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Research paper

Generalized free Gaussian white noise

Hakeem A. Othman [1,2*]

[1] *Department of Mathematics, Al-Qunfudah Center for Scientific Research, AL-Qunfudhah University college, Umm Al-Qura University, KSA*

[2] *Department of Mathematics, Rada'a College of Education and Science, Albida University, Albida, Yemen*

[*] *Corresponding author E-mail:hakim_albdoie@yahoo.com*

Abstract

Based on an adequate new Gel'fand triple, we construct the infinite dimensional free Gaussian white noise measure μ using the Bochner-Minlos theorem. Next, we give the chaos decomposition of an L^2 space with respect to the measure μ.

Keywords: *Chebychev polynomials, Wigner semicircle distribution, Fourier transform, Wigner semicircle white noise.*

1. Introduction

White noise analysis has been developed into a useful tool in infinite dimensional analysis and the interest in this field has grown at an explosive rate due to the large number of their application's domains such as quantum physics and in the theory of stochastic systems. It is a well-known fact that, in contrary to the finite dimensional analysis, no measure exists in infinite dimension which has the properties of the Lebesgue measure. However, the so-called Gaussian measures form one of the important and useful class of measures on infinite dimensional space, which have many similar properties to those of the Lebesgue one.

In the q-theory, the q-deformation of the Gaussian distribution was given in [5], in particular for q=0, we obtain the free case, $d\nu(x) = \frac{1}{2\Pi}\sqrt{4-x^2}dx$, $\mid x \mid\leq 2$, called the Wigner semicircle measure or the free Gaussian measure.

This paper is organized as follows: in Section 2, we give this measure with parameter ($\sigma\in\mathbb{R}$) in order to calculate its Fourier transform by using the Bessel function, we find

$$\widehat{\nu}_\sigma(x) = \int_{\mathbb{R}} e^{ixt}d\nu_\sigma(t) = j_1(2\sigma x), \;\; x\in\mathbb{R},$$

we use the Wigner semicircle functions and the differential equation satisfied by the Chebychev polynomials of the second kind to construct a standard nuclear triple

$$E := \bigcap_{p\geq 0} E_p \subset H \subset \bigcup_{p\geq 0} E_{-p} := E'.$$

Then by using the Bochner-Minlos theorem, we define the infinite dimensional Wigner semicircle (or free Gaussian) white noise measure μ on $(E', \mathfrak{B}(E'))$. Finally, in Section 3, we give the chaos decomposition of the space $L^2(E',\mu)$.

2. Wigner semicircle White Noise space

Let ν_σ be the Wigner semicircle distribution with parameter $\sigma\in\mathbb{R}$ (called also generalized free Gaussian distribution) given by

$$\begin{cases} d\nu_\sigma(x) = \frac{1}{2\Pi|\sigma|}\sqrt{4-(\frac{x^2}{\sigma^2})}\chi_{]-2|\sigma|,2|\sigma|[}dx \\ d\nu_0(x) = \delta_0(x), \end{cases} \quad (1)$$

where δ_0 is the Dirac measure on the point 0.

For $n = 0,1,2,...$, the Chebychev polynomials of the second Kind $U_n(x)$ are defined by the relation $\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta)$. For normalization we set:

$$P_n(x) = U_n(\frac{x}{2}), n = 0,1,2,....,$$

then $\{P_n(x)\}$ is a sequence of orthogonal polynomials associated with the Wigner semicircle law

$$d\nu(x) = \frac{1}{2\Pi}\sqrt{4-x^2}dx, \quad \mid x\mid\leq 2,$$

and satisfy the following recursion formula

$$xP_n(x) = P_{n+1}(x) + P_{n-1}(x).$$

The Bessel function of the Kind of order $\alpha > \frac{-1}{2}$ can be defined by

$$J_\alpha(x) = (\frac{x}{2})^\alpha \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)}(\frac{x}{2})^{2k}, \;\; x>0. \quad (2)$$

Moreover, we have the following Poisson-Mehler integral representation

$$J_\alpha(x) = \frac{1}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(\frac{x}{2})^\alpha \int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}e^{ixt}dt.$$

The normalized Bessel function of order $\alpha > \frac{-1}{2}$ is given by

$$j_\alpha(x) = \begin{cases} 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha} & if \;\; x\neq 0 \\ 1 & if \;\; x=0. \end{cases} \quad (3)$$

It is easy to see that the Fourier transform of the Wigner semicircle distribution with parameter σ is

$$\widehat{\nu}_\sigma(x) = \int_{\mathbb{R}} e^{ixt} d\nu_\sigma(t) = j_1(2\sigma x), \;\; x \in \mathbb{R}.$$

One can check the second equality by direct verification or by using (2) and (3). On the other hand, since $\int_{-2}^{2} P_n(x)P_m(x)d\nu(x) = \delta_{n,m}$, the corresponding Wigner semicircle functions $e_n(x)$ defined by

$$e_n(x) = (\frac{4-x^2}{4})^{\frac{1}{4}} P_n(x),$$

form an orthonormal basis $\{e_n, \;\; n \in \mathbb{N}\}$ of $H := L^2(I,dx)$, with $I =]-2,2[$. Define the operator A on H, by

$$A = (\frac{x^2-4}{4})\frac{d^2}{dx^2} + x\frac{d}{dx} - \frac{1}{4}(\frac{x^2}{x^2-4}) + 3.$$

Then the Wigner semicircle functions e_n are eigenvectors of A, namely,
$Ae_n = \lambda_n e_n$, where $\lambda_n = 4n(n+2) - \frac{1}{8}, \;\; n = 0,1,2,....$
This can be shown by using the differential equation

$$P_n''(x) - \frac{12x}{4-x^2}P_n'(x) + \frac{16n(n+2)}{4-x^2}P_n(x) = 0.$$

Moreover, for any $p > \frac{1}{4}, \;\; A^{-p}$ is Hilbert-shmidt operator satisfying

$$\| A^{-p} \|_{H.S} = \sum_{n=0}^{\infty} \lambda_n^{-2p} < \infty.$$

Now, for each $p \in \mathbb{R}$ define a norm $| \,.\, |_p$ on H by

$$| f |_p = | A^p f |_0 = (\sum_{n=0}^{\infty} \lambda_n^{2p} < f, e_n >^2)^{\frac{1}{2}}, \;\; f \in H$$

where $| \,.\, |_0$ and $< .,. >$ are, respectively, the norm and the inner product of H. For $p \geq 0$, let E_p be the Hilbert space consisting of all $f \in H$ with $| f |_p < \infty$ and E_{-p} the completion of H with respect to $| \,.\, |_{-p}$. Since A^{-1} is of Hilbert-Schmidt type, identifying H with its dual space we come to the real standard nuclear triple

$$E := \bigcap_{p\geq 0} E_p \subset H \subset \bigcup_{p \geq 0} E_{-p} := E'.$$

Being compatible to the inner product of H, the canonical bilinear form on $E' \times E$ is denoted by $< .,. >$ again.

Lemma 1. *The function*

$$C(\varphi) = j_1(2 < \varphi >), \varphi \in E \tag{4}$$

is a characteristic function on E, where $< \varphi > = \int_I \varphi(x)dx$.

Proof. Obviously C is continuous on E and $C(0) = 1$. We shall prove that C is positive definite. Given $\alpha_1, \alpha_2, ...\alpha_n \in C$ and $\varphi_1, \varphi_2, ..., \varphi_n \in E$ we have

$$\begin{aligned}\sum_{l,k=1}^{n} \alpha_l \overline{\alpha_k} C(\varphi_l - \varphi_k) &= \sum_{l,k=1}^{n} \alpha_l \overline{\alpha_k} j_1(2 < \varphi_l - \varphi_k >) \\ &= \sum_{l,k=1}^{n} \alpha_l \overline{\alpha_k} \frac{1}{\Pi} \int_{-1}^{-1} (1-t^2)^{\frac{1}{2}} e^{i2t<\varphi_l - \varphi_k>} dt \\ &= \frac{1}{\Pi} \int_{-1}^{-1} (1-t^2)^{\frac{1}{2}} \mid A_t \mid^2 dt \geq 0\end{aligned}$$

where $A_t = \sum_{k=1}^{n} \alpha_k exp(it < 2\varphi_k >)$. □

An application of the Bochner-Minlos theorem leads us to the following.

Definition 1. The probability measure μ on E', of which characteristic function is C given by equation (4), is called the Wigner semicircle white noise measure or the generalized free Gaussian white noise measure. The probability space (E', μ) is called the generalized free Gaussian white noise space.

Proposition 1. *For* $\xi \in E$, *let* X_ξ *be the random variable defined on* $(E', \mathfrak{B}(E'), \mu)$, *by* $X_\xi(\omega) := < \omega, \xi >$
where $\mathfrak{B}(E')$ *is the cylinder* σ*-algebra on* E'. *Then* X_ξ *has a Wigner semicircle distribution with parameter* $< \xi >$.

Proof. by using equations (2), (3) and (4), we have
$\int_{E'} e^{i\lambda X_\xi(\omega)} d\mu(\omega) = j_1(< 2\lambda\xi >) = \widehat{\nu}_{<\xi>}(\lambda), \;\; \lambda \in \mathbb{R} \;\; \lambda \in \mathbb{R}$
then the result is deduced. □

3. Chaos Decomposition of the white noise wigner semicircle space:

Let $L^2(E', \mu)$ be the real Hilbert space of square $\mu-$integrable function with norm denoted by $\| \,.\, \|_0$.

Lemma 2. *The measure* μ *satisfies the moment condition:*

$$\int_{E'} < \omega, \xi >^n d\mu(\omega) < \infty.$$

More precisely, for $\xi \in E$ *and* $n \in \mathbb{N}$, *we have:*

$$\int_{E'} < \omega, \xi >^{2n} d\mu(\omega) = \frac{(2n)!}{n!(n+1)!} < \xi >^{2n}$$

and

$$\int_{E'} < \omega, \xi >^{2n+1} d\mu(\omega) = 0.$$

Proof. By using Proposition 1 we get

$$\int_{E'} < \omega, \xi >^{2n} d\mu(\omega) = \int_{\mathbb{R}} x^{2n} d\nu_{<\xi>}(x) = \frac{(2n)!}{n!(n+1)!} < \xi >^{2n}.$$

This proves the first statement. The second is obvious from the symmetry of the measure $\nu_{<\xi>}$. □

From the above lemma, the linear function $\omega \longmapsto X_\xi(\omega), \;\; \xi \in E$ belongs to $L^2(E', \mu)$. Now introduce polynomial functions on the white noise Wigner semicircle space (E', μ). Let $\mathscr{P}_n(E')$ be the space of finite linear combinations of functions of the form
$\omega \longmapsto < \omega, \xi_1 > ... < \omega, \xi_n > = < \omega^{\otimes n}, \xi_1 \otimes ... \otimes \xi_n >, \;\; \omega \in E', \;\; \xi_1, ..., \xi_n \in E$.
An element of the algebraic sums
$\mathscr{P}(E') = \sum_{n=0}^{\infty} \mathscr{P}_n(E')$
is called a polynomial on the space (E', μ). From the equation one can see that the nuclear space E' is closed under the absolute value. This enable us to introduce a wick product in the following way

Definition 2. For $\omega \in E'$ and $n = 0,1,2,...$, we define the wick product $: \omega^{\otimes n} :\in E'^{\widehat{\otimes} n}$ as the linear functional on $E^{\widehat{\otimes} n}$ characterized by

$$<: \omega^{\otimes n} :, \varphi^{\otimes n} > = | \varphi |_0^n P_n(\frac{< \omega, | \varphi | >}{<| \varphi |>}), \;\; \varphi \in E' \tag{5}$$

and for any orthogonal vectors $\xi_1, ..., \xi_k \in E'$ and nonnegative integers n_j's such that $n_1 + ... + n_k = n$, we have

$$<: \omega^{\otimes n} :, \xi_1^{\otimes n_1} \widehat{\otimes} ... \widehat{\otimes} \xi_k^{\otimes n_k} > = <: \omega^{\otimes n_1} :, \xi_1^{\otimes n_1} > ... <: \omega^{\otimes n_k} :, \xi_k^{\otimes n_k} >. \tag{6}$$

Lemma 3. *We have the following statements:*

1. $\int_{E'} <:\omega^{\otimes n}:,\xi^{\otimes n}><:\omega^{\otimes m}:,\xi^{\otimes m}> d\mu(\omega) = |\xi|_0^{2n}\,\delta_{m,n}, \quad \xi\in E.$ (7)

2. $\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes n}:,\eta^{\otimes n}> d\mu(\omega) =$

$<\xi,\eta>^n\,\delta_{m,n}, \quad \xi,\eta\in E.$ (8)

3. *For all* $\phi_n,\psi_n\in E^{\widehat{\otimes} n}$*, we have*
$\int_{E'} <:\omega^{\otimes n}:,\phi_n><:\omega^{\otimes m}:,\psi_m> d\mu(\omega) = <\phi_n,\psi_n>\delta_{m,n}.$

Proof. 1. For $\xi\in E, \xi\neq 0$, the image of the Wigner semicircle white noise measure μ under the map

$$\omega \longmapsto <\omega, \frac{\xi}{<|\xi|>}>\in\mathbb{R}, \quad \omega\in E'$$

is the Wigner semicircle distribution ν. Then we have

$$\begin{aligned}\int_{E'} <:\omega^{\otimes n}:,\xi^{\otimes n} &>< :\omega^{\otimes m}:,\xi^{\otimes m}> d\mu(\omega)\\ &= |\xi|_0^{n+m}\int_I P_n(t)P_m(t)d\nu(t) = |\xi|_0^{2n}\,\delta_{m,n}.\end{aligned}$$

2. It is sufficient to prove the identity under the assumption $|\xi|_0 = |\eta|_0 = 1$. Taking a unit vector $\zeta\in E$ such that $<\xi,\zeta>=0$, we may write $\eta = \alpha\xi+\lambda\zeta, \quad \alpha^2+\lambda^2=1$, and we have

$$<:\omega^{\otimes n}:,\eta^{\otimes n}> =$$

$\sum_{k=0}^n \binom{n}{k}\alpha^{n-k}\lambda^k <:\omega^{\otimes(n-k)}:,\xi^{\otimes(n-k)}><:\omega^{\otimes k}>:,\zeta^{\otimes k}>$.
Then we get

$$\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes n}:,\eta^{\otimes n}> d\mu(\omega)$$
$$= \sum_{k=o}^{n}\binom{n}{k}\alpha^{n-k}\lambda^k$$
$$\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes(n-k)}:,\xi^{\otimes(n-k)}><:\omega^{\otimes k}:,\zeta^{\otimes k}> d\mu(\omega).$$

On the other hand, by using the independence of the two random variables $<.,\xi>$ and $<.,\zeta>$, we obtain

$$\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes(n-k)}:,\xi^{\otimes(n-k)}>$$
$$<:\omega^{\otimes k}:,\zeta^{\otimes k}> d\mu(\omega)$$
$$= \int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes(n-k)}:,\xi^{\otimes(n-k)}> d\mu(\omega)$$
$$\int_{E'} <:\omega^{\otimes k}:,\zeta^{\otimes(n-k)}> d\mu(\omega)\int_{E'} <:\omega^{\otimes k}:,\zeta^{\otimes k}> d\mu(\omega)$$

therefore the last integral is equal to 1 if k=0. Hence,

$$\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes n}:,\eta^{\otimes n}> d\mu(\omega)$$
$$= \alpha^n\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes n}:,\xi^{\otimes n}> d\mu(\omega).$$

Applying the statement 1 of Lemma 3, we conclude that

$$\int_{E'} <:\omega^{\otimes m}:,\xi^{\otimes m}><:\omega^{\otimes m}><:\omega^{\otimes n}:,\eta^{\otimes n}> d\mu(\omega) = \alpha^n\delta_{m,n},$$

since $\alpha = <\xi,\eta>$ we have complete the proof of the statement 2.

3. The statement follows from the second assertion by considering ϕ_n and ψ_n as linear combination of elements of the form $\xi^{\otimes n}, \xi\in E$. □

Proposition 2. *For two polynomials* ϕ,ψ*; given respectively by*

$$\phi(\omega) = \sum_{n=0}^{\infty} <:\omega^{\otimes n}:,\phi_n>, \qquad \psi(\omega) = \sum_{n=0}^{\infty} <:\omega^{\otimes n}:,\psi_n>,$$

it holds that

$$\int_{E'}\phi(\omega)\psi(\omega)d\mu(\omega) = \sum_{n=0}^{\infty}\langle\phi_n,\psi_n\rangle$$

In particular, the L^2-*norm of* ϕ*, with respect to* μ*, is to* μ*, is given by*

$$\|\phi\|_0^2 = \sum_{n=0}^{2} |\phi_n|_0^2 .$$

Proof. The statement follows from Lemma 3. □

The free Fock space $\mathfrak{F}(H)$ over H is defined as the direct sum of the n-th tensor power $H^{\otimes n}, \ n\in\mathbb{N}$,

$$\mathfrak{F}(H) := \bigoplus_{n=0}^{+\infty} H^{\otimes n}.$$

$\mathfrak{F}(H)$ consists of sequence $\overrightarrow{f} = (f^{(0)}, f^{(1)},\dots)$ such that, for any $n\in\mathbb{N}, \ f^{(n)}\in H^{\otimes n}$ and $\|\overrightarrow{f}\|_{\mathfrak{F}(H)}^2 = \sum_{n=0}^{+\infty}\|f^{(n)}\|_{H^{\otimes n}}^2 < \infty$.

Theorem 1. *For each* $F\in L^2(E',\mu)$*, there exists a unique sequence* $\overrightarrow{f} = (f^{(n)})_{n=0}^{\infty}\in\mathfrak{F}(H)$ *such that*

$$F = \sum_{n=0}^{+\infty} <:.^{\otimes n}:,f^{(n)}> \qquad (9)$$

in the L^2-*sense. Conversely, for any* $\overrightarrow{f} = (f^{(n)})_{n=0}^{\infty}\in\mathfrak{F}(H)$*, (9) defines a function on* $L^2(E',\mu)$*. In that case,*

$$\|F\|_{L^2(E',\mu)}^2 = \sum_{n=0}^{+\infty}\|f^{(n)}\|_{H^{\otimes n}}^2 .$$

The following unitary operator is called the Wiener-Ito- isometry:
$I:\mathfrak{F}(H)\longrightarrow L^2(E',\mu) \ (f^{(n)})_{n=0}^{\infty}\longmapsto F$

Proof. It is easy to see that the set

$$\mathscr{P}(E') =$$
$$\{\phi, \phi(\omega) = \sum_{k=0}^{n} <:\omega^{\otimes k}:,\phi^{(k)}>, \phi^{(k)}\in E^{\widehat{\otimes} n}, \ \ \omega\in E', \ \ n\in\mathbb{N}\}$$

of smooth continuous polynomials on E' is continuously and densely embedded in $L^2(E',\mu)$. Then, for any $F\in L^2(E',\mu)$, there exists a unique sequence $\overrightarrow{f} = (f^{(n)})_{n=0}^{\infty}\in\mathfrak{F}(H)$ such that
$F = \sum_{n=0}^{+\infty} <:.^{\otimes n}:,f^{(n)}>$. It follows

$$\begin{aligned}\|F\|_{L^2(E',\mu)}^2 &= \int_{E'}\Big(\sum_{n=0}^{+\infty} <:\omega^{\otimes n}:,f^{(n)}>\Big)^2 d\mu(\omega)\\ &= \sum_{n=0}^{+\infty} <f^{(n)},f^{(n)}>_{H^{\otimes n}} = \|\overrightarrow{F}\|_{\mathfrak{F}(H)}^2\end{aligned}$$

where Proposition 2 is used into account. The second part of the Theorem is straightforward. □

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