On a class of Leibniz algebras

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Abstract

We pointed out the class of Leibniz algebras such that the Killing form is non degenerate implies algebras are semisimple.

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1. Introduction

Throughout this paper, $F$ will be an algebraically closed field of characteristic zero. All vector spaces and algebras will be finite dimensional over $F$. Note the sum of two vector subspaces $V_1, V_2$ by $V_1 + V_2$ and direct sum by $V_1 \oplus V_2$. It is well-known that a Lie algebra is semisimple if and only if its Killing form is non degenerate. An equivalent criterion is found for Leibniz algebra $L$ which satisfies, for all $x, y$ in $L$, the trace of the endomorphism $(ad_x \circ ad_y)|_{\text{Ess}(L)}$ equals zero. Call such algebras “Killing-Leibniz-Algebra”.

Section 2 is devoted to basic facts. In Section 3, the links between radical and nilradical are set. Section 4 is devoted to the nilpotency of the ideal $\{\text{Rad}(L), L\}$. In Section 5, the main theorem is settled. For conclusion, we give an hierarchy of Leibniz algebras and two questions are done about Killing Leibniz Algebras.

2. Preliminary notes

Let us note that Leibniz algebras are defined in two classes:

- Right Leibniz algebras, with the rule
  \[[x, [y, z]] = [[x, y], z] - [[x, z], y]\] for any $x, y, z \in L$. (1)

- Left Leibniz algebras, with the rule
  \[[x, [y, z]] = [[x, y], z] + [y, [x, z]]\] for any $x, y, z \in L$. (2)

For an algebra $(A, [\cdot, \cdot])$ with vectors multiplication $[a, b]$, for all $a, b$ in $A$, define the algebra $(A, [\cdot, \cdot]^\text{op})$ as the underlying vector space $A$ where the vectors multiplication is defined by $[a, b]^\text{op} = [b, a]$. We have that:
Proposition 2.1. The algebra $(A,[ , ])$ is left Leibniz algebra if and only if the algebra $(A,[ , ]^{op})$ is right Leibniz algebra.

So results on Left Leibniz algebras are available on Right Leibniz algebras, (with minor variations). Here we write ”Leibniz algebras” for ”Right Leibniz algebras”.

It follows from the equation (1) called Leibniz identity that in any Leibniz algebra one has

$$[y,[x,x]] = 0, \ [z,[x,y]] + [z,[y,x]] = 0, \ \text{for all } x,y,z \in L.$$

Definition 2.2. (Ideal) A subspace $H$ of a Leibniz algebra $L$ is called left (respectively right) ideal if for $a \in H$ and $x \in L$ one has $[x,a] \in H$ (respectively $[a,x] \in H$). If $H$ is both left and right ideal, then $H$ is called (two-sided) ideal.

If $V$ is a vector space, let $\text{End}_F(V)$ denotes the set of all endomorphisms of $V$. An action of $L$ on $\text{End}_F(V)$ is a linear map of $L$ on $\text{End}_F(V)$.

Definition 2.3. (Representation) Let $L$ be a Leibniz algebra and $V$ a vector space. $V$ is an $L$-module if there are:

- a left action, $l : L \to \text{End}_F(V), x \mapsto l_x$,
- a right action, $r : L \to \text{End}_F(V), x \mapsto r_x$,

such that:

$$l_{[x,y]} = l_y l_x - l_x l_y, \quad l_{[x,y]} = r_y l_x - l_x r_y, \quad r_{[x,y]} = r_y r_x + r_x r_y.$$

For $x$ in $L$, $r_x(v)$ will be denoted by vx and $l_x(v)$ will be denoted by xv. The triplet $(l,r,V)$ is called a representation of $L$ on $V$. Now if $L$ is a Leibniz algebra, we have the adjoint representation “$(\text{Ad},\text{ad},L)$” defined as follows: for all $x$ and $y$ in $L$, $\text{ad}_x : L \to L, y \mapsto [y,x]$ and $\text{Ad}_x : L \to L, y \mapsto [x,y]$.

Remark 2.4.

For $x \in L$, $\text{ad}_x : L \to L$ is a derivation of $L$ i.e. for all $x,y,z \in L$, $\text{ad}_x([y,z]) = [\text{ad}_x(y),z] + [y,\text{ad}_x(z)]$.

For $x \in L$, $\text{Ad}_x : L \to L$ is an anti-derivation of $L$ i.e. for all $x,y,z \in L$, $\text{Ad}_x([y,z]) = [\text{Ad}_x(y),z] - [\text{Ad}_x(z),y]$.

For an arbitrary algebra and for all non negative integer $n$ let us define the sequences:

(i) $D^1 (L) = L^1 = L^2, \ D^{n+1} (L) = L^{n+1} = [L^{n},L^{n}]$;

(ii) $L^1 = L, \ L^{n+1} = [L^1, L^n] + [L^2, L^{n-1}] + \cdots + [L^{n-1}, L^2] + [L^n, L^1]$.

Definition 2.5. ([1]) An algebra $L$ is called solvable if there exists $m \in \mathbb{N}^*$ such that $D^m (L) = L^m = \{0\}$.

An algebra $L$ is called nilpotent if there exists $m \in \mathbb{N}^*$ such that $L^m = \{0\}$.

Definition 2.6. Let $A$ be a subspace of a Leibniz algebra $L$. The normalizer of $A$ is denoted by:

$$n_L(A) = \{ y \in L | [y,a] \in A \text{ and } [a,y] \in A \text{ for all } a \in A \}.$$

Definition 2.7. ([4]) A Leibniz algebra $L$ is said to be semisimple if $\text{Rad}(L) = \text{Ess}(L)$.

Equivalently, we can say that:

Leibniz algebra $L$ simple if $\{0\} \neq [L,L] \neq \text{Ess}(L)$ and every ideal of $L$ belongs to the set $\{L, \text{Ess}(L), \{0\}\}$. Since $D^1 = i^2$ is an ideal whenever $i$ is (by Equation 1), if $\text{rad}(L) \neq \text{Ess}(L)$ then $L$ contains an ideal $j$ which satisfies $j^2 \subseteq \text{Ess}(L) \subseteq j$.

So an other equivalent definition is:

Remark 2.8. $L$ is semisimple if it has no ideal $j$ which satisfies $j^2 \subseteq \text{Ess}(L) \subseteq j$.

Lemma 2.9. ([3]) Let $L$ be a Leibniz algebra and $(l,r,V)$ a representation of $L$. Let $A$ be a subspace of $L$, then $r_A = \{ r_x, \text{ for all } x \in A \}$ is a subspace of the vector space $\text{End}_F(V)$. In particular, $r_L$ is a Lie subalgebra of $\text{gl}(V)$ and $L$ is solvable (respectively nilpotent) if and only if $r_L$ is solvable (respectively nilpotent).

Proof. The results are clear since for all $x, y$ in $L$ and for all $\lambda$ in $F$, we have that $r_{x+\lambda y} = r_x + \lambda r_y$ and $[r_x,r_y] = r_{[y,x]}$. \(\square\)
Remark 2.10. Let $L$ be a Leibniz algebra and $(l, r, V)$ a representation of $L$. If for all $x$ in $L$, $r_x$ is nilpotent then $l_x$ is also nilpotent for all $x$. Since we have $t^k_x = (-1)^{k+1} t_x (r_x)^{k-1}$. Thus when $r_x$ is nilpotent for all $x$ in $L$, we can say that the representation $(l, r, V)$ of $L$ is nilpotent.

Lemma 2.11. Let $L$ be a Leibniz algebra and $(l, r, V)$ a representation of $L$. Let $A$ be a subspace of the vector space $L$ and let $x$ in the normalizer $n_L(A)$ of $A$. Then we have for all integer $k$ in $\mathbb{N}$ and for all $a$ in $A$:

i) $\delta_{k+1} = r_a^{k+1} r_x - r_x r_a^{k+1} \in r_A^k.$

ii) $\beta_{k+1} = r_x^{k+1} r_a - r_a r_x^{k+1} \in r_A^{k+1}.$

Proof. For i), since $[r, a] = [r, a]$, we have $\delta_1 = r_a r_x - r_x r_a = r_{[a, x]}$. Thus $\delta_1 \in r_A$ since $x \in n_L(A)$. And we have:

$\delta_2 = r_a^2 r_x - r_x r_a^2 = r_a (r_a r_x) - r_x r_a^2 = (r_a r_x + \delta_1) - r_x r_a^2 = (r_a r_x) + r_a \delta_1 - r_x r_a^2$.

With the hypothesis of recurrence: $\delta_k = r_a^k r_x - r_x r_a^k \in r_A^k$, we get:

$\delta_{k+1} = r_a^{k+1} r_x - r_x r_a^{k+1} = r_a (r_a^k r_x) - r_x r_a^{k+1} = (r_a r_x)^k + r_a \delta_k - r_x r_a^{k+1}$.

And for ii), we have $[r, a] = r_{[a, x]}$, so $\beta_1 = -\delta_1 \in r_A = r_{A^0}$ since $x \in n_L(A)$ (where $r_{A^0} = 1_V$). Note that we have:

$\beta_2 = r_a^{k+1} r_x - r_x r_a^{k+1} = r_a (r_a^k r_x) - r_x r_a^{k+1} = (r_a^k r_x) + r_a \beta_1 - r_x r_a^{k+1}$.

Set the hypothesis that $\beta_k = r_a^k r_x - r_x r_a^k \in r_A^{k-1} + \cdots + r_A r_x + r_A$, and then it will follow that:

$\beta_{k+1} = r_a^{k+1} r_x - r_x r_a^{k+1} = r_a (r_a^k r_x) - r_x r_a^{k+1} = (r_a^k r_x) + r_a \beta_k - r_x r_a^{k+1}$.

Proofs are done.

Lemma 2.12. Let $L$ be a Leibniz algebra and $(l, r, V)$ a representation of $L$. Let $A$ be a subspace of the vector space $L$ and $x$ in the normalizer $n_L(A)$ of $A$. Then we have for all integer $k$ and $p$ in $\mathbb{N}$:

$[r_{A^k}^p] \circ r_A \subseteq r_{A^k}^{p+1} r_x^k + \cdots + r_{A^p}^{p+1} r_x^p + r_A^{p+1}$.

Proof. We shall note that:

$[r_{A^k}^p] \circ r_A = r_A^p \circ [r_{A^k}^p \circ r_{A}] \subseteq r_A^p (r_{A^k}^{p+1} r_x^k + \cdots + r_{A}^{p+1} r_x^p + r_A) \subseteq r_A^{p+1} r_x^p + \cdots + r_A^{p+1} r_x^p + (r_x)^{p+1}$.

Thanks to the preceding lemma we have for all integer $k$, $l$, $p$ and $q$ in $\mathbb{N}$:

$r_{A^{k+l}}^p \circ r_A^q \subseteq r_{A^{k+l}}^{p+q} r_x^k + \cdots + r_{A}^{p+q} r_x^q$.

Lemma 2.13. Let $L$ be a Leibniz algebra and $(l, r, V)$ a representation of $L$. Let $A$ be a subspace of the vector space $L$ and $x$ in the normalizer $n_L(A)$ of $A$ and for a non negative integer $k$ let $E_k$ be the subspace $E_k = r_A^{k+1} + \cdots + r_A$.

Then we have for all integer $p$ in $\mathbb{N}^*$:

$E_p \subseteq r_{A^k}^p r_x^{p+1} + \cdots + r_{A^p}^{p+1} r_x^p + r_A^p$.

Proof. Let us compute $E_p$ for $p = 2, 3$; we have $[r_x, r_a] = [r_{[a, x]}], s o$

$E_2 = (r_{A^2}^2 + \cdots + r_A^2) r_{[a, x]} = (r_{A^2}^2 + \cdots + r_A^2) (r_{A^k} + \cdots + r_A) \subseteq (r_{A^2}^2 + \cdots + r_A^2) (r_{A^k} + \cdots + r_A) \subseteq r_{A^2}^{2k} + \cdots + r_{A^k}^{2k} + \cdots + r_A^{2k} r_x + r_A^2$,

$E_3 = (r_{A^3}^3 + \cdots + r_A^3) r_{[a, x]} = (r_{A^3}^3 + \cdots + r_A^3) (r_{A^k} + \cdots + r_A) \subseteq (r_{A^3}^3 + \cdots + r_A^3) (r_{A^k} + \cdots + r_A) \subseteq r_{A^3}^{3k} + \cdots + r_{A^k}^{3k} + \cdots + r_A^{3k} r_x + r_A^3$.

and set by hypothesis that we have $E_{k-1} \subseteq r_{A^k}^{p-1} r_{[a, x]}^p + \cdots + r_A^{p-1} r_{[a, x]}^p$.
So we will get $E^p_k = (r_A r_x^k + \cdots + r_A)^p = (r_A r_x^k + \cdots + r_A)^{p-1} (r_A r_x^k + \cdots + r_A) \subset (r_A r_x^{(p-1)k} + \cdots + r_A r_x^k + r_A^{p-1}) (r_A r_x^k + \cdots + r_A) \subset (r_A^{p-1} r_x^{(p-1)k}) (r_A r_x^k) + \cdots + r_A^{p-1} (r_A r_x^k) + (r_A^{p-1} r_x) r_A + r_A^{p-1} r_A \subset r_A^{p} r_x^p + \cdots + r_A^{p} r_x^k + \cdots + r_A^{p} r_x + r_A$.

The proof is then done. □

**Lemma 2.14.** Let $L$ be a Leibniz algebra and $(l,r,V)$ a representation of $L$. Let $A$ be a subspace of the vector space $L$ and $x$ in the normalizer $n_L(A)$ of $A$. Let $m$ be a non negative integer. Then for all $(\lambda, a) \in F \times A$,

$$f_m = (r_a + \lambda x)^m - \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} r_x^k \in r_A r_x^m + \cdots + r_A.$$

**Proof.** By induction:

$$f_1 = (r_a + \lambda x)^1 - \sum_{k=0}^{1} \binom{1}{k} \lambda^k r_a^{1-k} r_x^k = r_a + \lambda x - (r_a + \lambda x) = 0 \in r_A r_x^1 + r_A.$$ 

And if by hypothesis we have: $f_m = (r_a + \lambda x)^m - \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} r_x^k \in r_A r_x^m + \cdots + r_A$, we can write:

$$f_{m+1} = (r_a + \lambda x)^{m+1} - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k = (r_a + \lambda x)^{m} (r_a + \lambda x) - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} r_x^k + f_m (r_a + \lambda x) - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} r_x^k + f_m r_a + \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k + \lambda f_m r_x - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k.$$

Then we have

$$f_{m+1} = \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} (r_a r_x^k + \beta_k) + f_m r_a + \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k + \lambda f_m r_x - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k+1} r_x^k + \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} \beta_k + f_m r_a$$

$$+ \lambda f_m r_x - \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k r_a^{m+1-k} r_x^k$$

$$= r_a^{m+1} + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m-k} \beta_k + f_m r_a$$

$$= r_a^{m+1} + \sum_{j=1}^{m+1} \binom{m}{j} \lambda^k r_a^{m-j+1} r_x^k + \sum_{j=0}^{m+1} \binom{m}{j} \lambda^k r_a^{m-j} \beta_j + f_m r_a$$

$$= r_a^{m+1} + \sum_{j=1}^{m+1} \binom{m}{j} \lambda^k r_a^{m-j+1} r_x^k + \sum_{j=0}^{m+1} \binom{m}{j} \lambda^k r_a^{m-j} \beta_j + f_m r_a$$

$$= r_a^{m+1} + \sum_{j=1}^{m+1} \binom{m}{j} \lambda^k r_a^{m-j+1} r_x^k + \sum_{j=1}^{m+1} \binom{m}{j-1} \lambda^k r_a^{m-j+1} r_x^k + \sum_{j=1}^{m+1} \lambda^j r_a^{m+1} r_x^k$$

$$= r_a^{m+1} + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k$$

$$= \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k + \sum_{k=0}^{m+1} \binom{m}{k} \lambda^k r_a^{m+1-k} r_x^k.$$
Finally we have
\[ f_{m+1} = \sum_{k=0}^{m} \binom{m}{k} \lambda^k \beta_{m-k} r_x + f_m r_x + \lambda f_m r_x \]
\[ \in \sum_{k=0}^{m} \binom{m}{k} \lambda^k (r_A)^{m-k} \left( r_A r_x^m + \cdots + r_A \right) + \left( r_A r_x^m + \cdots + r_A \right) r_x + \lambda \left( r_A r_x^m + \cdots + r_A \right) r_x \]
\[ \in r_A r_x^{m+1} + \cdots + r_A r_x + r_A. \]

**Definition 2.15.** Call \( x \in \text{End}(V) \) semisimple if the roots of its minimum polynomial over \( F \) are all distinct, or equivalently, if \( x \) is diagonalizable.

**Remark 2.16.**
1. Two commuting semisimple endomorphisms are simultaneously diagonalizable, so their sum and difference are both semisimple.
2. If \( x \) is semisimple and \( x \) leaves a subspace \( W \) invariant, then the restriction of \( x \) to \( W \) denoted by \( x|_W \) is semisimple.

**Definition 2.17.** Call \( x \in L \) ad-semisimple (respectively \( \text{Ad}-\text{semisimple} \)) if the endomorphisms \( \text{ad}_x \) is semisimple (respectively \( \text{Ad}_x \) is semisimple).

Call \( x \in L \) ad-nilpotent (respectively \( \text{Ad}-\text{nilpotent} \)) if the endomorphisms \( \text{ad}_x \) is nilpotent (respectively \( \text{Ad}_x \) is nilpotent).

**Lemma 2.18.** Let \( V = V_1 \oplus V_2 \) be a direct sum of two vector spaces \( V_1, V_2 \), an non negative integer \( p \) and \( \sigma \) an endomorphism of \( V \) such that \( \sigma^p(V) \subseteq V_1 \), then the trace of \( \sigma \) denoted by \( \text{tr}(\sigma) = \text{tr}(\sigma|_{V_1}) \), where \( \sigma|_{V_1} \) is the restriction of \( \sigma \) to \( V_1 \).

**Proof.** Since we have an algebraically closed field, we can find a basis \( \{v_1, \cdots, v_m, \cdots, v_n\} \) of \( V \) with \( \{v_1, \cdots, v_m, \} \) is a basis of \( V_1 \) and scalars \( \lambda_1, \cdots, \lambda_n \) such that the matrix of \( \sigma \) in this basis is
\[ N_{0k} = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & \lambda_2 & a_{2,3} & \cdots & a_{2,n} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \lambda_{n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} \]

For \( m+1 \leq i \leq n \), we have a vector \( 0 \neq v_i \in V_2 \) such that \( \sigma(v_i) = \lambda_i v_i \). Then \( \sigma^p(v_i) = \lambda^p_i v_i \in V_2 \cap V_1 = \{0\} \). So \( \lambda_i = 0 \) for \( m+1 \leq i \leq n \), and
\[ \text{tr}(\sigma) = \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{m} \lambda_j = \text{tr}(\sigma|_{V_1}). \]

**3. Radical and Nilradical**

The proof of following proposition can be found in [5].

**Proposition 3.1.** Let \( \mathfrak{W} \) be a Lie subalgebra of \( \text{End}_F(V) \) where \( V \) is an \( F \)-vector space. Then \( \mathfrak{W} \) is solvable if and only if \( tr(x \circ y) = 0 \) for all \( x \in \mathfrak{W} \) and \( y \in [\mathfrak{W}, \mathfrak{W}] \).

**Theorem 3.2.** [1, Theorem 3.7] Let \( L \) be a Leibniz algebra. Then \( L \) is solvable if and only if for all \( x \in L \) and all \( y \in [L,L], tr(ad_x \circ ad_y) = 0 \).

If \( i \) is an ideal of \( L \) and \( L/i \) is solvable (respectively nilpotent), then \( D^{(m)}(L/i) = 0 \) (respectively \( (L/i)^n = 0 \)) implies that \( D^{(m)}(L) \subset i \) (respectively \( L^n \subset i \) nilpotent). If \( i \) itself is solvable with \( D^{(m)}(i) = 0 \) (respectively nilpotent with \( i^n = 0 \)), then \( D^{(m+n)}(L) = 0 \) (respectively \( L^{m+n} = 0 \)).

So we have proved:
Proposition 3.3. If $\iota \subset L$ is an ideal, and both $\iota$ and $L/\iota$ are solvable (respectively nilpotent), so is $L$ solvable (respectively nilpotent).

If $\iota$ and $\jmath$ are solvable ideals, then $(\iota + \jmath)/\jmath \equiv \iota/(\iota \cap \jmath)$ is solvable, being the homomorphic image of a solvable algebra. So, by the previous proposition, we have the

Proposition 3.4. If $\iota$ and $\jmath$ are solvable ideals (respectively nilpotent ideals) in $L$ so $\iota + \jmath$ is solvable (respectively nilpotent). In particular, every Leibniz algebra $L$ has a largest solvable ideal which contains all other solvable ideals and a largest nilpotent ideal which contains all other nilpotent ideals.

The largest solvable one is denoted by $\operatorname{Rad}(L)$.

The largest nilpotent one is denoted by $\operatorname{Nil}(L)$.

Remark 3.5. Note that $\operatorname{Ess}(L) \subseteq \operatorname{Nil}(L) \subseteq \operatorname{Rad}(L)$.

4. The ideal $\{\operatorname{Rad}(L), L\}$

Let us denote the subspace $[\operatorname{Rad}(L), L] + [L, \operatorname{Rad}(L)]$ by $\{\operatorname{Rad}(L), L\}$.

Lemma 4.1. Let $L$ be a Leibniz algebra and $(l,r,V)$ a representation of $L$. Let $A$ be a subspace of $L$ for which there exists an integer $n \in \mathbb{N}^*$ with $r_A^n = \{0\}$ and let $x$ be in $n_L(A)$ such that $r_x$ is nilpotent. Then there exists an integer $N \in \mathbb{N}^*$ with $(r_{A+F x})^N = \{0\}$.

Proof. Let us notice that for any non negative integer $p$ we have

$$(r_{a+x})^p = \sum_{k=0}^{p} \binom{p}{k} r_a^k r_x^{p-k} + f_p \text{ where } f_p \in E_p = r_A r_{x}^p + \cdots + r_{A}.$$

Let $m$ an integer with $(r_x)^m = 0.$ Then with $p = 2 \sup (m, n) + 1 > m + n$ we have that $(r_{a+x})^p = f_p \in E_p$. And so

$$(r_{a+x})^p = (f_p)^n = (r_A r_{x}^p + \cdots + r_{A})^n \subseteq r_A n r_{x}^p + \cdots + r_{A} r_{x}^{2p} + \cdots + r_{A} r_{x}^n.$$

Since $r_A^n = \{0\}$, $(r_{a+x})^{pn} = 0.$ So $r_{a+x}$ is nilpotent for all $a = \lambda x$ in $A+F x$. By [7, Theorem 3.2], page 41] the associative algebra $r_{A+F x}$ is nilpotent algebra. So there is some integer $N \in \mathbb{N}^*$ such that $(r_{A+F x})^N = \{0\}.$

Proposition 4.2. For any representation $(l,r,V)$ of the Leibniz algebra $L$, the restriction of $r$ to the ideal $\{\operatorname{Rad}(L), L\}$ is nilpotent, i.e. there exists an integer $m \in \mathbb{N}^*$ with $(r_{\{\operatorname{Rad}(L), L\}})^m = \{0\}$.

Proof. According to [3, Corollary 4.4] the representation of $V$ is nilpotent on the ideal $[L,L]$. Now let $T \subseteq \{\operatorname{Rad}(L), L\}$ be a subspace containing $[\operatorname{Rad}(L), \operatorname{Rad}(L)]$, which is maximal with respect to the property that the representation of $V$ is nilpotent on $T$. Note that $T$ always is an ideal of $\operatorname{Rad}(L)$, hence in particular a subalgebra, because it contains $[\operatorname{Rad}(L), \operatorname{Rad}(L)]$.

Assume that $T \neq \{\operatorname{Rad}(L), L\}$. Then there exist at least an $x$ in $\operatorname{Rad}(L)$ and $y$ in $L$ with $[x,y] \notin T$ or $[y,x] \notin T$.

If $[x,y] \notin T$, the subspace $B = \operatorname{Rad}(L) + F x$ is a subalgebra of $L$, $\operatorname{Rad}(L)$ is a solvable ideal of $B$ and $B/\operatorname{Rad}(L) \approx F$ is abelian. Therefore $B$ is a solvable ideal by Proposition 3.3.

Again we use [3, Corollary 4.4] to see that the representation of $V$ is nilpotent on $[B,B]$ and hence that $r_{[x,y]}$ is nilpotent.

Since $T \subseteq \operatorname{Rad}(L)$ and $[x,y] \subseteq [\operatorname{Rad}(L), y] \subseteq \operatorname{Rad}(L)$, we have $[[x,y], T] \subseteq [\operatorname{Rad}(L), T] \subseteq T$ and $[y,x], [T, [x,y]] \subseteq [T, \operatorname{Rad}(L)] \subseteq T$.

Finally the preceding lemma show that the representation of $V$ is nilpotent on the subspace $T \oplus F [x,y]$. This contradicts the maximality of $T$.

If $[y,x] \notin T$, the subspace $B = \operatorname{Rad}(L) + F x$ is a subalgebra of $L$, $\operatorname{Rad}(L)$ is a solvable ideal of $B$ and $B/\operatorname{Rad}(L) \approx F$ is abelian. Therefore $B$ is a solvable ideal by Proposition 3.3.

Again we use [3, Corollary 4.4] to see that the representation of $V$ is nilpotent on $[B,B]$ and hence that $r_{[y,x]}$ is nilpotent.

Since $T \subseteq \operatorname{Rad}(L)$ and $[y,x] \subseteq [y, \operatorname{Rad}(L)] \subseteq \operatorname{Rad}(L)$, we have $[[y,x], T] \subseteq [\operatorname{Rad}(L), T] \subseteq T$ and $[[y,x], T] \subseteq [\operatorname{Rad}(L), T] \subseteq T$.

Finally the preceding lemma show that the representation of $V$ is nilpotent on the subspace $T \oplus F [x,y]$. This contradicts the maximality of $T$. 


We conclude that $T$ must be equal to $\{\text{Rad}(L), L\}$, so the representation of $V$ is nilpotent on $\{\text{Rad}(L), L\}$.

Applying the preceding proposition to the adjoint representation $(\text{Ad}, \text{ad}, L)$ of the Leibniz algebra $L$ and using Engel’s Theorem [2], we get the:

**Corollary 4.3.** The ideal $\{\text{Rad}(L), L\}$ is nilpotent. In particular, $x$ is ad-nilpotent for every $x$ in $\{\text{Rad}(L), L\}$.

**Corollary 4.4.** Let $L$ be a Leibniz algebra and $D$ a derivation of $L$. Then $D(\text{Rad}(L)) \subseteq \text{Nil}(L)$. In particular $\text{Nil}(L)$ is a characteristic ideal.

**Proof.** For a derivation $D$ of $L$, define the Leibniz algebra $\tilde{L} = L \times |D| F$ with the bracket $((x, l), (y, l)) = (I(D)(x) - tD(y) + [x, y], 0)$. Then, $(D(\text{Rad}(L)), 0) = [(\text{Rad}(L), 0)(0, 1)] \subseteq (L, 0) \cap \text{rad}(\tilde{L}) \subseteq \text{Nil}(L) = \text{nil}(L), 0)$. So $D(\text{Rad}(L)) \subseteq \text{Nil}(L)$.

\[\Box\]

## 5. Main theorem

We deal in this section with Leibniz algebras which satisfy equation

$$\forall x, y \in L, tr(ad_x \circ ad_y) = 0$$

Call such Leibniz algebras: Killing Leibniz Algebras.

A bilinear form $(-, -): L \times L \rightarrow F$ is called invariant if $([x, y], z) + (y, [x, z]) = 0$ for all $x, y, z$ in $L$.

Notice that $(x, y)$ is an invariant form, and its orthogonal $x^\perp$ is again an ideal.

One way of producing invariant forms is from representations: if $(l, r, V)$ is a representation of $L$, then $(x, y)_r = tr(r(x \circ r(y))$ is invariant. Indeed,

$$\langle x, y \rangle + (y, [x, z]) = tr((r_y \circ r_x) - r_x \circ r(y)) = tr((r_y \circ r_z) - r_z \circ r(y)) = 0.$$

In particular, if we take $l = \text{Ad}$, $r = \text{ad}$, $V = L$ the corresponding bilinear form is called the Killing form and will be denoted by $\mathcal{R} = (-, -)_R$.

**Remark 5.1.** for all $x$ in $\text{Ess}(L)$, $y, z$ in $L$ we have: $(ad_x \circ ad_y)(z) = (ad_x)([z, y]) = [z, y, x] = 0$. Then $ad_x \circ ad_y \equiv 0$ and $(x, y)_R = tr(ad_x \circ ad_y) = 0$, so $\text{Ess}(L) \subseteq \text{ker}(\mathcal{R})$.

**Theorem 5.2.** Let $L$ be a Leibniz algebra of a class Killing Leibniz Algebras and $\text{ker}(\mathcal{R})$ the kernel of its Killing form.

$$\text{ker}(\mathcal{R}) = \text{Ess}(L) \iff L \text{ is semisimple}.$$
Remark 5.3. • I. Demir et al. give another proof of ⇒. (see [4, Theorem 5.8]).
• In Lie algebras case, Theorem 5.2 is the well known Cartan’s criterion for semisimplicity.

6. Conclusion

Let us cite an example of Leibniz algebra which is solvable and the kernel of it’s Killing form is $\text{Ess}(L)$.

Example 6.1. [6]
Let $L = Cx + Cy$ be the two dimensional complex Leibniz algebra which generators satisfy $[x, y] = x$; $[x, x] = [y, y] = [y, x] = 0$.

Let us find the kernel of the Killing form of the non lie leibniz algebra $L = Fx \oplus Fy$ defined in Example 6.1. Let $a = a_{11}x + a_{12}y$ and $b = a_{21}x + a_{22}y$ be two elements of algebra. The matrix of the endomorphism $ad_a$ is

$\begin{pmatrix}
a_{12} & 0 \\
0 & 0
\end{pmatrix}$

and the matrix of the endomorphism $ad_b$ is

$\begin{pmatrix}
a_{22} & 0 \\
0 & 0
\end{pmatrix}$.

Then the Killing form is defined by $(a, b)_R = a_{12}a_{22}$ for all $a, b$ in $L$.

Since $\text{Ess}(L) = \{0\}$ for any Lie algebra; Lie algebras are Killing Leibniz algebras and the Theorem 5.2 is knowned for Lie algebras (cf. [5]).

"Left central Leibniz" are also Killing Leibniz algebras.

Example 6.1 is an algebra not in a class of Killing Leibniz algebras.

We claim that

Claim: The class of Killing Leibniz Algebras is a widest class wich satisfies Theorem 5.2.

In [6], the authors call an algebra that is both a left and right Leibniz algebra a symmetric Leibniz algebra. they call L a left central Leibniz algebra if it is a left Leibniz algebra that also satisfies $[[a, a], b] = 0$, $a \in L$, $b \in L$.

There is a hierarchy of algebras

$\{\text{leftLeibniz}\} \supseteq \{\text{leftcentralLeibniz}\} \supseteq \{\text{symmetricLeibniz}\} \supseteq \{\text{Lie}\}$.

We call a right central Leibniz algebra if it is a right Leibniz algebra that also satisfies $[b, [a, a]] = 0$, $a \in L$, $b \in L$ and there is a hierarchy of algebras

$\{\text{rightLeibniz}\} \supseteq \{\text{rightcentralLeibniz}\} \supseteq \{\text{symmetricLeibniz}\} \supseteq \{\text{Lie}\}$.

So we can complete the hierarchy of Leibniz algebras as

$\{\text{rightLeibniz}\} \supseteq \{\text{rightKillingLeibniz}\} \supseteq \{\text{rightcentralLeibniz}\} \supseteq \{\text{symmetricLeibniz}\} \supseteq \{\text{Lie}\}$.

and

$\{\text{leftLeibniz}\} \supseteq \{\text{leftKillingLeibniz}\} \supseteq \{\text{leftcentralLeibniz}\} \supseteq \{\text{symmetricLeibniz}\} \supseteq \{\text{Lie}\}$.

Questions:

• Can we prove the Weyl’s theorem on complete reducibility for Killing Leibniz Algebras?

• In [6], the authors show that "left central Leibniz algebras" satisfy a version of the Malcev theorem. Do the Killing Leibniz Algebras also satisfy this theorem?

References


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