

Stability Analysis of a two-strained Dengue Infection Dynamic Model Incorporating vector Controls and Vaccination

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Abstract

A modified mathematical model for the transmission dynamics of a two-strain dengue infection in a population is designed and vividly analysed. The model incorporates vector control measures such as good environmental sanitation and use of mosquito treated nets to reduce mosquito density and bite respectively. Also incorporated in the work is the addition of vaccines τ to treatment γ at secondary infection stage to ensure recovery with long term immunity. The reproduction number was computed using the next generation matrix method. Bifurcation analysis reveals a backward bifurcation at $R_0 = 1$, indicating that the disease persistence is possible even when $R_0 < 1$. The results of the analysis show that the Disease-Free Equilibrium (DFE) is both locally and globally asymptotically stable via Routh-Hurwitz and Lyapunov criteria. In the absence of the vector control measures and vaccination, the model exhibits a unique endemic equilibrium whenever the reproduction threshold exceeds unity ($R_0 > 1$); while in the presence of the control measures, the reproduction number reduces below unity ($R_e < 1$).

Keywords: Dengue; perturbation; reproduction number; optimal control.

1. Introduction

Dengue Fever is a vector-borne infectious disease spread by female *Aedes aegypti* mosquito [1]. The disease has the characteristics of re-infecting an individual because of Temporal Cross-Immunity (TCI) factor in the primary infection. It has four serotypes that co-circulate within the hosts viz: Deng-1 to Deng-4 [3]. These serotypes are associated with severe disease occasioned by the Antibody-Dependent Enhancement (ADE).

The primary dengue infections are often mild [5]. However, the secondary dengue infection is characterized by a severe disease with hemorrhagic symptoms [2]. In this way persons that have experienced primary infection with one Dengue type has life-long immunity against the particular type suffered. But immediately the temporary cross-immunity (TCI) time period expires, the person becomes prone to a distinct dengue virus type [6]. Hence, it is only after a patient recovers from the secondary infection that such patient will have lasting immunity [9, 7].

Prevention involves mosquito control measures and avoiding mosquito bites by using mosquito treated nets and good environmental sanitation [8]. However, because of the complex nature of dengue disease, vaccines have been developed to provide long term protection against every dengue virus serotypes [6].

2. Materials and Methods

We formulate a modified SIR-SIR model by extending an existing one to incorporate optimal vector control and preventive vaccination. We let N and M be the Total human and Mosquito populations respectively, S be the Susceptible humans with no history of Dengue infection, I_p be the Primary Infected individuals, R_p be Recovered individuals from primary infection with temporary immunity, S_p be Susceptible persons with history of Dengue infection, I_s be the Secondary infected individuals due to disease enhancement factor, R be the Recovered individuals from secondary infection with long-term immunity, S_v be Susceptible Mosquitoes, I_v be the Infected Mosquitoes, Λ be the

Recruitments into the human population, Ω be the Recruitment into the mosquito population, δ be the Use of mosquito treated nets, θ be the good environmental sanitation, τ be Effective Vaccine, β be Transmission rate or interaction rate from vector to humans, ϑ be Transmission rate or interaction rate from humans to vector, γ be Recovery rate of humans, α be the Temporary immunity rate crossed to secondary infections, Φ be the Rate of secondary infection to overall force of infection, μ be the Natural mortality rate of humans, ν be the Natural mortality rate of mosquitoes.

2.1. Assumption of the Model

1. Recruitment into the human population are both children and Adults,
2. Natural mortality and birth rates are not equal in both human and vector populations,
3. Individuals die due to the disease at the rate η and σ and naturally at the rate μ ,
4. Mosquitoes die due to the disease at the rate ω and naturally at the rate ν ,
5. Recovery is by clinical treatment,
6. Individuals recover from primary infection with temporal immunity,
7. Severe disease occurs at secondary infection due to ADE process,
8. Vector controls are by the use of mosquito treated nets and good environmental sanitation while prevention is by vaccination.

However, the total population as function of t for both humans and mosquitoes are $N(t) = S(t) + I_p(t) + R_p(t) + S_p(t) + I_s(t) + R(t)$ and $M(t) = S_v(t) + I_v(t)$ respectively. The Extended Model Equations are as

$$\left. \begin{aligned} \frac{dS}{dt} &= \Lambda - G_1(1 - \delta - \theta)S - \mu S, \\ \frac{dI_p}{dt} &= G_1(1 - \delta - \theta)S - (\mu + \gamma + \eta)I_p, \\ \frac{dR_p}{dt} &= \gamma I_p - (\mu + \alpha)R_p, \\ \frac{dS_p}{dt} &= \alpha R_p - G_1 S_p - \mu S_p, \\ \frac{dI_s}{dt} &= G_1 S_p - (\gamma + \tau)I_s - \mu I_s - \sigma I_s, \\ \frac{dR}{dt} &= (\gamma + \tau)I_s - \mu R, \\ \frac{dS_v}{dt} &= \Omega - G_2(1 - \delta - \theta)S_v - \nu S_v, \\ \frac{dI_v}{dt} &= G_2(1 - \delta - \theta)S_v - (\nu + \omega)I_v. \end{aligned} \right\} \quad (1)$$

Where

$$G_1 = \frac{\beta}{M} I_v,$$

and

$$G_2 = \frac{\vartheta}{N} I_p + \frac{\vartheta}{N} \Phi I_s.$$

The extended schematic diagram is given as

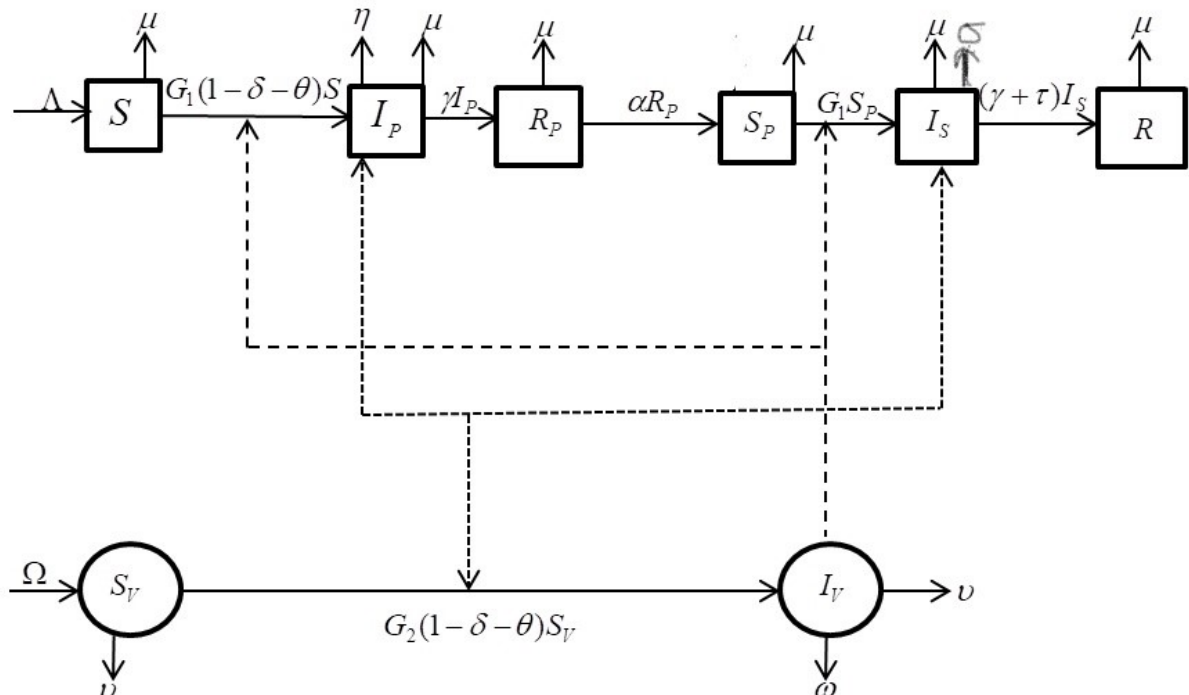


Figure 1: The schematic diagram of Dengue, [20].

2.2. Existence and Uniqueness of Solution

Here we show on analysis that the solutions $S(t), I_p(t), R_p(t), S_p(t), I_s(t), R(t), S_v(t)$ and $I_v(t)$ of the model exists and are unique following the theorem and proofs in [16] and [17]. Let $\Omega = ((t, y_i), a < t < b \text{ and } -\infty < y_i < +\infty)$ and that $f(t, y_i)$ is continuous on the domain $\Omega \in \mathbb{R}^8$, if the function f satisfies the Lipschitz condition on $\Omega \in \mathbb{R}^8$ on the variable $y_i \subset \Omega \in \mathbb{R}^8$, with $y_1 = S, y_2 = I_p, y_3 = R_p, y_4 = S_p, y_5 = I_s, y_6 = R, y_7 = S_v$ and $y_8 = I_v$, then the system of the model equations have unique solutions $y_i(t)$ for $a < t < b$.

Proof. Let $y_i^1 = f_i(t, y_i(t))$, where

$$y_1^1 = f_1(t, y_1(t)) = \frac{dS}{dt} = \Lambda - G_1 S(1 - \delta - \theta) - \mu S$$

$$y_2^1 = f_2(t, y_2(t)) = \frac{dI_p}{dt} = G_1 S(1 - \delta - \theta) - (\mu + \gamma + \eta) I_p$$

$$y_3^1 = f_3(t, y_3(t)) = \frac{dR_p}{dt} = \gamma I_p - (\mu + \alpha) R_p$$

$$y_4^1 = f_4(t, y_4(t)) = \frac{dS_p}{dt} = \alpha R_p - G_1 S_p - \mu S_p$$

$$y_5^1 = f_5(t, y_5(t)) = \frac{dI_s}{dt} = G_1 S_p - (\gamma + \tau) I_s - (\mu + \sigma) I_s$$

$$y_6^1 = f_6(t, y_6(t)) = \frac{dR}{dt} = (\gamma + \tau) I_s - \mu R$$

$$y_7^1 = f_7(t, y_7(t)) = \frac{dS_v}{dt} = \Omega - G_2(1 - \delta - \theta) S_v - \nu S_v$$

$$y_8^1 = f_8(t, y_8(t)) = \frac{dI_v}{dt} = G_2(1 - \delta - \theta) S_v - (\nu + \omega) I_v$$

That the function $f_i(t, y_i)$ is continuous means that the partial derivatives w.r.t each of the state variables exists.

Hence,

$$\frac{\partial f_1(t, S)}{\partial S} = -G_1(1 - \delta - \theta) - \mu$$

$$\frac{\partial f_2(t, I_p)}{\partial I_p} = -(\mu + \gamma + \eta)$$

$$\frac{\partial f_3(t, R_p)}{\partial R_p} = -(\mu + \alpha)$$

$$\frac{\partial f_4(t, S_p)}{\partial S_p} = -(G_1 + \mu)$$

$$\frac{\partial f_5(t, I_s)}{\partial I_s} = -(\gamma + \tau + \mu + \sigma)$$

$$\frac{\partial f_6(t, R)}{\partial R} = -\mu$$

$$\frac{\partial f_7(t, S_v)}{\partial S_v} = -G_2(1 - \delta - \theta) - \nu$$

$$\frac{\partial f_8(t, I_v)}{\partial I_v} = -(\nu + \omega)$$

Then, applying the mean value theorem (Stewart, 2015), we have from the first equation

$$\frac{f_1(t, S_2) - f_1(t, S_1)}{S_2 - S_1} = \frac{\partial(t, S^*)}{\partial S}$$

Where $S^* \in (S_2, S_1)$ is any point between S_1 and S_2 .

$$\frac{f_1(t, S_2) - f_1(t, S_1)}{S_2 - S_1} = \frac{\partial(t, S)}{\partial S} = -G_1(1 - \delta - \theta) - \mu$$

$$f_1(t, S_2) - f_1(t, S_1) = |S_2 - S_1| |(-G_1(1 - \delta - \theta) - \mu)| = |-1| |S_2 - S_1| |\mu + G_1(1 - \delta - \theta)|$$

$$f_1(t, S_2) - f_1(t, S_1) = |S_2 - S_1| |\mu + G_1(1 - \delta - \theta)| \leq \mu + |S_2 - S_1|$$

From the second equation,

$$\frac{f_2(t, I_{p2}) - f_2(t, I_{p1})}{I_{p2} - I_{p1}} = \frac{\partial(t, I_p)}{\partial I_p} = -(\mu + \gamma + \eta)$$

$$f_2(t, I_{p2}) - f_2(t, I_{p1}) = |-1| |\mu + \gamma + \eta| |I_{p2} - I_{p1}| \leq |\mu + \gamma + \eta| |I_{p2} - I_{p1}|$$

From the third equation

$$f_3(t, R_{p2}) - f_3(t, R_{p1}) = |-1| |\mu + \alpha| |R_{p2} - R_{p1}| \leq |\mu + \alpha| |R_{p2} - R_{p1}|$$

From the fourth equation

$$\frac{f_4(t, S_{p2}) - f_4(t, S_{p1})}{S_{p2} - S_{p1}} = -(\mu + G_1)$$

$$f_4(t, S_{p2}) - f_4(t, S_{p1}) = |-1||\mu + G_1||S_{p2} - S_{p1}| \leq |\mu + G_1||S_{p2} - S_{p1}|$$

From the fifth equation

$$\begin{aligned} \frac{f_5(t, I_{s2}) - f_5(t, I_{s1})}{I_{s2} - I_{s1}} &= \frac{\partial f_5(t, I_s)}{\partial I_s} = -(\gamma + \tau + \mu + \sigma) \\ f_5(t, I_{s2}) - f_5(t, I_{s1}) &= |-1||\gamma + \tau + \mu + \sigma||I_{s2} - I_{s1}| \leq |\gamma + \tau + \mu + \sigma||I_{s2} - I_{s1}| \end{aligned}$$

From the sixth equation

$$\begin{aligned} \frac{f_6(t, R_2) - f_6(t, R_1)}{R_2 - R_1} &= \frac{\partial f_6(t, R)}{\partial R} = -\mu \\ f_6(t, R_2) - f_6(t, R_1) &= |-1||\mu||R_2 - R_1| \leq |\mu||R_2 - R_1| \end{aligned}$$

From the seventh equation

$$\begin{aligned} f_7(t, S_{v2}) - f_7(t, S_{v1}) &= |-1||v + G_2(1 - \delta - \theta)||S_{v2} - S_{v1}| \\ &\leq |v + G_2(1 - \delta - \theta)||S_{v2} - S_{v1}| \end{aligned}$$

And equation eight becomes

$$\begin{aligned} \frac{f_8(t, I_{v2}) - f_8(t, I_{v1})}{I_{v2} - I_{v1}} &= \frac{\partial f_8(t, I_v)}{\partial I_v} = -(v + \omega) \\ &= |-1||v + \omega||I_{v2} - I_{v1}| \leq |v + \omega||I_{v2} - I_{v1}| \end{aligned}$$

□

Therefore, the model has unique solutions since the partial derivatives of the states variables exists.

2.3. Positivity of Solution

The positivity in this model indicates a fundamental constraint which ensures that all state variables remain greater than or equal to zero for all time $t \geq 0$. It is a critical requirement for a model to be physically meaningful. Let the initial data $(S(0), I_p(0), R_p(0), S_p(0), I_s(0), R(0), S_v(0), I_v(0)) \geq 0 \in \mathbb{R}_+^8$. Then, the solution set $(S(t), I_p(t), R_p(t), S_p(t), I_s(t), R(t), S_v(t), I_v(t))$ of the model are positive for all $t \geq 0$.

Proof. From the model equation (1)

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - G_1(1 - \delta - \theta)S - \mu S, \\ &\geq -(G_1(1 - \delta - \theta) + \mu)S, \\ \int \frac{1}{S} dS &\geq -(G_1(1 - \delta - \theta) + \mu) \int dt, \\ \ln S &\geq -(G_1(1 - \delta - \theta) + \mu)t + c, \\ S &\geq A_1 e^{-(G_1(1 - \delta - \theta) + \mu)t}. \end{aligned}$$

Hence the solution with initial condition becomes $S(t) \geq S(0)e^{-(G_1(1 - \delta - \theta) + \mu)t} > 0$, which implies that $S(t) > 0$. Applying similar method to other state variables, we have that $I_p(t) \geq 0, R_p(t) \geq 0, S_p(t) \geq 0, I_s(t) \geq 0, R(t) \geq 0$ and $S_v(t) > 0$. □

2.4. Boundedness of Solution

In this section, we show that the model system of equations have differentiable properties and bounded above by $\frac{\Lambda}{\mu}$. Let $D \subset \mathbb{R}_+^6$ be the set, then the solution set $X(t) = (S + I_p + R_p + S_p + I_s + R) \in D \subset \mathbb{R}_+^6$ of model equation (1) are feasible for all $t \geq 0$.

Proof. We require to prove that the solution set $X(t)$ are attainable in the Domain $D \subset \mathbb{R}_+^6$. Let $X(t) = (S + I_p + R_p + S_p + I_s + R) \in D \subset \mathbb{R}_+^6$ be the solution of the system with positive initial condition.

The total population of human host is given by: $N = S + I_p + R_p + S_p + I_s + R$.

$$\begin{aligned} \frac{dN}{dt} &= \frac{dS}{dt} + \frac{dI_p}{dt} + \frac{dR_p}{dt} + \frac{dS_p}{dt} + \frac{dI_s}{dt} + \frac{dR}{dt}, \\ &= \Lambda - \mu(S + I_p + R_p + S_p + I_s + R) - \eta I_p - \sigma I_s, \\ &= \Lambda - \mu N - \eta I_p - \sigma I_s. \end{aligned}$$

That of vector population is given by: $M = S_v + I_v$,

$$\frac{dM}{dt} = \frac{dS_v}{dt} + \frac{dI_v}{dt} = \Omega - vM.$$

In the absence of the disease infection, the human population is resolved as follows:

$$\frac{dN}{dt} = \Lambda - \mu N - \eta I_p - \sigma I_s \leq \Lambda - \mu N,$$

$$\frac{dN}{dt} \leq \Lambda - \mu N.$$

By separation of variables,

$$\ln(\Lambda - \mu N) \geq -\mu(t + c),$$

then

$$N(t) \leq \frac{\Lambda - Ae^{-\mu t}}{\mu}.$$

Applying the initial condition $N(0) = N_0$, the solution becomes

$$N(t) \leq \frac{\Lambda - N(0)e^{-\mu t}}{\mu} \text{ as } t \rightarrow \infty.$$

Hence,

$$0 < N(t) \leq \frac{\Lambda}{\mu}.$$

Similarly, that of mosquito population gives

$$0 < M(t) \leq \frac{\Omega}{\nu}.$$

Therefore, the population is analytic since the upper bound limit $\frac{\Lambda}{\mu}$ and $\frac{\Omega}{\nu}$ exists. $\square$

2.5. Existence of Disease-Free and Endemic Equilibria

The extended model admits disease-free and endemic equilibria.

$$\begin{aligned} \frac{dS}{dt} = 0 &\Rightarrow \Lambda - \left(\frac{\beta}{M} (1 - \delta - \theta) I_V + \mu \right) S = 0, \\ \Lambda &= \left(\frac{\beta}{M} (1 - \delta - \theta) I_V + \mu \right) S. \end{aligned}$$

In the absence of the infection, $I_V = 0$, hence $S = \frac{\Lambda}{\mu}$. Similarly, $I_p = 0, R_p = 0, S_p = 0, I_s = 0, R = 0, S_v = \frac{\Omega}{\nu}$ and $I_v = 0$.

On the other hand, in the presence of the dengue infection then $I_p \neq 0, R_p \neq 0, S_p \neq 0, I_s \neq 0, R \neq 0, S \neq 0, S_v \neq 0, I_v \neq 0, N \neq 0, M \neq 0$. Hence, the steady state of the model at Endemic Equilibrium (EE) state is given by $(x_*) = (S^*, I_p^*, R_p^*, S_p^*, I_s^*, R^*, I_v^*, S_v^*)$,

$$S^* = \frac{\Lambda}{\mu + G_1(1 - \delta - \theta)},$$

$$I_p^* = \frac{\Lambda G_1(1 - \delta - \theta)}{(\mu + \gamma + \eta)(\mu + G_1(1 - \delta - \theta))},$$

$$R_p^* = \frac{\gamma \Lambda G_1(1 - \delta - \theta)}{(\mu + \alpha)(\mu + \gamma + \eta)(\mu + G_1(1 - \delta - \theta))},$$

$$S_p^* = \frac{\alpha \gamma \Lambda}{(G_1 + \mu)(\mu + \alpha)(\mu + \gamma + \eta)(\mu + G_1(1 - \delta - \theta))},$$

$$I_s^* = \frac{G_1 \alpha \gamma \Lambda}{(\mu + \gamma + \tau + \sigma)(G_1 + \mu)(\mu + \alpha)(\mu + \gamma + \eta)(\mu + G_1(1 - \delta - \theta))},$$

$$R^* = \frac{G_1 \alpha \gamma \Lambda (\gamma + \tau)}{\mu(\mu + \gamma + \tau + \sigma)(G_1 + \mu)(\mu + \alpha)(\mu + \gamma + \eta)(\mu + G_1(1 - \delta - \theta))},$$

$$S_v^* = \frac{\Omega}{G_2(1 - \delta - \theta) + \nu},$$

$$I_v^* = \frac{\Omega G_2(1 - \delta - \theta)}{(\nu + \omega)(G_2(1 - \delta - \theta) + \nu)}.$$

2.6. Computation of the Effective and Basic Reproduction Number

In this section we use the next generation matrix in [12]. The Model is decomposed into two transition matrices F and V representing the inflow and outflow of the system as follows:

$$x'_j = f_i(x) - v_i(x),$$

$$F = \left(\frac{\partial f_i}{\partial x_j} \right) \Big|_{x^0}, \quad V = \left(\frac{\partial v_i}{\partial x_j} \right) \Big|_{x^0}.$$

Where F and V are $m \times m$ matrices of partial derivatives of the transition matrices f_i and v_i respectively, and represent the infectious compartments [14]

$x_j = I_p, I_s, I_v$ and $x'_j = \partial I_p, \partial I_s, \partial I_v$. Where $1 \leq i, j \leq m$.

$$f_i = \begin{pmatrix} \frac{\beta}{M} I_v (1 - \delta - \theta) S \\ \frac{\beta}{M} I_v S_p \\ \frac{\vartheta}{N} S_v (I_p + \Phi I_s) \end{pmatrix}, \quad v_i = \begin{pmatrix} (\mu + \gamma + \eta) I_p \\ (\gamma + \tau + \mu + \sigma) I_s \\ (v + \omega) I_v \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 & \frac{\beta}{M} S(1 - \delta - \theta) \\ 0 & 0 & 0 \\ \frac{\vartheta}{N} S_v & \frac{\vartheta}{N} \Phi S_v & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\mu + \gamma + \eta) & 0 & 0 \\ 0 & (\gamma + \tau + \mu + \sigma) & 0 \\ 0 & 0 & (v + \omega) \end{pmatrix},$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu + \gamma + \eta} & 0 & 0 \\ 0 & \frac{1}{\gamma + \tau + \mu + \sigma} & 0 \\ 0 & 0 & \frac{1}{v + \omega} \end{pmatrix} \quad \text{then} \quad FV^{-1} = \begin{pmatrix} 0 & 0 & \frac{\beta S(1 - \delta - \theta)}{M(v + \omega)} \\ 0 & 0 & 0 \\ \frac{\vartheta S_v}{N(\mu + \gamma + \eta)} & \frac{\vartheta \Phi S_v}{N(\gamma + \tau + \mu + \sigma)} & 0 \end{pmatrix}.$$

The basic reproduction number R_0 is the spectral radius of FV^{-1} :

$$R_0 = \sqrt{\frac{\beta \vartheta}{(\mu + \gamma + \eta)(v + \omega)}} = 2.309114 > 1.$$

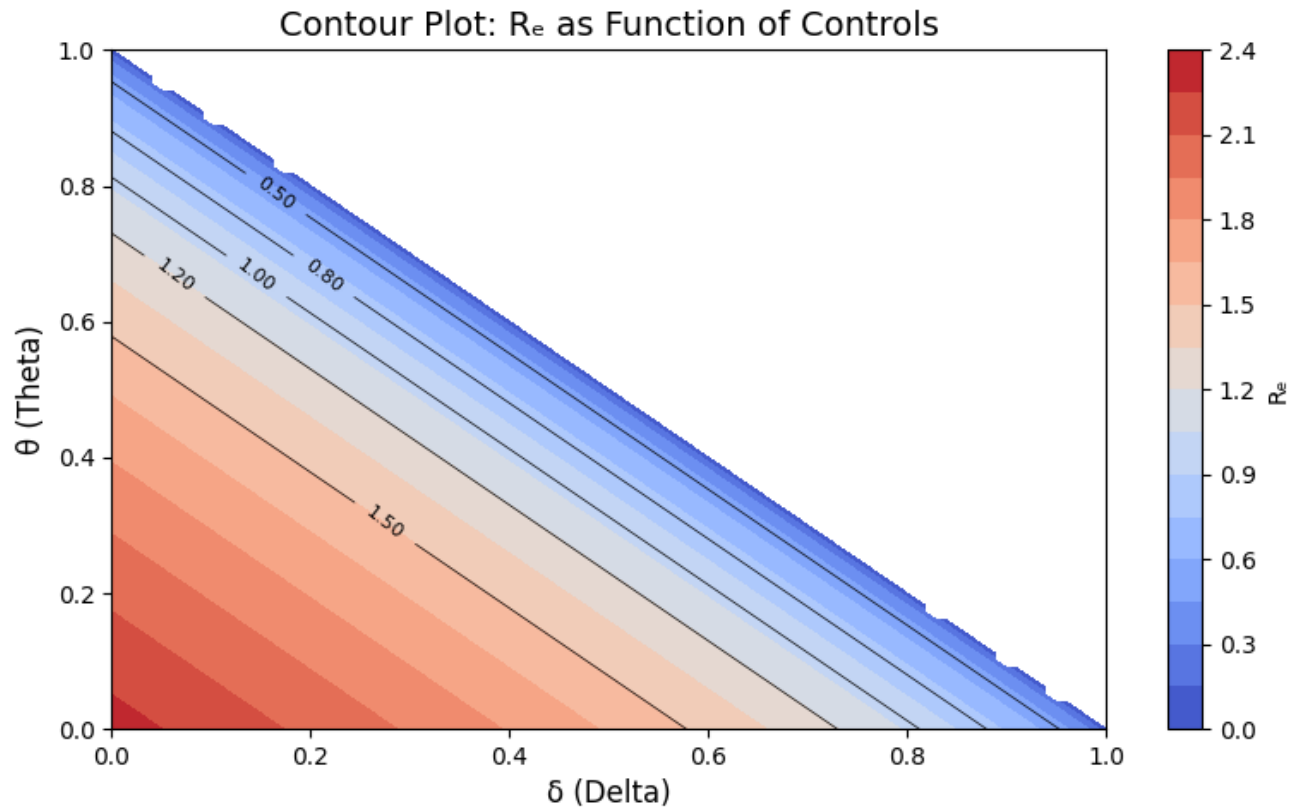
The effective reproduction number with control is:

$$R_e = \sqrt{\frac{\beta \vartheta (1 - \delta - \theta)}{(\mu + \gamma + \eta)(v + \omega)}} = 0.326557996 < 1.$$

It is observed that $R_e < R_0$ whenever controls are effective. These results demonstrate that with the integrated control measures (use of mosquito treated nets $\delta = 0.44$, good environmental sanitation $\theta = 0.54$, and vaccination $\tau = 0.95$), the effective reproduction number is significantly reduced below the threshold value of 1, indicating that the disease can be eradicated from the population. Figure 2 shows that as δ and θ increase the reproduction number decreases. It is observed that bifurcation point occurs at $\delta = \theta = 0.8$. Beyond this control value, the system exhibits disease-free equilibrium, and below 0.8, the system exhibits Endemic equilibrium. Therefore, it is necessary and sufficient to increase controls above 0.8 for total eradication of the disease.

Table 1: Table of parameters.

parameters	Description	values	Source
Λ	Recruitment into susceptible population	100	[1]
Ω	Recruitment into vector population	1000	[1]
δ	Use of ITNs	0.44	Assumed
θ	Good environmental sanitation	0.54	Assumed
τ	vaccination	0.95	[10]
β	Transmission rate from vector to human	0.5	[1]
ϑ	Transmission rate from human to vectors	0.75	[10]
γ	Recovery rate of human	0.52	[1]
α	Temporal immunity rate	0.5	[1]
Φ	Rate of secondary infection to overall force of infection	2.6	[1]
μ	Natural mortality rate for humans	0.02	[1]
ν	Natural mortality for mosquito	0.03	[1]
ω	Induced death rate	0.1	[10]
η	Induced death rate of I_p class	0.001	[10]
σ	Induced death rate for I_s	0.002	Assumed

Figure 2: R_e as a function controls

2.7. Local Stability Analysis

Next, we use the classical linearization Theorem- The Routh-Hurwitz Criterion to assess the local stability of the model. Given a coefficient Matrix (M), the Equilibrium point is called Asymptotically stable if: $Tr(J(M))_{DFE} < 0$ and $Det(J(M))_{DFE} > 0$ [13].

Proof. We set the equations of the model as

$$\left. \begin{aligned} f_1 &= \Lambda - G_1(1 - \delta - \theta)S - \mu S, \\ f_2 &= G_1(1 - \delta - \theta)S - (\mu + \gamma + \eta)I_p, \\ f_3 &= \gamma I_p - (\mu + \alpha)R_p, \\ f_4 &= \alpha R_p - G_1 S_p - \mu S_p, \\ f_5 &= G_1 S_p - (\gamma + \tau)I_s - \mu I_s - \sigma I_s, \\ f_6 &= (\gamma + \tau)I_s - \mu R, \\ f_7 &= \Omega - G_2(1 - \delta - \theta)S_v - v S_v, \\ f_8 &= G_2(1 - \delta - \theta)S_v - (v + \omega)I_v. \end{aligned} \right\} \quad (2)$$

The Jacobian Matrix is given below

$$J = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\beta\Lambda(1-\delta-\theta)}{M\mu} \\ 0 & -(\mu + \gamma + \eta) & 0 & 0 & 0 & 0 & 0 & \frac{\beta\Lambda(1-\delta-\theta)}{M\mu} \\ 0 & \gamma & -(\mu + \alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\gamma + \tau + \mu + \sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma + \tau) & -\mu & 0 & 0 \\ 0 & -\frac{\partial\Omega(1-\delta-\theta)}{Nv} & 0 & 0 & -\frac{\partial\Phi\Omega(1-\delta-\theta)}{Nv} & 0 & -v & 0 \\ 0 & \frac{\partial\Omega(1-\delta-\theta)}{Nv} & 0 & 0 & \frac{\partial\Phi\Omega(1-\delta-\theta)}{Nv} & 0 & 0 & -(v + \omega) \end{pmatrix}$$

The trace of the above Jacobian matrix is given as: $Tr(J) = -5\mu - 2\gamma - \eta - \tau - \sigma - 2v - \omega$ and substituting the parameters values, we

have $Tr(J) = -2.253 < 0$. The characteristic equation is given by:

$$\det(J - \lambda I) = \begin{vmatrix} -\mu - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\beta\Lambda(1-\delta-\theta)}{M\mu} \\ 0 & -m - \lambda & 0 & 0 & 0 & 0 & 0 & \frac{\beta\Lambda(1-\delta-\theta)}{M\mu} \\ 0 & \gamma & -(\mu + \alpha) - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\mu - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma + \tau) & -\mu - \lambda & 0 & 0 \\ 0 & -\frac{\vartheta\Omega(1-\delta-\theta)}{N\gamma} & 0 & 0 & -\frac{\vartheta\Phi\Omega(1-\delta-\theta)}{N\gamma} & 0 & -v - \lambda & 0 \\ 0 & \frac{\vartheta\Omega(1-\delta-\theta)}{N\gamma} & 0 & 0 & \frac{\vartheta\Phi\Omega(1-\delta-\theta)}{N\gamma} & 0 & 0 & -(v + \omega) - \lambda \end{vmatrix} = 0$$

Where $m = \mu + \gamma + \eta$ and $q = \gamma + \tau + \mu + \sigma$. The eigenvalues are: $\lambda_1 = -\mu < 0$, $\lambda_2 = -(\mu + \alpha) < 0$, $\lambda_3 = -\mu < 0$, $\lambda_4 = -(\gamma + \tau + \mu + \sigma) < 0$, $\lambda_5 = -\mu < 0$, $\lambda_6 = -v < 0$ while the two other eigenvalues are determined as the roots of the quadratic equation $a_1\lambda^2 + a_2\lambda + a_3 = 0$, where $a_1 = 1 > 0$, $a_2 = (v + \omega + \gamma + \tau + \mu + \sigma) > 0$ and $a_3 = (v + \omega)(\gamma + \tau + \mu + \sigma)(1 - R_e) > 0$, for $R_e < 1$. According to Routh-Hurwitz Criterion, $a_1 > 0$, $a_1a_2 - a_3 > 0$ ensures that the DFE is asymptotically stable whenever $R_e < 1$. $\square$

2.8. Global Stability Analysis via Lyapunov Function

Let D be an open subset of R^8 . The smooth function $V : D \rightarrow R^8$ is said to be positive definite on the set D if the following conditions hold:

1. $V(x) > 0$, for all $x \in D$ and
2. $V(x) = 0$ at DFE.

The equilibrium point solution is globally asymptotically stable if in addition to conditions (1) and (2) above, the time derivative $\frac{dV}{dt} < 0$. The smooth function V is called Lyapunov function.

Proof. Let the Lyapunov function be defined as

$$V = c_1 I_p + c_2 I_s + c_3 I_v,$$

where $c_1, c_2, c_3 > 0$ are constants to be determined.

Differentiating V along the system trajectories:

$$\dot{V} = c_1 \dot{I}_p + c_2 \dot{I}_s + c_3 \dot{I}_v.$$

Substituting the expressions for $\dot{I}_p$, $\dot{I}_s$, and $\dot{I}_v$:

$$\begin{aligned} \dot{V} = & c_1 (G_1(1 - \delta - \theta)S - (\mu + \gamma + \eta)I_p) \\ & + c_2 (G_1S_p - (\gamma + \tau + \mu + \sigma)I_s) \\ & + c_3 (G_2(1 - \delta - \theta)S_v - (v + \omega)I_v). \end{aligned}$$

At the DFE

Since $S \leq S^0 = \frac{\Lambda}{\mu}$ and $S_v \leq S_v^0 = \frac{\Omega}{v}$, we have:

$$\begin{aligned} \dot{V} \leq & c_1 (G_1(1 - \delta - \theta)S^0 - (\mu + \gamma + \eta)I_p) \\ & + c_2 (G_1S_p - (\gamma + \tau + \mu + \sigma)I_s) \\ & + c_3 (G_2(1 - \delta - \theta)S_v^0 - (v + \omega)I_v). \end{aligned}$$

Choosing Coefficients

To eliminate cross-terms, we set:

$$c_1 = 1, \quad c_3 = \frac{G_1(1 - \delta - \theta)S^0}{v + \omega}, \quad c_2 = 0.$$

This simplifies $\dot{V}$ to:

$$\dot{V} \leq (G_1(1 - \delta - \theta)S^0 - (\mu + \gamma + \eta)I_p) + \frac{G_1G_2(1 - \delta - \theta)^2S^0S_v^0}{v + \omega} - (v + \omega)I_v.$$

Since $\mathcal{R}_e < 1$, then:

$$G_1(1 - \delta - \theta)S^0 < (\mu + \gamma + \eta), \quad \text{and} \quad \frac{G_1G_2(1 - \delta - \theta)^2S^0S_v^0}{v + \omega} < (v + \omega),$$

Hence, $\dot{V} \leq 0$

$\square$

2.9. Bifurcation Analysis

At bifurcation, $R_e = 1$. Reproduction number of infection is essential in this section because it represents the spreading of infection [1]. Recall that

$$R_e = \sqrt{\frac{\beta \vartheta (1 - \delta - \theta)}{(\mu + \gamma + \eta)(\nu + \omega)}} \leq \frac{\beta \vartheta (1 - \delta - \theta)}{(\mu + \gamma + \eta)(\nu + \omega)}.$$

Let $\Psi = \beta$ be the bifurcation parameter and $R_e = 1$ be the bifurcation point.

Then

$$R_e = 1 = \frac{\Psi \vartheta (1 - \delta - \theta)}{(\mu + \gamma + \eta)(\nu + \omega)},$$

$$\Psi = \frac{(\mu + \gamma + \eta)(\nu + \omega) \mu \nu}{\vartheta (1 - \delta - \theta)}.$$

From the foregoing, R_e is directly proportional to the bifurcation parameter Ψ , because increase in the value of the bifurcation parameter will bring a considerable increase in the effective reproduction number R_e of the Dengue infectious disease and vice versa [18]. Now, the differential equations of the model is expressed as

$$\frac{dx}{dt} = f(x, \Psi).$$

Where x is an 8-dimensional vector with compartments

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T,$$

corresponding to the State variables $(S, I_p, R_p, S_p, I_s, R, S_v, I_v)$ respectively. Hence the function

$$f : R^8 \times R \rightarrow R^8.$$

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8).$$

Hence, the model equations can be re-written as

$$\left. \begin{aligned} f_1 &= \Lambda - \frac{\Psi}{M} (1 - \delta - \theta) x_1 x_8 - \mu x_1, \\ f_2 &= \frac{\Psi}{M} (1 - \delta - \theta) x_1 x_8 - (\mu + \gamma + \eta) x_2, \\ f_3 &= \gamma x_2 - (\mu + \alpha) x_3, \\ f_4 &= \alpha x_3 - \frac{\Psi}{M} x_4 x_8 - \mu x_4, \\ f_5 &= \frac{\Psi}{M} x_4 x_8 - (\gamma + \tau) x_5 - \mu x_5 - \sigma x_5, \\ f_6 &= (\gamma + \tau) x_5 - \mu x_6, \\ f_7 &= \Omega - \left(\frac{\vartheta}{N} x_2 + \frac{\Phi \vartheta}{N} x_5 \right) (1 - \delta - \theta) x_7 - \nu x_7, \\ f_8 &= \left(\frac{\vartheta}{N} x_2 + \frac{\Phi \vartheta}{N} x_5 \right) (1 - \delta - \theta) x_7 - (\nu + \omega) x_8. \end{aligned} \right\} \quad (3)$$

The Jacobian Matrix at DFE with $\beta = \Psi$ is given as

$$J = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu} \\ 0 & -(\mu + \gamma + \eta) & 0 & 0 & 0 & 0 & 0 & \frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu} \\ 0 & \gamma & -(\mu + \alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\gamma + \tau + \mu + \sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma + \tau) & -\mu & 0 & 0 \\ 0 & -\frac{\vartheta \Omega (1 - \delta - \theta)}{N \nu} & 0 & 0 & -\frac{\vartheta \Phi \Omega (1 - \delta - \theta)}{N \nu} & 0 & -\nu & 0 \\ 0 & \frac{\vartheta \Omega (1 - \delta - \theta)}{N \nu} & 0 & 0 & \frac{\vartheta \Phi \Omega (1 - \delta - \theta)}{N \nu} & 0 & 0 & -(\nu + \omega) \end{pmatrix}$$

The Matrix has a right eigen-vectors given as

$$h = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8)^T.$$

The zero eigenvalues are calculated as follows

$$\begin{aligned} -\mu h_1 - \frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu} h_8 &= 0, \\ h_1 &= \frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu^2} h_8, \\ -(\mu + \gamma + \eta) h_2 + \frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu} h_8 &= 0, \\ h_2 &= \frac{\Psi \Lambda (1 - \delta - \theta)}{M \mu (\mu + \gamma + \eta)} h_8, \\ \gamma h_2 - (\mu + \alpha) h_3 &= 0, \\ h_3 &= \frac{\gamma \Psi \Lambda (1 - \delta - \theta)}{M \mu (\mu + \gamma + \eta) (\mu + \alpha)} h_8, \end{aligned}$$

$$\begin{aligned}
h_4 &= \frac{\alpha}{\mu} h_3 = \frac{\alpha \gamma \Psi \Lambda (1 - \delta - \theta)}{M \mu (\mu + \gamma + \eta) \mu (\mu + \alpha)} h_8, \\
-(\mu + \gamma + \tau + \sigma) h_5 &= 0 \Rightarrow h_5 = 0, \\
(\gamma + \tau) h_5 - \mu h_6 &= 0 \Rightarrow h_6 = 0, \\
h_7 &= \frac{-\vartheta \Omega (1 - \delta - \theta)}{v^2} h_2 = \frac{-\vartheta \Omega \Psi \Lambda (1 - \delta - \theta)^2}{v^2 \mu M (\mu + \gamma + \eta)} h_8 \Rightarrow h_7 = -\frac{\vartheta \Omega \Psi \Lambda (1 - \delta - \theta)^2}{v^2 \mu M (\mu + \gamma + \eta)} h_8, \\
h_8 &= h_8.
\end{aligned}$$

$$h = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8)^T.$$

Similarly, the Left eigenvectors of the Matrix is given as

$$v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8).$$

The zero eigenvalues are obtained as follows

$$\begin{aligned}
-\mu v_1 &= 0 \Rightarrow v_1 = 0, \\
-\mu v_4 &= 0 \Rightarrow v_4 = 0, \\
-(\mu + \alpha) v_3 + \alpha v_4 &= 0 \Rightarrow v_3 = 0, \\
-(\mu + \gamma + \eta) v_2 + \gamma v_3 - \frac{\vartheta \Omega}{N v} (1 - \delta - \theta) v_7 + \frac{\vartheta \Omega}{N v} (1 - \delta - \theta) v_8 &= 0, \\
v_2 &= \frac{\vartheta \Omega}{N v (\mu + \gamma + \eta)} (1 - \delta - \theta) (v_8 - v_7), \\
-(\mu + \gamma + \tau + \sigma) v_5 - \frac{\vartheta \Phi \Omega}{N v} (1 - \delta - \theta) v_7 + \frac{\vartheta \Phi \Omega}{N v} (1 - \delta - \theta) v_8 &= 0, \\
v_5 &= \frac{\Phi \vartheta \Omega}{N v (\mu + \gamma + \tau + \sigma)} (1 - \delta - \theta) (v_8 - v_7), \\
-\mu v_6 &= 0 \Rightarrow v_6 = 0, \\
-v v_7 &= 0 \Rightarrow v_7 = 0, \\
v_8 &= v_8,
\end{aligned}$$

$$v = \left(0, \frac{\vartheta \Omega}{N v (\mu + \gamma + \eta)} (1 - \delta - \theta), 0, 0, \frac{\Phi \vartheta \Omega}{N v (\mu + \gamma + \tau + \sigma)} (1 - \delta - \theta), 0, 0, 1 \right)^T.$$

Consider the following general system of ordinary differential equations with parameter Ψ , $\frac{dx}{dt} = f(x, \Psi)$, $f : R^n \times R \rightarrow R^n$ such that $\frac{\partial f_i}{\partial x_j}(0, 0)$ is the linearization matrix of the system around the equilibrium point $(0, 0)$ with Ψ evaluated at DFE. The local dynamics or the nature of the equilibria is totally governed by the numbers a and b [19].

Proof.

$$\begin{aligned}
a &= \sum v_k h_i h_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (0, \Psi), \\
b &= \sum v_k h_i \frac{\partial^2 f_k}{\partial x_i \partial \Psi} (0, \Psi).
\end{aligned}$$

Where a and b are constant coefficients that determine the nature of the equilibria near to the bifurcation point, and f_k is the k th component of f such that $f \in C^2$.

$k = 2, 5$.

$$\begin{aligned}
a &= v_2 h_1 h_8 \frac{\partial^2 f_2}{\partial x_1 \partial x_8} + v_2 h_8 h_1 \frac{\partial^2 f_2}{\partial x_8 \partial x_1} + v_5 h_4 h_8 \frac{\partial^2 f_5}{\partial x_4 \partial x_8} + v_5 h_8 h_4 \frac{\partial^2 f_5}{\partial x_8 \partial x_4}, \\
\frac{\partial^2 f_2}{\partial x_1 \partial x_8} &= \frac{\Psi}{M} (1 - \delta - \theta) = \frac{\partial^2 f_2}{\partial x_8 \partial x_1}, \\
\frac{\partial^2 f_5}{\partial x_4 \partial x_8} &= \frac{\Psi}{M} = \frac{\partial^2 f_5}{\partial x_8 \partial x_4}, \\
a &= 2 v_2 h_1 h_8 \frac{\Psi}{M} (1 - \delta - \theta) + 2 v_5 h_4 h_8 \frac{\Psi}{M},
\end{aligned}$$

substituting the expressions for v_2 , v_5 , h_1 , h_4 , and setting $v_7 = 0$, $v_8 = 1$:

$$a = \frac{2 \Psi}{M} h_8 \left[\frac{\vartheta \Omega}{N v (\mu + \gamma + \eta)} (1 - \delta - \theta)^2 \frac{\Psi \Lambda}{M \mu^2} + \frac{\Phi \vartheta \Omega}{N v (\mu + \gamma + \tau + \sigma)} (1 - \delta - \theta) \frac{\alpha \gamma \Psi \Lambda}{M \mu (\mu + \gamma + \eta) \mu (\mu + \alpha)} \right] h_8,$$

$$a = \frac{2\Psi^2 \vartheta \Omega \Lambda}{M^2 N \nu \mu^2} h_8^2 \left[\frac{(1 - \delta - \theta)^2}{(\mu + \gamma + \eta)} + \frac{\Phi \alpha \gamma (1 - \delta - \theta)}{\mu(\mu + \gamma + \tau + \sigma)(\mu + \gamma + \eta)(\mu + \alpha)} \right].$$

The sign of a depends on the term in brackets. Define the critical value Φ_c as:

$$\Phi_c = \frac{\mu(\mu + \gamma + \tau + \sigma)(\mu + \alpha)(1 - \delta - \theta)}{\alpha \gamma}.$$

$$b = v_2 h_8 \frac{\partial^2 f_2}{\partial x_8 \partial \Psi} = v_2 h_8 \frac{\Lambda}{M} (1 - \delta - \theta) > 0.$$

Existence equilibria in word will depend on the parameter ϕ . The system exhibit a backward bifurcation at $R_e = 1$ if and only if $\phi > \phi_c$ and forward bifurcation if and only if $R_e = 1$ if and only if $\phi < \phi_c$. Therefore, the system exhibits:

- Forward bifurcation at $R_e = 1$ if $\Phi < \Phi_c$ ($a < 0$)
- Backward bifurcation at $R_e = 1$ if $\Phi > \Phi_c$ ($a > 0$)

□

Using the parameter values, $a = 0.01966992 > 0$, $b = 0.9242 > 0$, $\phi_c = 0.05968$ and $\phi_0 = 1.084$

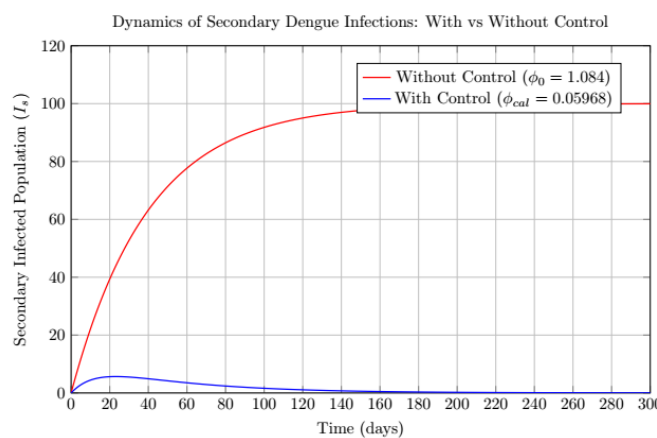


Figure 3: Dynamics of Secondary Dengue Infection

3. Graphical Interpretation of Reproduction Number (R_e) and R_0

Figure 2 shows that as δ and θ increase, the reproduction number decreases. It is observed that bifurcation point occurred $\delta = \theta = 0.8$. Beyond this control value, the system exhibit forward bifurcation, and below 0.8 the system exhibit backward bifurcation. There, it is necessary and sufficient to increase controls above 0.8 for total eradication of the disease.

Figure 3 illustrate the rate of secondary infection with control and without control measures. Without control, the infection grows monotonically to a high endemic equilibrium level. Since $\phi_0 > 1$, secondary infections are amplified, leading to sustained high-level transmission. The disease becomes highly endemic with a large reservoir of infected individuals. With control, the infection shows a small, contained out break and then dies out completely, because $\phi_c < 1$ means the disease can not sustain itself.

3.1. Conclusion

The rate of secondary infection with control significantly less than the rate of secondary infection without control, confirming that the controls reduce severity of secondary infection. The control measure reduced the reproduction number below 1.

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