 1. ■

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