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Research paper

Well-Posedness of the Variable Exponent Ricci Flow

Mykola Yaremenko[1*]

[1]*National Technical University of Ukraine, "Igor Sikorsky Kyiv Polytechnic Institute"*
[*]*Corresponding author E-mail: math.kiev@gmail.com*

Abstract

We establish local well-posedness for the variable exponent Ricci flow

$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_{g(t)}|^{p(x)-2}\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

on closed Riemannian manifolds, where $p : M \to (1,\infty)$ is a Hölder continuous function. We identify precise conditions on the exponent function $p(x)$ and initial metric g_0 that guarantee existence, uniqueness, and continuous dependence on initial data. The main challenges arise from the degenerate/singular parabolic nature of the equation when $p(x) \neq 2$ and the spatially varying nonlinearity. Our results reveal a trichotomy: for $p(x) \geq 2$, we obtain strong solutions under mild conditions; for $1 < p(x) < 2$, well-posedness requires a curvature gap condition; and at the critical interface where $p(x)$ crosses 2, we demonstrate potential non-uniqueness. Applications to adaptive geometric regularization and connections to variable exponent Sobolev spaces are discussed.

Keywords: *Variable exponent Ricci flow; degenerate parabolic equations; well-posedness; geometric analysis; parabolic Hölder spaces*

1. Introduction

Richard Hamilton introduced the Ricci flow in his seminal 1982 paper [5], and the flow has since revolutionized geometric analysis over the past four decades. Originally conceived as a tool to deform Riemannian metrics in the direction of their Ricci curvature, the flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)},$$

has yielded profound insights into the structure and classification of manifolds. Its most celebrated application remains Grigori Perelman's resolution of the Poincaré and Thurston's Geometrization conjectures [11, 14, 15], demonstrating the flow's remarkable power to untangle the topological and geometric complexity of three-manifolds through controlled singularity formation and surgical decomposition.
From a variational perspective, the classical Ricci flow can be viewed as the L^2-gradient flow of the Einstein-Hilbert functional

$$\mathscr{E}(g) = \int_M R_g \, dV_g,$$

with respect to the natural L^2-metric on the space of Riemannian metrics [13]. This interpretation has motivated numerous generalizations where the L^2-gradient structure is replaced by other metrics, particularly those arising from L^p-norms of curvature. The present work investigates a substantial generalization where the exponent 2 in the curvature norm is replaced by a spatially varying function $p(x)$, yielding the *variable exponent Ricci flow*

$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_{g(t)}|^{p(x)-2}\mathrm{Ric}_{g(t)}, \quad g(0) = g_0, \tag{1}$$

where $p : M \to (1,\infty)$ is a Hölder continuous function and $|\mathrm{Ric}|$ denotes the pointwise tensor norm. This equation interpolates between different parabolic regimes depending on the local value of $p(x)$, introducing adaptive geometric regularization with mathematically novel and physically meaningful properties.

1.1. Motivation and Background

The study of equations with variable exponents has emerged as a vibrant area in partial differential equations and calculus of variations over the past two decades, driven by applications in materials science, image processing, and continuum mechanics [4, 12, 1]. In electrorheological fluids, for instance, the effective viscosity depends on the electric field strength, leading to power-law behavior with spatially varying exponents [12]. Similarly, in image restoration, variable exponent $p(x)$-Laplacian models achieve edge preservation while maintaining smoothness in homogeneous regions [16].
In geometric analysis, several motivations justify the investigation of (1):

1. **Adaptive Geometric Regularization:** The exponent $p(x)$ functions as a *local regularization parameter*. In regions where $p(x) > 2$, the flow exhibits **sublinear growth** in curvature magnitude, tending to preserve geometric features such as edges, ridges, or regions of high curvature concentration. This behavior is analogous to total variation regularization in image processing. Conversely, where $1 < p(x) < 2$, the flow exhibits **superlinear growth**, promoting aggressive smoothing and faster dissipation of curvature anomalies. The critical case $p(x) \equiv 2$ recovers the classical Ricci flow with its well-known balance between smoothing and singularity formation.
2. **Singularity Modulation and Control:** One of the central challenges in Ricci flow theory is the analysis and classification of finite-time singularities [6]. The variable exponent framework provides a mechanism to *modulate singularity formation* by locally adjusting $p(x)$. For instance, one might choose $p(x) > 2$ near anticipated singular regions to suppress curvature blow-up, or $p(x) < 2$ elsewhere to accelerate convergence to canonical metrics. This adaptive control is unavailable in constant-exponent flows.
3. **Connection to Variable Exponent Sobolev Geometry:** The functional

$$\mathscr{F}_{p(\cdot)}(g) = \int_M \frac{1}{p(x)} |\mathrm{Ric}_g|^{p(x)}\, dV_g$$

generates (1) as its formal L^2-gradient flow. While this functional is not diffeomorphism invariant unless p is constant, the connection to the burgeoning theory of variable exponent Sobolev spaces [4] raises natural questions about the geometric implications of such non-standard growth conditions.
4. **Physical and Materials Modeling:** In theories of elastic manifolds or biological membranes, the elastic energy density may exhibit power-law hardening or softening depending on local strain or curvature. Equation (1) provides a geometrically intrinsic model for such materials, where $p(x)$ encodes spatially varying elastic properties.

Despite these motivations, the analytical foundations of (1) remain largely unexplored. The equation presents significant challenges:

1. **Degenerate/Singular Parabolicity:** For $p(x) > 2$, the equation becomes degenerate when $|\mathrm{Ric}| = 0$; for $1 < p(x) < 2$, it becomes singular at zero curvature. The type changes spatially depending on $p(x)$, creating interfaces between different parabolic regimes.
2. **Loss of Scaling Symmetry:** Unlike the classical Ricci flow, which is invariant under $g \mapsto \lambda g$, $t \mapsto \lambda t$, the variable exponent flow breaks this scaling unless p is constant, complicating blow-up analysis and singularity classification.
3. **Nonlocal Coupling:** Although the equation is formally local, the variable exponent $p(x)$ introduces implicit nonlocal effects: evolution at a point depends on $p(x)$, which may itself be determined by global geometric or topological data.

These features place (1) within the broader context of non-standard growth parabolic equations, a subject with substantial technical developments in recent years [1, 9, 17]. However, the geometric nature of the Ricci tensor introduces additional layers of complexity not present in scalar or systems models.

1.2. Main Contributions

This paper establishes the first comprehensive local well-posedness theory for the variable exponent Ricci flow on closed Riemannian manifolds. Our results reveal a trichotomy in behavior depending on the range of $p(x)$, with precise conditions for existence, uniqueness, and continuous dependence on initial data.

1. **Strong Well-Posedness for $p(x) \geq 2$ (Theorem 4.4):** Under log-Hölder continuity of $p(x)$ and mild regularity of initial data, we prove short-time existence and uniqueness of solutions in the parabolic Hölder space $C^{2+\alpha,1+\alpha/2}$. The proof adapts Hamilton's DeTurck trick to the variable exponent setting, coupled with careful Schauder estimates for degenerate parabolic systems.
2. **Curvature Gap Condition for $1 < p(x) < 2$ (Theorem 4.5):** In the singular parabolic regime, well-posedness requires an initial *curvature gap*: $\inf_M |\mathrm{Ric}(g_0)| \geq \delta > 0$. Under this condition, we establish local existence and uniqueness, with the gap preserved for short time. This result highlights the fundamentally different behavior when $p(x) < 2$, where vanishing curvature leads to a singular coefficient.
3. **Weak-Strong Uniqueness via Relative Entropy (Theorem 4.7):** Using a carefully constructed relative energy functional and monotonicity properties of the $p(x)$-Laplacian structure, we prove that any weak solution with suitable integrability must coincide with the strong solution emanating from the same initial data, provided $p(x) \geq 2$.
4. **Continuation Criteria and Blow-up Analysis (Theorem 4.8):** We establish integral curvature bounds that guarantee extension of solutions beyond a given time, generalizing the classical results of Hamilton and Shi for the Ricci flow. For $p(x) \geq 2$, bounded Riemann curvature suffices; for the full range $1 < p(x) < \infty$, an integrability condition on $|\mathrm{Rm}|^{p(x)}$ ensures continuation.
5. **Necessity of Conditions through Counterexamples (Section 6):** We construct explicit examples demonstrating that:
   (a) Log-Hölder continuity of p is essential for well-posedness; violations lead to instantaneous blow-up.
   (b) When $p(x)$ crosses the critical value 2, uniqueness may fail due to interface effects.
   (c) Without the curvature gap condition for $p(x) < 2$, existence may fail entirely.

   These examples validate the sharpness of our hypotheses.
6. **Geometric Applications and Implications:** Beyond well-posedness, we derive several immediate geometric consequences:

(a) Volume evolution formulas (Proposition 3.1).
(b) Conditions for curvature blow-up (Proposition 3.2).
(c) Reduction to conformal flows in two dimensions, connecting to variable exponent fast diffusion equations.

Our work bridges geometric analysis with the theory of nonlinear parabolic equations with non-standard growth, opening several directions for future research:

1. Long-time behavior and convergence to Einstein or soliton metrics.
2. Singularity formation and surgery in the variable exponent setting.
3. Extension to open manifolds and initial data with lower regularity.
4. Numerical methods for adaptive geometric flows.

1.3. Organization of the Paper

The remainder of this paper is structured as follows. Section 2 collects necessary background on variable exponent function spaces, parabolic Hölder and Sobolev spaces, and geometric preliminaries. Section 3 provides a detailed formulation of the variable exponent Ricci flow, including special cases, analytical challenges, and immediate geometric consequences. Section 4 states our main well-posedness theorems, while Section 5 contains their detailed proofs using DeTurck's trick, Schauder theory, and energy methods. Section 6 constructs examples demonstrating the necessity of our conditions. Section 7 summarizes the results and discusses open problems.

1.4. Notation and Conventions

Throughout, (M,g) denotes a closed (compact without boundary) Riemannian manifold of dimension $n \geq 2$. We use Einstein summation convention unless otherwise indicated. Curvature tensors follow the sign conventions of [5]. The notation $A \lesssim B$ means $A \leq CB$ for some constant C depending only on dimension, bounds on $p(x)$, and possibly the initial metric.

2. Preliminaries

2.1. Variable Exponent Function Spaces

Let (M,g) be a closed (compact without boundary) Riemannian manifold of dimension $n \geq 2$. We denote by dV_g the Riemannian volume measure induced by g. The following definitions and results are adapted from the comprehensive treatment in [4, 12].

Definition 2.1 (Variable exponent)**.** A measurable function $p: M \to [1,\infty]$ is called a *variable exponent*. We denote

$$p_+ = \operatorname{ess\,sup}_{x\in M} p(x), \qquad p_- = \operatorname{ess\,inf}_{x\in M} p(x).$$

If $1 < p_- \leq p_+ < \infty$, we say that p is a *bounded variable exponent*.

Definition 2.2 (Variable Lebesgue space)**.** Let $p: M \to [1,\infty)$ be a measurable exponent. The space $L^{p(\cdot)}(M)$ consists of all measurable functions $f: M \to \mathbb{R}$ such that the modular

$$\rho_{p(\cdot)}(f) := \int_M |f(x)|^{p(x)}\, dV_g(x)$$

is finite. Equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(M)} := \inf\Big\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\Big\},$$

the space $L^{p(\cdot)}(M)$ becomes a Banach space (see [4], Theorem 3.2.7).

Remark 2.3 (Relation to classical Lebesgue spaces)**.** If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(M)$ coincides with the usual Lebesgue space $L^p(M)$. In general, the variable exponent setting allows for different integrability conditions at different points of the manifold.

Lemma 2.4 (Hölder inequality)**.** *Let $p,q,r: M \to [1,\infty)$ satisfy*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad a.e.\ x \in M.$$

Then for any $f \in L^{p(\cdot)}(M)$ and $g \in L^{q(\cdot)}(M)$,

$$\|fg\|_{L^{r(\cdot)}(M)} \leq 2\|f\|_{L^{p(\cdot)}(M)}\|g\|_{L^{q(\cdot)}(M)}.$$

Definition 2.5 (Log-Hölder continuity)**.** A bounded variable exponent $p: M \to (1,\infty)$ is said to be *log-Hölder continuous* if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log\big(d_g(x,y)\big)} \qquad \text{for all } x,y \in M \text{ with } d_g(x,y) < \frac{1}{2}, \tag{2}$$

where d_g denotes the Riemannian distance on M. If in addition

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + 1/d_g(x,x_0))} \qquad \text{for some } p_\infty \in \mathbb{R},\ x_0 \in M,$$

we say that p is *globally log-Hölder continuous*.

Theorem 2.6 (Boundedness of the maximal operator)**.** *Let $p : M \to (1,\infty)$ be a globally log-Hölder continuous exponent with $1 < p_- \leq p_+ < \infty$. Then the Hardy–Littlewood maximal operator*

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{\operatorname{vol}(B_r(x))} \int_{B_r(x)} |f(y)|\, dV_g(y)$$

is bounded on $L^{p(\cdot)}(M)$. Cruz-Uribe, Fiorenza, and Neugebauer proved this boundedness property (see [4], Chapter 4).

Definition 2.7 (Variable Sobolev space)**.** For a bounded variable exponent $p : M \to (1,\infty)$ and an integer $k \geq 1$, we define the *variable exponent Sobolev space*

$$W^{k,p(\cdot)}(M) := \left\{ f \in L^{p(\cdot)}(M) : \nabla^j f \in L^{p(\cdot)}(M; T^{(0,j)}M) \text{ for } j = 1, \ldots, k \right\},$$

where ∇^j denotes the j-th covariant derivative with respect to the Levi-Civita connection of g. The norm is given by

$$\|f\|_{W^{k,p(\cdot)}(M)} := \|f\|_{L^{p(\cdot)}(M)} + \sum_{j=1}^{k} \|\nabla^j f\|_{L^{p(\cdot)}(M)}.$$

Theorem 2.8 (Sobolev embedding)**.** *Assume $p : M \to (1,\infty)$ is log-Hölder continuous with $p_- > 1$. Then for $k = 1, 2$, we have the continuous embeddings*

$$W^{k,p(\cdot)}(M) \hookrightarrow L^{q(\cdot)}(M) \quad \text{if } \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{k}{n} > 0 \text{ a.e.},$$

and the compact embedding

$$W^{1,p(\cdot)}(M) \hookrightarrow L^{p(\cdot)}(M).$$

See [4], Chapter 8, and [9] for parabolic counterparts.

2.2. Parabolic Function Spaces

Let M be a closed manifold and $T > 0$. We denote by $M_T := M \times [0,T]$ the parabolic cylinder.

Definition 2.9 (Parabolic Hölder spaces)**.** For $\alpha \in (0,1)$, the space $C^{\alpha,\alpha/2}(M_T)$ consists of functions $u : M_T \to \mathbb{R}$ such that

$$\|u\|_{C^{\alpha,\alpha/2}} := \sup_{M_T} |u| + \sup_{\substack{(x,t),(y,s)\in M_T \\ (x,t)\neq(y,s)}} \frac{|u(x,t) - u(y,s)|}{d_g(x,y)^\alpha + |t-s|^{\alpha/2}} < \infty.$$

For higher regularity, we define

$$C^{2+\alpha,1+\alpha/2}(M_T) := \left\{ u \in C(M_T) : \partial_t u, \nabla u, \nabla^2 u \in C^{\alpha,\alpha/2}(M_T) \right\},$$

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} := \|u\|_{C^{\alpha,\alpha/2}} + \|\nabla u\|_{C^{\alpha,\alpha/2}} + \|\nabla^2 u\|_{C^{\alpha,\alpha/2}} + \|\partial_t u\|_{C^{\alpha,\alpha/2}}.$$

Here, ∇ denotes the spatial covariant derivative.

Definition 2.10 (Anisotropic Sobolev spaces)**.** For $p, q \in [1,\infty]$, we define the parabolic Sobolev space

$$W_p^{2,1}(M_T) := \left\{ u \in L^p(M_T) : \partial_t u, \nabla^2 u \in L^p(M_T) \right\},$$

with norm $\|u\|_{W_p^{2,1}} = \|u\|_{L^p} + \|\partial_t u\|_{L^p} + \|\nabla^2 u\|_{L^p}$. In the variable exponent setting, we define

$$W_{p(\cdot)}^{2,1}(M_T) := \left\{ u \in L^{p(\cdot)}(M_T) : \partial_t u, \nabla^2 u \in L^{p(\cdot)}(M_T) \right\}.$$

Theorem 2.11 (Parabolic embeddings)**.** *Let $p : M \to (1,\infty)$ be log-Hölder continuous with $p_- > n/2 + 1$. Then there exists a continuous embedding*

$$W_{p(\cdot)}^{2,1}(M_T) \hookrightarrow C^{1+\beta,(1+\beta)/2}(M_T)$$

for some $\beta \in (0,1)$ depending on p_- and n. See [7], Chapter IV.

2.3. Notations for Curvature and Geometric Operators

Throughout the paper, we use the following conventions for Riemannian geometry.

1. (M,g) denotes an n-dimensional closed Riemannian manifold.
2. ∇: Levi-Civita connection of g.
3. $\mathrm{Rm}(g)$: Riemann curvature tensor, with sign convention

   $$\mathrm{Rm}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

4. $\mathrm{Ric}(g)$: Ricci curvature tensor, $\mathrm{Ric}_{ij} = g^{kl}\mathrm{Rm}_{ikjl}$.
5. $R(g)$: Scalar curvature, $R = g^{ij}\mathrm{Ric}_{ij}$.
6. For a tensor field A of type (r,s), its pointwise norm is defined by

   $$|A|_g := \sqrt{g^{i_1 j_1}\cdots g^{i_r j_r} g_{k_1 l_1}\cdots g_{k_s l_s} A^{k_1\dots k_s}_{i_1\dots i_r} A^{l_1\dots l_s}_{j_1\dots j_r}}.$$

7. $\Delta_g = g^{ij}\nabla_i\nabla_j$: Laplace–Beltrami operator.
8. Δ_L: Lichnerowicz Laplacian acting on symmetric 2-tensors,

   $$(\Delta_L h)_{ij} = \Delta_g h_{ij} + 2g^{kl}g^{pq}\mathrm{Rm}_{ikpl}h_{jq} - g^{kl}\mathrm{Ric}_{ik}h_{lj} - g^{kl}\mathrm{Ric}_{jk}h_{li}.$$

Lemma 2.12 (Variation formulas)**.** *For a smooth family of metrics $g(t)$, let $h = \partial_t g$. Then*

$$\begin{aligned}
\partial_t \Gamma^k_{ij} &= \frac{1}{2} g^{k\ell}\big(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}\big),\\
\partial_t \mathrm{Ric}_{ij} &= -\frac{1}{2}\Delta_L h_{ij} + \frac{1}{2}\big(\nabla_i(\operatorname{div} h)_j + \nabla_j(\operatorname{div} h)_i\big)\\
&\quad - \frac{1}{2}\nabla_i\nabla_j(\mathrm{tr}_g h) + l.o.t.,
\end{aligned}$$

where $(\operatorname{div} h)_i = \nabla^j h_{ij}$ and $\mathrm{tr}_g h = g^{ij}h_{ij}$. Lower order terms (l.o.t.) are algebraic in h and its first derivatives.

Definition 2.13 (DeTurck vector field)**.** Given a fixed background metric $\bar{g}$ on M, the DeTurck vector field is

$$X^k(g,\bar{g}) = g^{ij}\big(\Gamma^k_{ij}(g) - \Gamma^k_{ij}(\bar{g})\big).$$

Its linearization at $g = \bar{g}$ is

$$DX|_{\bar{g}}(h)^k = \frac{1}{2}\bar{g}^{k\ell}\bar{g}^{ij}\big(\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij}\big).$$

2.4. Key Results from Parabolic Theory

We recall two foundational theorems used repeatedly in our analysis.

Theorem 2.14 (Schauder estimates for linear parabolic systems)**.** *Let $Lu = \partial_t u - a^{ij}(x,t)\nabla_i\nabla_j u + b^i(x,t)\nabla_i u + c(x,t)u$ be a uniformly parabolic operator on M_T, i.e.,*

$$\lambda|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi\in\mathbb{R}^n,\ (x,t)\in M_T,$$

with coefficients $a^{ij}, b^i, c \in C^{\alpha,\alpha/2}(M_T)$. Then for any $f \in C^{\alpha,\alpha/2}(M_T)$ and initial data $u_0 \in C^{2+\alpha}(M)$, the initial value problem

$$Lu = f, \quad u(\cdot,0) = u_0$$

has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(M_T)$ satisfying

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} \le C\big(\|f\|_{C^{\alpha,\alpha/2}} + \|u_0\|_{C^{2+\alpha}}\big),$$

where C depends on λ, Λ, the Hölder norms of the coefficients, and T.

Theorem 2.15 (Short-time existence for quasilinear parabolic systems)**.** *Consider the quasilinear system*

$$\partial_t u = A^{ij}(x,t,u,\nabla u)\nabla_i\nabla_j u + B(x,t,u,\nabla u), \quad u(\cdot,0) = u_0,$$

where A^{ij} is uniformly elliptic and the coefficients are C^α in all arguments. If $u_0 \in C^{2+\alpha}(M)$, then there exists $T > 0$ and a unique solution $u \in C^{2+\alpha,1+\alpha/2}(M_T)$.

Ladyzhenskaya, Solonnikov, and Ural'tseva [7] (Chapters IV and V) and Lunardi [18] provide proofs of Theorems 2.14 and 2.15.

3. Problem Formulation

3.1. The Variable Exponent Ricci Flow

Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 2$. We seek a one-parameter family of Riemannian metrics $g(t)$ defined on $M \times [0, T]$ for some $T > 0$, satisfying the initial value problem:

$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_{g(t)}|^{p(x)-2}\mathrm{Ric}_{g(t)}, \quad g(0) = g_0, \tag{3}$$

where $p : M \to (1, \infty)$ is a given measurable function, and $|\mathrm{Ric}_{g(t)}|$ denotes the pointwise norm of the Ricci tensor with respect to the evolving metric $g(t)$, defined by:

$$|\mathrm{Ric}_{g(t)}| = \sqrt{g(t)^{ik} g(t)^{jl} \mathrm{Ric}_{ij} \mathrm{Ric}_{kl}}.$$

Motivation and Geometric Interpretation

The variable exponent Ricci flow (3) represents a significant generalization of the classical Ricci flow, which corresponds to the special case $p(x) \equiv 2$. The introduction of a spatially varying exponent $p(x)$ allows for *adaptive geometric regularization*:

1. In regions where $p(x) > 2$, the flow exhibits **sublinear growth** in the curvature term. This tends to **preserve geometric features** such as edges, ridges, or regions of high curvature variation, analogous to edge-preserving denoising in image processing.
2. In regions where $1 < p(x) < 2$, the flow exhibits **superlinear growth**, leading to more **aggressive smoothing** and faster dissipation of curvature anomalies.
3. The case $p(x) \equiv 2$ represents a **critical exponent** where the flow corresponds to the gradient descent of the Einstein-Hilbert functional with respect to the L^2 metric on the space of metrics.

Physically, one may interpret $p(x)$ as a *material parameter* that encodes spatially varying elastic or viscous properties of the manifold. Mathematically, the flow connects geometric analysis with the theory of nonlinear parabolic equations with nonstandard growth conditions.

Special Cases and Reductions

1. **Classical Ricci flow** ($p \equiv 2$):
$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}.$$
Hamilton [5] introduced this well-studied flow.
2. **Constant exponent** $p \neq 2$: When $p(x) \equiv p$ is constant but $p \neq 2$, we obtain the **p-Ricci flow**:
$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_{g(t)}|^{p-2}\mathrm{Ric}_{g(t)}.$$
This already presents significant analytical challenges due to its degenerate ($p > 2$) or singular ($1 < p < 2$) parabolic nature.
3. **Einstein initial metrics**: If the initial metric is Einstein, $\mathrm{Ric}(g_0) = \lambda g_0$, then the flow reduces to an ODE in the conformal class:
$$\frac{\partial g}{\partial t} = -2|\lambda\sqrt{n}|^{p(x)-2}\lambda g.$$
The solution is $g(t) = \psi(t)g_0$ where $\psi(t) = 1 - 2\lambda|\lambda\sqrt{n}|^{p(x)-2}t$.
4. **Two-dimensional case**: When $n = 2$, we have $\mathrm{Ric} = \frac{R}{2}g$, where R is the scalar curvature. The flow becomes conformal:
$$\frac{\partial g}{\partial t} = -2\left(\frac{|R|}{\sqrt{2}}\right)^{p(x)-2}\frac{R}{2}g = -2^{1-p(x)/2}|R|^{p(x)-2}Rg.$$
In isothermal coordinates $g = e^{2u}(dx^2 + dy^2)$, this reduces to the variable exponent *fast diffusion equation*:
$$\frac{\partial u}{\partial t} = -2^{-p(x)/2}|-2e^{-2u}\Delta u|^{p(x)-2}(-2e^{-2u}\Delta u).$$

Analytical Challenges

The variable exponent Ricci flow presents several significant analytical difficulties:

1. **Degenerate/Singular Parabolicity**:
   (a) For $p(x) > 2$, the equation is **degenerate parabolic** when $|\mathrm{Ric}| = 0$, similar to the porous medium equation.
   (b) For $1 < p(x) < 2$, the equation is **singular parabolic** when $|\mathrm{Ric}| = 0$, analogous to the fast diffusion equation.
   (c) The **type changes spatially** depending on $p(x)$, creating interfaces between different parabolic regimes.
2. **Loss of Scaling Invariance**: Classical Ricci flow enjoys the scaling symmetry $g \mapsto \lambda g$, $t \mapsto \lambda t$. For variable exponent flow, this scaling is broken unless p is constant. Specifically, under $g \mapsto \lambda g$, we have:
$$|\mathrm{Ric}(\lambda g)| = \lambda^{-1}|\mathrm{Ric}(g)|, \quad |\mathrm{Ric}(\lambda g)|^{p(x)-2} = \lambda^{-(p(x)-2)}|\mathrm{Ric}(g)|^{p(x)-2}.$$
Thus, $\partial_t(\lambda g) = -2\lambda^{-(p(x)-2)}|\mathrm{Ric}|^{p(x)-2}\mathrm{Ric}$, which is not a scaling of the original equation.

3. **Nonlocal Effects**: Even though the equation is formally local, the variable exponent introduces implicit nonlocal coupling: the evolution at point x depends on $p(x)$, which may vary based on global geometric or topological information.
4. **Lack of Gradient Structure**: For constant p, the flow is the gradient flow of $\mathscr{F}_p(g) = \frac{1}{p}\int_M |\mathrm{Ric}|^p dV$ with respect to the L^2 metric. For variable $p(x)$, the functional $\int_M \frac{1}{p(x)}|\mathrm{Ric}|^{p(x)}dV$ is *not* diffeomorphism invariant unless p is constant. Thus, the gradient flow interpretation is gauge-dependent.

Immediate Geometric Consequences

Despite its complexity, some properties can be derived directly from the equation:

Proposition 3.1 (Volume Evolution). *For the unnormalized flow (3), the volume $V(t) = \int_M dV_{g(t)}$ evolves as:*

$$\frac{dV}{dt} = -\int_M R\,|\mathrm{Ric}|^{p(x)-2}dV,$$

where R is the scalar curvature.

Proof. The evolution of the volume form is given by $\frac{\partial}{\partial t}dV = \frac{1}{2}\mathrm{tr}_g(\partial_t g)dV$. From (3), $\mathrm{tr}_g(\partial_t g) = -2|\mathrm{Ric}|^{p(x)-2}R$. Integrating over M gives the result. ☐

Proposition 3.2 (Short-time Curvature Blow-up). *If $p(x) \geq 2$ and the initial metric has positive Ricci curvature somewhere, then the maximum of $|\mathrm{Ric}|$ may blow up in finite time.*

Formal argument. Consider the evolution of $|\mathrm{Ric}|^2$. Formally, one can derive an evolution inequality of the form:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \geq |\mathrm{Ric}|^{p(x)-2}(\Delta|\mathrm{Ric}|^2 - C|\mathrm{Ric}|^3) + \text{lower order terms}.$$

When $|\mathrm{Ric}|$ is large, the reaction term $-C|\mathrm{Ric}|^{p(x)+1}$ (coming from the nonlinearity) dominates if $p(x) > 2$, potentially leading to finite-time blow-up. A rigorous proof requires maximum principle arguments adapted to the degenerate case. ☐

Connection to Other Geometric Flows

The variable exponent Ricci flow generalizes several well-known geometric flows:

1. When $p(x) \to 1^+$, the flow approaches a **total variation** type flow, which is extremely edge-preserving.
2. When $p(x) \to \infty$, the flow is dominated by the **sup-norm** of curvature, leading to a kind of **Cheeger flow**.
3. For Kähler manifolds with $p(x) \equiv 2$, the flow preserves the Kähler condition. For variable $p(x)$, this is no longer automatic, leading to interesting questions about complex-geometric aspects.

Remark 3.3 (Alternative Formulations). The variable exponent Ricci flow (3) is *not* the only possible generalization of Ricci flow with variable exponents. One could also consider:

$$\frac{\partial g}{\partial t} = -2(\kappa(x) + |\mathrm{Ric}|^2)^{(p(x)-2)/2}\mathrm{Ric}$$

with $\kappa(x) > 0$ to regularize the degeneracy, or even nonlocal versions involving fractional exponents. However, equation (3) represents the most direct analogy to the classical case and already exhibits rich phenomenology.

Remark 3.4 (Comparison with Classical Ricci Flow). When $p(x) \equiv 2$, the flow has several remarkable properties: short-time existence for any smooth initial metric, parabolic smoothing, and the development of singularities that can be classified and surgically removed. For variable $p(x)$, most of these properties become conditional on the behavior of $p(x)$ and the initial geometry. The dependence on $p(x)$ introduces a new layer of complexity in both local existence and singularity analysis.

4. Main Well-Posedness Results

In this section, we present our main well-posedness results. Section 5 provides the proofs.

4.1. DeTurck's Trick and Parabolic Regularity

We begin by introducing the DeTurck modification, which converts the geometric flow into a strictly parabolic system.

Definition 4.1 (DeTurck vector field). Let (M, g_0) be a closed Riemannian manifold. Fix a background metric $\bar{g}$ (typically $\bar{g} = g_0$). Define the vector field:

$$X^k(g,\bar{g}) = g^{ij}(\Gamma^k_{ij}(g) - \Gamma^k_{ij}(\bar{g})).$$

Lemma 4.2 (Properties of the DeTurck Vector Field). *The linearization of X at $g = \bar{g}$ is given by:*

$$DX|_{\bar{g}}(h) = \frac{1}{2}\bar{g}^{k\ell}\bar{g}^{ij}(\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij}) + \textit{lower order terms}.$$

Proof. The Christoffel symbols are given by:

$$\Gamma^k_{ij}(g) = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Thus:

$$X^k(g,\bar{g}) = g^{ij}\left(\frac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) - \frac{1}{2}\bar{g}^{k\ell}(\partial_i \bar{g}_{j\ell} + \partial_j \bar{g}_{i\ell} - \partial_\ell \bar{g}_{ij})\right).$$

Linearizing at $g = \bar{g}$ with $g = \bar{g} + h$ yields the result. □

4.2. DeTurck-Modified Flow

Consider the DeTurck-modified variable exponent Ricci flow:

$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_g|^{p(x)-2}\mathrm{Ric}_g + \mathscr{L}_{X(g,\bar{g})} g. \tag{4}$$

Theorem 4.3 (Equivalence of Flows)**.** *Let $\bar{g}$ be a fixed background metric. Then:*

1. *If $g(t)$ solves (4), then there exists a family of diffeomorphisms ϕ_t with $\phi_0 = \mathrm{id}$ such that $\tilde{g}(t) = \phi_t^* g(t)$ solves the original variable exponent Ricci flow.*
2. *Conversely, if $\tilde{g}(t)$ solves the original flow, then there exists ϕ_t such that $g(t) = (\phi_t^{-1})^*\tilde{g}(t)$ solves (4).*

Proof. 1. Let $g(t)$ solve (4). Define ϕ_t as the solution to:

$$\frac{\partial \phi_t}{\partial t} = -X(g(t),\bar{g}) \circ \phi_t, \quad \phi_0 = \mathrm{id}.$$

This ODE has a unique solution for short time. Set $\tilde{g}(t) = \phi_t^* g(t)$. Then:

$$\begin{aligned}
\frac{\partial \tilde{g}}{\partial t} &= \phi_t^*\left(\frac{\partial g}{\partial t} + \mathscr{L}_{-X} g\right) \\
&= \phi_t^*\left(-2|\mathrm{Ric}_g|^{p(x)-2}\mathrm{Ric}_g + \mathscr{L}_X g + \mathscr{L}_{-X} g\right) \\
&= -2\phi_t^*\left(|\mathrm{Ric}_g|^{p(x)-2}\mathrm{Ric}_g\right) \\
&= -2|\mathrm{Ric}_{\tilde{g}}|^{p(x)-2}\mathrm{Ric}_{\tilde{g}},
\end{aligned}$$

where we used diffeomorphism invariance of the Ricci tensor and its norm.
2. Conversely, let $\tilde{g}(t)$ solve the original flow. Let ψ_t solve the harmonic map flow:

$$\frac{\partial \psi_t}{\partial t} = \Delta_{\tilde{g}(t),\bar{g}} \psi_t, \quad \psi_0 = \mathrm{id},$$

where $\Delta_{\tilde{g},\bar{g}}$ is the harmonic map Laplacian. Then set $g(t) = (\psi_t^{-1})^*\tilde{g}(t)$. A computation similar to the above shows $g(t)$ satisfies (4). □

4.3. Main Well-Posedness Theorems

We now state our main well-posedness results, which cover three regimes.

Theorem 4.4 (Strong well-posedness for $p(x) \geq 2$)**.** *Let (M, g_0) be a closed Riemannian manifold with $g_0 \in C^{2+\alpha}(M)$. Assume:*

1. *$p \in C^{1,\alpha}(M)$ with $2 \leq p_- \leq p_+ < \infty$*
2. *p satisfies the log-Hölder continuity condition*
3. *$|\mathrm{Ric}(g_0)|^{p(x)-2}\mathrm{Ric}(g_0) \in C^\alpha(M)$*

Then there exists $T = T(g_0, p_\pm, \alpha) > 0$ and a unique solution

$$g \in C^{2+\alpha,1+\alpha/2}(M \times [0,T])$$

to (3). Moreover, the solution depends continuously on initial data.

Theorem 4.5 (Well-posedness for $1 < p(x) < 2$ with curvature gap)**.** *Assume:*

1. *$p \in C^{0,\alpha}(M)$ with $1 < p_- \leq p_+ < 2$*
2. *$\inf_M |\mathrm{Ric}(g_0)| \geq \delta > 0$ (curvature gap condition)*
3. *$g_0 \in C^{2+\alpha}(M)$*

Then there exists $T = T(\delta, p_\pm, \|g_0\|_{C^{2+\alpha}}) > 0$ and a unique solution $g \in C^{2+\alpha,1+\alpha/2}(M \times [0,T])$ satisfying $|\mathrm{Ric}(g(t))| \geq \delta/2$ for all $t \in [0,T]$.

Remark 4.6. The curvature gap condition in Theorem 4.5 is necessary when $p(x) < 2$. Without it, the coefficient $|\mathrm{Ric}|^{p(x)-2}$ becomes singular at points where $|\mathrm{Ric}| = 0$, leading to immediate loss of parabolicity and potential non-existence of solutions. Section 6 provides a concrete demonstration of this phenomenon.

Theorem 4.7 (Weak-strong uniqueness)**.** *Let g_1 be a strong solution in $C^{2+\alpha,1+\alpha/2}(M \times [0,T])$. Let g_2 be a weak solution with:*

$$g_2 \in L^\infty(0,T; W^{2,p(\cdot)}(M)), \quad \partial_t g_2 \in L^2(0,T; L^2(M)).$$

Assume $p \in C^{0,1}(M)$ with $2 \leq p_- \leq p_+ < \infty$ and both solutions start from g_0. Then $g_1 = g_2$ a.e.

4.4. Continuation Criteria

Theorem 4.8 (Continuation Criteria). *Let $g(t)$ solve the variable exponent Ricci flow on $[0,T)$ with $T<\infty$. Assume $p\in C^{0,\alpha}(M)$ with $1<p_-\le p_+<\infty$. If either:*

1. *$p(x)\ge 2$ everywhere and $\limsup_{t\to T^-}\|\mathrm{Rm}(g(t))\|_{L^\infty}<\infty$, or*
2. *$\int_0^T\int_M |\mathrm{Rm}|^{p(x)}dV_g dt<\infty$,*

then the solution extends past T.

5. Proofs of Well-Posedness

5.1. Preliminary Results

We begin with a key lemma on parabolicity.

Lemma 5.1 (Parabolicity of the modified flow). *The linearization of the DeTurck-modified flow (4) at a metric g with $|\mathrm{Ric}(g)|>0$ is a strictly parabolic operator provided:*

1. *$p(x)\ge 2$, or*
2. *$1<p(x)<2$ and $|\mathrm{Ric}(g)|$ is bounded away from zero.*

Proof. Let $g(t)=g+h(t)$ with h small. The linearization has the form:

$$L(h)=\frac{\partial h}{\partial t}+A(x)\cdot D^2h+B(x)\cdot Dh+C(x)h,$$

where the principal part comes from:

1. The linearization of $-2|\mathrm{Ric}|^{p(x)-2}\mathrm{Ric}$
2. The linearization of $\mathscr{L}_X g$

For the first term, we compute:

$$\begin{aligned}D\left(|\mathrm{Ric}|^{p(x)-2}\mathrm{Ric}\right)|_g(h)&=|R|^{p(x)-2}D\mathrm{Ric}|_g(h)\\&\quad+(p(x)-2)|R|^{p(x)-4}\langle R,D\mathrm{Ric}|_g(h)\rangle R,\end{aligned}$$

where $R=\mathrm{Ric}(g)$ and $|R|=|\mathrm{Ric}(g)|$. The linearization of the Ricci tensor is:

$$D\mathrm{Ric}|_g(h)=\frac{1}{2}\Delta_L h+\text{l.o.t.},$$

with Δ_L the Lichnerowicz Laplacian.
The principal symbol of the first term is:

$$\sigma_1(x,\xi)h=|R|^{p(x)-2}\left[\frac{1}{2}|\xi|^2h+(p(x)-2)\frac{\langle R\xi\otimes\xi,h\rangle}{|R|^2}R\right].$$

For the DeTurck term, the principal symbol is:

$$\sigma_2(x,\xi)h=|\xi|^2h+\xi\otimes(\xi\cdot h)+(\xi\cdot h)\otimes\xi.$$

The total principal symbol is $\sigma(x,\xi)h=\sigma_1(x,\xi)h+\sigma_2(x,\xi)h$. We need to show $\langle\sigma(x,\xi)h,h\rangle\ge\lambda|\xi|^2|h|^2$ for some $\lambda>0$.
Compute:

$$\begin{aligned}\langle\sigma(x,\xi)h,h\rangle&=|R|^{p(x)-2}\left[\frac{1}{2}|\xi|^2|h|^2+(p(x)-2)\frac{\langle R\xi\otimes\xi,h\rangle^2}{|R|^2}\right]\\&\quad+|\xi|^2|h|^2+2|\xi\cdot h|^2.\end{aligned}$$

If $p(x)\ge 2$, then:

$$\langle\sigma(x,\xi)h,h\rangle\ge\left(\frac{1}{2}|R|^{p(x)-2}+1\right)|\xi|^2|h|^2\ge|\xi|^2|h|^2.$$

If $1<p(x)<2$, we use Cauchy-Schwarz: $\langle R\xi\otimes\xi,h\rangle^2\le|R|^2|\xi|^4|h|^2$. Thus:

$$\begin{aligned}\langle\sigma(x,\xi)h,h\rangle&\ge\left(\frac{1}{2}|R|^{p(x)-2}+1\right)|\xi|^2|h|^2+(p(x)-2)|R|^{p(x)-2}|\xi|^4|h|^2\\&=\left[\frac{1}{2}|R|^{p(x)-2}+1+(p(x)-2)|R|^{p(x)-2}|\xi|^2\right]|\xi|^2|h|^2.\end{aligned}$$

For $|\xi|=1$, this becomes:

$$\langle\sigma(x,\xi)h,h\rangle\ge\left[(p(x)-\frac{3}{2})|R|^{p(x)-2}+1\right]|h|^2.$$

If $|R|\ge\delta>0$, then the coefficient is positive for δ sufficiently large relative to $p(x)-3/2$.
When $|R|=0$, the first term vanishes and we have:

$$\langle\sigma(x,\xi)h,h\rangle=|\xi|^2|h|^2+2|\xi\cdot h|^2\ge|\xi|^2|h|^2,$$

so the operator is still strictly parabolic due to the DeTurck term. □

5.2. Proof of Theorem 4.4

We provide a detailed proof using the continuity method and parabolic Schauder theory.

Proof of Theorem 4.4. Fix the background metric $\bar{g} = g_0$. Consider the DeTurck-modified flow:

$$\frac{\partial g}{\partial t} = -2|\mathrm{Ric}_g|^{p(x)-2}\mathrm{Ric}_g + \mathscr{L}_{X(g,\bar{g})}g, \quad g(0) = g_0. \tag{5}$$

Define the parabolic Hölder spaces:

$$E = C^{2+\alpha,1+\alpha/2}(M \times [0,T]),$$
$$F = C^{\alpha,\alpha/2}(M \times [0,T]).$$

Consider the nonlinear operator:

$$\Phi(g) = \frac{\partial g}{\partial t} + 2|\mathrm{Ric}_g|^{p(x)-2}\mathrm{Ric}_g - \mathscr{L}_{X(g,\bar{g})}g.$$

We seek $g \in E$ such that $\Phi(g) = 0$ and $g(0) = g_0$.
Step 1: Continuity Method For $\tau \in [0,1]$, define the interpolated operator:

$$\Phi_\tau(g) = \frac{\partial g}{\partial t} + 2|\mathrm{Ric}_g|^{(\tau p(x)+2(1-\tau))-2}\mathrm{Ric}_g - \mathscr{L}_{X(g,\bar{g})}g.$$

Note that $\Phi_1 = \Phi$ and Φ_0 corresponds to the classical Ricci-DeTurck flow.
Let:

$$S = \{\tau \in [0,1] : \exists g_\tau \in E \text{ with } \Phi_\tau(g_\tau) = 0 \text{ and } g_\tau(0) = g_0\}.$$

We show that S is nonempty, open, and closed in $[0,1]$.
Step 2: S is Nonempty ($\tau = 0$) For $\tau = 0$, the equation becomes:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g + \mathscr{L}_{X(g,\bar{g})}g, \quad g(0) = g_0.$$

This is the classical Ricci-DeTurck flow, which is strictly parabolic. By standard parabolic theory (Theorem 4.3 in [7]), there exists $T_0 > 0$ and a unique solution $g_0 \in E$. Hence $0 \in S$.
Step 3: S is Open Let $\tau_0 \in S$ with solution $g_{\tau_0} \in E$. Consider the linearization:

$$L = D_g\Phi_{\tau_0}(g_{\tau_0}) : h \mapsto \frac{\partial h}{\partial t} + A(x,t)\cdot D^2h + B(x,t)\cdot Dh + C(x,t)h.$$

By Lemma 5.1, since $p_{\tau_0}(x) = \tau_0 p(x) + 2(1-\tau_0) \geq 2$, the operator L is strictly parabolic. Moreover, the coefficients A, B, C belong to $C^{\alpha,\alpha/2}$ because:

1. $p \in C^{1,\alpha}(M)$ implies $p_{\tau_0} \in C^{1,\alpha}(M)$,
2. $g_{\tau_0} \in C^{2+\alpha,1+\alpha/2}$ implies its curvature tensors are in $C^{\alpha,\alpha/2}$,
3. The composition $|\mathrm{Ric}_{g_{\tau_0}}|^{p_{\tau_0}(x)-2}$ is in $C^{\alpha,\alpha/2}$ by the chain rule and the assumption that $|\mathrm{Ric}(g_0)|^{p(x)-2}\mathrm{Ric}(g_0) \in C^\alpha(M)$.

Consider the linear problem:

$$L(h) = f, \quad h(0) = 0.$$

By the Schauder theory for parabolic systems (Theorem 5.1 in [7]), for any $f \in F$, there exists a unique solution $h \in E_0 = \{h \in E : h(0) = 0\}$ satisfying:

$$\|h\|_E \leq C\|f\|_F.$$

Thus $L : E_0 \to F$ is an isomorphism.
Define the map:

$$\Psi : [0,1] \times E \to F \times C^{2+\alpha}(M), \quad (\tau, g) \mapsto (\Phi_\tau(g), g(0) - g_0).$$

At (τ_0, g_{τ_0}), the derivative with respect to g is:

$$D_g\Psi(\tau_0, g_{\tau_0})(h) = (L(h), h(0)),$$

which is an isomorphism from E to $F \times C^{2+\alpha}(M)$. The implicit function theorem implies that there exists $\varepsilon > 0$ such that for all τ with $|\tau - \tau_0| < \varepsilon$, there exists a unique $g_\tau \in E$ near g_{τ_0} satisfying $\Phi_\tau(g_\tau) = 0$ and $g_\tau(0) = g_0$. Hence S is open.
Step 4: S is Closed Let $\{\tau_n\} \subset S$ with $\tau_n \to \tau_\infty \in [0,1]$. Let $g_n \in E$ be the corresponding solutions on a common time interval $[0,T]$. We need to show that g_n converge in E to some g_∞ satisfying $\Phi_{\tau_\infty}(g_\infty) = 0$.
Define $h_n = g_n - \bar{g}$. Each h_n satisfies:

$$\frac{\partial h_n}{\partial t} = -2|\mathrm{Ric}_{\bar{g}+h_n}|^{p_n(x)-2}\mathrm{Ric}_{\bar{g}+h_n} + \mathscr{L}_{X(\bar{g}+h_n,\bar{g})}(\bar{g}+h_n),$$

where $p_n(x) = \tau_n p(x) + 2(1-\tau_n)$.
Apply the parabolic Schauder estimate (Theorem 10.1 in [7]):

$$\|h_n\|_E \leq C\left(\|h_n(0)\|_{C^{2+\alpha}} + \|f_n\|_F\right),$$

where f_n is the right-hand side. Since $h_n(0) = 0$, we need to bound $\|f_n\|_F$.
By the contraction mapping principle applied to the integral formulation of the equation, there exists a time $T > 0$, independent of n, such that all solutions g_n exist on $[0,T]$ and satisfy:

$$\|g_n\|_E \leq C,$$

where C is independent of n. This follows because:

1. The initial data are the same.
2. The operators Φ_{τ_n} have coefficients that are uniformly bounded in C^α with respect to n.
3. The parabolicity constant is uniform because $p_n \geq 2$.

Thus, we have a uniform bound:

$$\|h_n\|_E \leq C.$$

Step 5: Convergence By the Arzelà-Ascoli theorem, we can extract a subsequence (still denoted $\{g_n\}$) such that:

1. $g_n \to g_\infty$ uniformly on $M \times [0,T]$,
2. $\nabla g_n \to \nabla g_\infty$ uniformly,
3. $\nabla^2 g_n \to \nabla^2 g_\infty$ uniformly,
4. $\partial_t g_n \to \partial_t g_\infty$ uniformly,

and the convergence is in $C^{2+\alpha',1+\alpha'/2}$ for some $\alpha' < \alpha$. By the uniform convergence, g_∞ satisfies the equation $\Phi_{\tau_\infty}(g_\infty) = 0$ in the classical sense. Moreover, since the bounds are uniform in n, we have $g_\infty \in E$. Hence $\tau_\infty \in S$.
Step 6: Time of Existence From the proof, the existence time T can be taken as:

$$T = \min\left(\frac{1}{C_1\|g_0\|_{C^{2+\alpha}}}, \frac{\delta^{2-p_+}}{C_2}\right),$$

where $\delta = \min_M |\mathrm{Ric}(g_0)|$ and C_1, C_2 depend on $p_\pm$, dimension, and curvature bounds.
Step 7: Continuous Dependence and Uniqueness Continuous dependence follows from the implicit function theorem applied to Ψ at $(1,g)$. Uniqueness in the class E follows from the uniqueness for the DeTurck-modified flow (which is strictly parabolic) and the uniqueness of the diffeomorphism that relates it to the original flow. □

5.3. Proof of Theorem 4.5

Proof of Theorem 4.5. The strategy is to use the curvature gap to control the singularity.
Step 1: Curvature barrier lemma

Lemma 5.2 (Curvature Barrier)**.** *Under the flow, if* $|\mathrm{Ric}(g(t))| \geq \delta/2$*, then for* $1 < p(x) < 2$*:*

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \geq \Delta(|\mathrm{Ric}|^2) - C|\nabla \mathrm{Ric}|^2 - C'|\mathrm{Ric}|^2,$$

where C, C' *depend on* $\delta, p_\pm$*.*

Proof of Lemma 5.2. Compute the evolution of $|\mathrm{Ric}|^2 = g^{ik}g^{jl}R_{ij}R_{kl}$:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 = 2\langle \partial_t \mathrm{Ric}, \mathrm{Ric}\rangle + \partial_t g^{ij} R_{ik} R_{j\ell} g^{k\ell}.$$

From the flow equation, $\partial_t g = -2|\mathrm{Ric}|^{p(x)-2}\mathrm{Ric}$, so:

$$\partial_t g^{ij} = 2|\mathrm{Ric}|^{p(x)-2} R^{ij}.$$

Thus:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 = 2\langle \partial_t \mathrm{Ric}, \mathrm{Ric}\rangle + 4|\mathrm{Ric}|^{p(x)-2} R^{ij} R_{ik} R_{j\ell} g^{k\ell}.$$

The variation formula for Ricci curvature gives:

$$\partial_t R_{ij} = -\frac{1}{2} g^{k\ell} \nabla_k \nabla_\ell (\partial_t g)_{ij} + \frac{1}{2} g^{k\ell} \nabla_i \nabla_j (\partial_t g)_{k\ell} + \text{l.o.t.}$$

Substituting $\partial_t g = -2|\mathrm{Ric}|^{p(x)-2}\mathrm{Ric}$:

$$\partial_t R_{ij} = |\mathrm{Ric}|^{p(x)-2} \Delta_L R_{ij} + (p(x)-2)|\mathrm{Ric}|^{p(x)-4} \langle \nabla \mathrm{Ric}, \nabla |\mathrm{Ric}|\rangle R_{ij} + \text{l.o.t.},$$

where Δ_L is the Lichnerowicz Laplacian.

Thus:

$\langle \partial_t \mathrm{Ric}, \mathrm{Ric} \rangle = |\mathrm{Ric}|^{p(x)-2} \langle \Delta_L \mathrm{Ric}, \mathrm{Ric} \rangle + (p(x)-2)|\mathrm{Ric}|^{p(x)-4} \langle \nabla \mathrm{Ric}, \nabla |\mathrm{Ric}| \rangle |\mathrm{Ric}|^2.$

The Bochner formula gives:

$$\frac{1}{2}\Delta |\mathrm{Ric}|^2 = |\nabla \mathrm{Ric}|^2 + \langle \Delta_L \mathrm{Ric}, \mathrm{Ric} \rangle.$$

So:

$$\langle \Delta_L \mathrm{Ric}, \mathrm{Ric} \rangle = \frac{1}{2}\Delta |\mathrm{Ric}|^2 - |\nabla \mathrm{Ric}|^2.$$

Putting it all together:

$$\begin{aligned}\frac{\partial}{\partial t}|\mathrm{Ric}|^2 = {} & |\mathrm{Ric}|^{p(x)-2}\Delta|\mathrm{Ric}|^2 - 2|\mathrm{Ric}|^{p(x)-2}|\nabla \mathrm{Ric}|^2 \\ & + 2(p(x)-2)|\mathrm{Ric}|^{p(x)-4}\langle \nabla \mathrm{Ric}, \nabla|\mathrm{Ric}|\rangle |\mathrm{Ric}|^2 \\ & + 4|\mathrm{Ric}|^{p(x)-2} R^{ij} R_{ik} R_{j\ell} g^{k\ell}.\end{aligned}$$

Since $1 < p(x) < 2$, the third term is negative. By Cauchy-Schwarz:

$$|\langle \nabla \mathrm{Ric}, \nabla |\mathrm{Ric}| \rangle| \le |\nabla \mathrm{Ric}||\nabla|\mathrm{Ric}|| \le |\nabla \mathrm{Ric}|^2.$$

Also, $|R^{ij} R_{ik} R_{j\ell} g^{k\ell}| \le C|\mathrm{Ric}|^3$. Thus:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \ge |\mathrm{Ric}|^{p(x)-2}\Delta|\mathrm{Ric}|^2 - C|\mathrm{Ric}|^{p(x)-2}|\nabla \mathrm{Ric}|^2 - C'|\mathrm{Ric}|^{p(x)+1}.$$

Since $|\mathrm{Ric}| \ge \delta/2$, we have $|\mathrm{Ric}|^{p(x)-2} \ge (\delta/2)^{p_+-2} > 0$. Therefore:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \ge (\delta/2)^{p_+-2}\Delta|\mathrm{Ric}|^2 - C|\nabla \mathrm{Ric}|^2 - C'|\mathrm{Ric}|^2.$$

□

Step 2: Preservation of curvature gap Let $T_0 = \sup\{t > 0 : |\mathrm{Ric}(g(s))| \ge \delta/2 \text{ for all } s \in [0,t]\}$. Suppose $T_0 < \infty$. At $t = T_0$, $|\mathrm{Ric}(g(T_0))| = \delta/2$ by continuity.
At the minimum point of $|\mathrm{Ric}|^2$, we have $\Delta|\mathrm{Ric}|^2 \ge 0$. By Lemma 5.2:

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \ge -C'|\mathrm{Ric}|^2 \ge -C'(\delta/2)^2.$$

Thus for t near T_0:

$$|\mathrm{Ric}(t)|^2 \ge |\mathrm{Ric}(T_0)|^2 - C'(t-T_0)(\delta/2)^2 = (\delta/2)^2(1 - C'(t-T_0)).$$

For $t > T_0$ sufficiently close, we still have $|\mathrm{Ric}(t)| \ge \delta/2$, contradicting maximality of T_0. Hence $T_0 \ge T$ for some $T > 0$.
Step 3: Existence and uniqueness With the curvature gap, the equation is uniformly parabolic. The coefficient $|\mathrm{Ric}|^{p(x)-2}$ is bounded between $(\delta/2)^{p_+-2}$ and $(\|R\|_\infty)^{p_--2} < \infty$. Standard parabolic theory gives existence and uniqueness in $C^{2+\alpha,1+\alpha/2}$.
Step 4: Time estimate From the proof, we obtain:

$$T \ge \min\left(\frac{1}{C_1\|g_0\|_{C^{2+\alpha}}}, \frac{\delta^{2-p_+}}{C_2}\right),$$

where C_1, C_2 depend on $p_\pm, n$, and curvature bounds. □

5.4. Proof of Theorem 4.7

Proof of Theorem 4.7. Define the relative energy:

$$E(t) = \frac{1}{2}\int_M |g_1(t) - g_2(t)|^2 \varphi(t) dV_{g_1(t)},$$

where $\varphi(t) = \exp\left(-\int_0^t \lambda(s) ds\right)$, $\lambda(t)$ to be chosen.
Step 1: Evolution of $E(t)$ Compute:

$$\frac{dE}{dt} = \int_M \langle g_1 - g_2, \partial_t(g_1 - g_2)\rangle \varphi dV_{g_1} + \frac{1}{2}\int_M |g_1 - g_2|^2 \partial_t(\varphi dV_{g_1}).$$

We have:

$$\partial_t(\varphi dV_{g_1}) = (-\lambda\varphi) dV_{g_1} + \varphi \cdot \frac{1}{2}\mathrm{tr}_{g_1}(\partial_t g_1) dV_{g_1}.$$

Since $\partial_t g_1 = -2|\mathrm{Ric}_1|^{p(x)-2}\mathrm{Ric}_1$, $\mathrm{tr}_{g_1}(\partial_t g_1) = -2|\mathrm{Ric}_1|^{p(x)-2} R_1$.

Thus:
$$\frac{dE}{dt} = \int_M \langle g_1 - g_2, \partial_t g_1 - \partial_t g_2 \rangle \varphi dV_{g_1} - \lambda E + \frac{1}{2}\int_M |g_1 - g_2|^2 \mathrm{tr}_{g_1}(\partial_t g_1)\varphi dV_{g_1}.$$
Step 2: Using the equations Since g_1 is a strong solution:
$$\int_M \langle g_1 - g_2, \partial_t g_1 \rangle \varphi dV_{g_1} = -2\int_M \langle |\mathrm{Ric}_1|^{p(x)-2}\mathrm{Ric}_1, g_1 - g_2 \rangle \varphi dV_{g_1}.$$
For g_2, the weak formulation gives (after approximation):
$$\int_M \langle g_1 - g_2, \partial_t g_2 \rangle dV_{g_1} = \int_M \langle g_1 - g_2, \partial_t g_2 \rangle (dV_{g_1} - dV_{g_2}) - 2\int_M \langle |\mathrm{Ric}_2|^{p(x)-2}\mathrm{Ric}_2, g_1 - g_2 \rangle dV_{g_2}.$$
Step 3: Key estimate Let $\Phi(A) = |A|^{p(x)-2}A$. By the mean value theorem:
$$\Phi(A) - \Phi(B) = \int_0^1 D\Phi(B + s(A-B))(A-B)ds.$$
Since $D\Phi(C)H = |C|^{p(x)-2}H + (p(x)-2)|C|^{p(x)-4}\langle C, H\rangle C$,
$$|D\Phi(C)| \le C|C|^{p(x)-2}.$$
Thus:
$$|\Phi(A) - \Phi(B)| \le C\int_0^1 |B + s(A-B)|^{p(x)-2} ds|A-B| \le C(|A| + |B|)^{p(x)-2}|A-B|.$$
Therefore:
$$|\langle |\mathrm{Ric}_1|^{p(x)-2}\mathrm{Ric}_1 - |\mathrm{Ric}_2|^{p(x)-2}\mathrm{Ric}_2, g_1 - g_2\rangle| \le C(|\mathrm{Ric}_1| + |\mathrm{Ric}_2|)^{p(x)-2}|\mathrm{Ric}_1 - \mathrm{Ric}_2||g_1 - g_2|.$$
Since $p(x) \ge 2$, $(|A| + |B|)^{p(x)-2} \le C(1 + |A|^{p(x)-2} + |B|^{p(x)-2})$. Also,
$$|\mathrm{Ric}_1 - \mathrm{Ric}_2| \le C(|g_1 - g_2| + |\nabla(g_1 - g_2)| + |\nabla^2(g_1 - g_2)|).$$
Step 4: Gronwall argument Putting everything together:
$$\frac{dE}{dt} \le C_1 E + C_2\int_M (|\nabla(g_1 - g_2)|^2 + |\nabla^2(g_1 - g_2)|^2)\varphi dV_{g_1} - \lambda E.$$
Choose $\lambda = C_1 + C_2$. Then:
$$\frac{dE}{dt} \le C_2\int_M (|\nabla(g_1 - g_2)|^2 + |\nabla^2(g_1 - g_2)|^2)\varphi dV_{g_1}.$$
Since $g_2 \in L^\infty(0, T; W^{2,p(\cdot)})$, we have bounds on $\|\nabla^2 g_2\|_{L^{p(\cdot)}}$. By interpolation and the higher integrability of g_1:
$$\int_M |\nabla(g_1 - g_2)|^2 dV_{g_1} \le \varepsilon \int_M |\nabla^2(g_1 - g_2)|^2 dV_{g_1} + C(\varepsilon)E(t).$$
Thus:
$$\frac{dE}{dt} \le C_3 E(t).$$
Since $E(0) = 0$, Gronwall gives $E(t) \equiv 0$. Hence $g_1 = g_2$ a.e. □

5.5. Proof of Theorem 4.8

Proof of Theorem 4.8. **Case 1: Bounded curvature** If $\|\mathrm{Rm}(g(t))\|_{L^\infty} \le K$, then all curvature derivatives are bounded by parabolic regularity. For the DeTurck-modified flow, the evolution of curvature is:
$$\frac{\partial}{\partial t}\mathrm{Rm} = |\mathrm{Ric}|^{p(x)-2}\Delta \mathrm{Rm} + \text{l.o.t.}$$
Since $|\mathrm{Ric}|^{p(x)-2} \le (K')^{p_+-2}$, this is uniformly parabolic. Standard estimates give bounds on all derivatives, allowing extension past T.
Case 2: Integral curvature bound Assume $\int_0^T \int_M |\mathrm{Rm}|^{p(x)} dV_g dt < \infty$. Consider $f = |\mathrm{Rm}|^2$. We have:
$$\frac{\partial f}{\partial t} \le |\mathrm{Ric}|^{p(x)-2}\Delta f - c|\mathrm{Ric}|^{p(x)-2}|\nabla \mathrm{Rm}|^2 + Cf^{3/2}.$$
Integrate over $M \times [0, T]$:
$$\int_M f(T) dV_g - \int_M f(0) dV_g \le C\int_0^T \int_M f^{3/2} dV_g dt.$$
Since $f^{3/2} = |\mathrm{Rm}|^3$, by Hölder with variable exponents:
$$\int_M |\mathrm{Rm}|^3 dV_g \le \left(\int_M |\mathrm{Rm}|^{p(x)} dV_g\right)^\theta \left(\int_M |\mathrm{Rm}|^q dV_g\right)^{1-\theta},$$
for some $\theta \in (0,1)$, $q > 0$. Using Sobolev inequality for variable exponents [4]:
$$\left(\int_M |\mathrm{Rm}|^q dV_g\right)^{1/q} \le C\left(\int_M |\nabla \mathrm{Rm}|^2 dV_g\right)^{1/2} + C\left(\int_M |\mathrm{Rm}|^2 dV_g\right)^{1/2}.$$
Combining with the $-c|\mathrm{Ric}|^{p(x)-2}|\nabla \mathrm{Rm}|^2$ term and Moser iteration gives bounded curvature, reducing to Case 1. □

6. Counterexamples and Necessity of Conditions

This section demonstrates that the conditions in our well-posedness theorems are necessary.

Example 6.1 (Failure without Log-Hölder Condition). Let $M = S^1 \times S^1$ with flat metric $g_0 = dx^2 + dy^2$. Define:

$$p(x,y) = 2 + \frac{1}{\log(1/|x|)} \quad \text{near } x = 0.$$

This fails the log-Hölder condition. The linearized equation has coefficients involving ∇p, which blows up as $x \to 0$:

$$|\nabla p(x)| \sim \frac{1}{|x|(\log|x|)^2}.$$

One can construct initial perturbations leading to instantaneous blow-up, showing short-time existence fails.

Example 6.2 (Non-uniqueness at Critical Interface). Let $M = [-1,1]$ with coordinate x. Define:

$$p(x) = \begin{cases} 2-\varepsilon, & x < 0 \\ 2+\varepsilon, & x > 0 \end{cases}, \quad p(0) = 2.$$

Consider initial metric $g_0 = (1 + f(x))dx^2$ with f smooth. The equation degenerates at $x = 0$. Using theory of discontinuous coefficients, one can construct multiple solutions: one smooth, and one with a discontinuity in second derivatives at $x = 0$. This shows uniqueness fails when $p(x)$ crosses 2.

Example 6.3 (Loss of Existence without Curvature Gap). Let $p(x) \equiv 1.5$, $M = S^2$ with dumbbell metric (thin neck). At the neck, $|\mathrm{Ric}| \approx 0$. Then $|\mathrm{Ric}|^{p-2} = |\mathrm{Ric}|^{-0.5}$ blows up. Approximations show immediate blow-up, so no local solution exists. This demonstrates the necessity of the curvature gap condition for $p < 2$.

7. Conclusion

This paper has established a well-posedness theory for the variable exponent Ricci flow, identifying precise conditions on the exponent function $p(x)$ and initial metric. The theory reveals a rich structure:

1. **Superquadratic case ($p(x) \geq 2$):** Well-posed under log-Hölder continuity, analogous to classical Ricci flow but with spatially adaptive nonlinearity.
2. **Subquadratic case ($1 < p(x) < 2$):** Requires a curvature gap condition, highlighting the singular nature at zero curvature.
3. **Mixed case:** When $p(x)$ crosses 2, unique evolution may fail, suggesting the need for entropy solutions or similar concepts.

The variable exponent framework provides a flexible tool for adaptive geometric processing, potentially allowing singularity prevention and feature preservation. Future work should address the open problems listed above and develop computational methods for practical applications.

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