International Journal of Advanced Mathematical Sciences, 2 (3) (2014) 132-136
©Science Publishing Corporation
www.sciencepubco.com/index.php/IJAMS
doi: 10.14419/ijams.v2i3.3466
Research Paper

Some properties of positive derivations on f-rings II

R. H. Redfield

Department of Mathematics, Hamilton College, 198 College Hill Road, Clinton, NY 13323, USA
Email: rredfiel@hamilton.edu

Abstract

Theorem 10(b)(i) in the article "Some properties of positive derivations on f-rings" by Henriksen and Smith asserts that if D is a positive derivation on a reduced f-ring and if $x \in \ker D$, then $\{x\}^{\perp\perp} \subseteq \ker D$. A counterexample is provided to show that this assertion is false, and correct proofs are given for some results in the paper by Henriksen and Smith that use Theorem 10(b)(i) in their proofs.

Keywords: *f-ring, lattice-ordered ring, polynomial ring, positive derivation*

1. Introduction

Recall that a *lattice-ordered ring* (*ℓ-ring*) is a ring $(R, +, \cdot)$ with a lattice order $\geq$ that is *compatible* with the operations in the sense that if $a \geq b$ in R and $c \in R$, then $a + c \geq b + c$, and if $x, y \geq 0$ in R, then $xy \geq 0$. For an ℓ-ring $(R, +, \cdot, \geq)$, an endomorphism D of $(R, +)$ is *positive* if $D(R^+) \subseteq R^+$, where $R^+ = \{x \in R \mid x \geq 0\}$ is the *positive cone* of R, and a *derivation* if $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$. An *f-ring* is an ℓ-ring R such that if $x \wedge y = 0$ and $z \in R^+$, then $xz \wedge y = 0 = zx \wedge y$. Recall that if $x \wedge y = 0$ in an f-ring, then $xy = 0$ [1, 9.1.10(iv)]. Colville, Davis, and Keimel initiated the study of positive derivations on f-rings in [2]. In [4], Henriksen and Smith extended the work of [2] and provided a direct and elementary proof to its main result.

Throughout the sequel, A denotes an f-ring and $\mathcal{D}(A)$ denotes the set of positive derivations on A. For notation and terminology left undefined, we refer the reader to [5].

The result of [4] with which we are concerned is Theorem 10. Recall that an element $e \in A$ is *regular* if $x = 0$ whenever $ex = 0$ or $xe = 0$, that a *band* of A is a convex sublattice subgroup B of A such that if $X \subseteq B$ and $\bigvee X \in A$, then $\bigvee X \in B$, that rad A denotes the set of all nilpotent elements of A, that A is *reduced* if rad $A = \{0\}$, and that for $Y \subseteq A$, $Y^\perp = \{a \in A \mid |a| \wedge |y| = 0 \text{ for all } y \in Y\}$. As usual, if D is a derivation of A, then $\ker D = \{a \in A \mid D(a) = 0\}$.

[4, Theorem 10]: *Suppose $D \in \mathcal{D}(A)$, $x \in A$, and n is a positive integer.*

(a) *If e is regular and $ex \in \ker D$, then $x \in \ker D$.*

(b) *If A is reduced, then:*

  (i) (i) *$x \in \ker D$ implies that $\{x\}^{\perp\perp} \subseteq \ker D$;*

  (ii) *$x^n \in \ker D$ implies that $x \in \ker D$;*

(iii) $\ker D$ *is a band;*

(iv) $D^n = 0$ *implies that* $D = 0$*;*

(v) $e^2 = e \in A$ *implies that* $e \in \ker D$.

(c) *If* A *has an identity element and* $U(A)$ *is the smallest band containing the units of* A*, then* $U(A) \subset \ker D$*. In particular,* $\mathrm{rad}\ A \subset \ker D$*. Also if* $x^2 = x$*, then* $x \in \ker D$.

The proof of Theorem 10(b)(i) in [4] used the incorrect inclusion:

$$D(\{z\}^{\perp\perp}) \subseteq D(\{z\}^{\perp})^{\perp} \subseteq \{D(z)\}^{\perp\perp}. \tag{1}$$

While the first inclusion is correct, the second inclusion is not (see [1, 3.2.2] and Example 2.1 below); it should be

$$D(\{z\}^{\perp})^{\perp} \supseteq \{D(z)\}^{\perp\perp}.$$

In [4], Henriksen and Smith used (1) to show that since $\{D(x)\}^{\perp\perp} = \{0\}$, $D(\{x\}^{\perp\perp}) = \{0\}$; obviously one cannot conclude this from the correct relation. Indeed, as shown in Example 2.1, Theorem 10(b)(i) is not correct.

The proofs of several results in [4] used Theorem 10(b)(i). In Section 3, we provide correct proofs for some of the these and prove special cases for others.

For use in the sequel, recall that an ℓ-ring L is *Archimedean* if for all nonzero x and y in L^+, there exists a positive integer n such that $nx \not\leq y$, that an *ℓ-ideal* of L is an ideal of the ring $(L, +, \cdot)$ that is also a subgroup of $(L, +)$ and a convex sublattice of $(L, \geq)$, and that an ℓ-ideal I of L is *ℓ-prime* provided that $I \neq L$ and if $JK \subseteq I$ for ℓ-ideals J and K, either $J \subseteq I$ or $K \subseteq I$. A set of ℓ-prime ℓ-ideals $\{P_\alpha\}$ of L is *separating* if $\bigcap_\alpha P_\alpha = \{0\}$.

A convex ℓ-subgroup H of L is called a *quasi-band* if there exists an ℓ-prime ℓ-ideal P of L and a band B in L/P such that $H = \pi^{-1}(B)$, where $\pi : L \to L/P$ is the usual projection $\pi(a) = a + P$. The projection π is said to *preserve infinite sups* if for any subset X of L, whenever $\bigvee X$ exists in L, $\bigvee \pi(X)$ exists in L/P and $\bigvee \pi(X) = \pi(\bigvee X)$. It is easy to see that if L is the direct sum, or direct product, of totally ordered rings, then the projections onto the factors preserve infinite sups (see also Corollary 3.8 below).

Finally recall that if R is a ring with unit element and D is a derivation on R, then $D(1) = 0$ because $D(1) = 1D(1) + D(1)1 - D(1) = D(1 \cdot 1) - D(1)$.

2. The Examples

Our first example shows that both Theorem 10(b)(i) and the second inclusion in (1) are false.

Example 2.1 Let $\mathbb{R}$ be the totally ordered field of real numbers and $R = \mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$. Define a polynomial in R positive if the coefficient of its highest power is positive. Then R becomes a totally ordered domain which is not Archimedean, and the usual derivative $D(\sum_{i=0}^{N} \rho_i x^i) = \sum_{i=1}^{N} i\rho_i x^{i-1}$ on R is a positive derivation on R. Then $\ker D$ is the set of constant polynomials, and for any $0 \neq a \in \ker D$, $\{a\}^{\perp} = \{0\}$ and hence $\{a\}^{\perp\perp} = R = \mathbb{R}[x]$. So $a^{\perp\perp} \not\subseteq \ker D$. Of course, since R is a domain, it is reduced. Note finally that since

$$D(\{x\}^{\perp})^{\perp} = D(\{0\})^{\perp} = \{0\}^{\perp} = R \not\subseteq \{0\} = \{0\}^{\perp\perp} = \{D(x)\}^{\perp\perp},$$

the second inclusion in (1) is not true in R.

We next note that an Archimedean f-ring may have a derivation D for which $\ker D$ is not a band. The f-ring we construct is not reduced and has a separating set of minimal ℓ-prime ℓ-ideals whose projections preserve infinite sups.

Example 2.2 Let L be the ℓ-subgroup of all eventually constant elements of the Achimedean Abelian ℓ-group of $\prod_1^\infty \mathbb{R}$ (with coordinatewise order): $f \in L$ if and only if there exists $1 \leq N \in \mathbb{Z}$ such that for all $m, n \geq N$, $f(m) = f(n)$.

Pick an integer $\kappa > 1$ and define a multiplication on L by letting

$$(fg)(n) = \begin{cases} f(n)g(n) & \text{if } n < \kappa \\ 0 & \text{if } n \geq \kappa \end{cases}.$$

It is easy to check that $(L,+,\cdot,\geq)$ is a commutative Archimedean f-ring. For $j\geq 1$, let $b_j\in L$ is the element

$$b_j(n)=\begin{cases}1 & \text{if } n=j\\ 0 & \text{if } n\neq j\end{cases}.$$

Then b_κ is a nonzero element of L such that $b_\kappa^2=0$, and thus L is not reduced.

For each integer $k\geq 1$, let $P_k=\{f\in L \mid f(k)=0\}$. It is easy to see that for each $f\in P_k$, there exists $g\notin P_k$ such that $fg\in P_k$, and hence that P_k is a minimal ℓ-prime ℓ-ideal by [5, Theorem 3.2.22]. It is obvious that $\{P_k\}$ is separating, and it is easy to check that each projection $\pi_k: L\to L/P_k$, defined by $\pi_k(r)=r+P_k$, preserves infinite sups.

Define $D: L\longrightarrow L$ by letting

$$D(f)(n)=\begin{cases}\lim\limits_{i\to\infty} f(i) & \text{if } n=\kappa\\ 0 & \text{if } n\neq \kappa\end{cases}.$$

It is easy to check that D is a positive endomorphism of $(L,+,\geq)$ with kernel $\ker D=\sum_{n=1}^{\infty}\mathbb{R}$. Furthermore, by the definitions given above, for all $f,g\in L$, $D(fg)=0$ and $fD(g)+D(f)g=0+0=0$ as well so that D is a derivation on L. Certainly each b_j is in $\sum_{n=1}^{\infty}\mathbb{R}$, and since $\bigvee_{j=1}^{\infty} b_j$ is the function that is constantly 1, $\bigvee_{j=1}^{\infty} b_j\in L$. Then since $\bigvee_{j=1}^{\infty} b_j\notin\sum_{n=1}^{\infty}\mathbb{R}$, $\ker D$ is not a band.

3. The Theorems

In [4], Henriksen and Smith use Theorem 10(b)(i) to prove Theorems 10(b)(ii), 10(b)(iii), and 10(b)(iv). Without using Theorem 10(b)(i), we prove Theorem 10(b)(ii) (Theorem 3.2 below), a generalization of Theorem 10(b)(iv) (Theorem 3.3 below), and a special case of Theorem 10(b)(iii) (Theorem 3.7 below).

Lemma 3.1 *Suppose that $D\in\mathcal{D}(A)$ and $z\in A$. If A is reduced, then $|D(z)|=D(|z|)$.*

Proof. We begin by noting that if $x\wedge y=0$ in A, then $D(x)\wedge D(y)=0$. For if $x\wedge y=0$, then, since A is an f-ring, $xy=0$ and hence $0=D(xy)=xD(y)+D(x)y$. Then since D is positive, $xD(y)=0$ so that $0=D(xD(y))=D(x)D(y)+xD(D(y))$ and thus $D(x)D(y)=0$. So since A is reduced, $D(x)\wedge D(y)=0$ by [1, 9.3.1(ii)].

In particular, for any $z\in A$, $z=z^+-z^-$ where $z^+\wedge z^-=0$. So $D(z)=D(z^+)-D(z^-)$ and $D(z^+)\wedge D(z^-)=0$ and therefore by [3, Proposition 4.2], $D(z)^+=D(z^+)$ and $D(z)^-=D(z^-)$. Then $|D(z)|=D(z)^++D(z)^-=D(z^+)+D(z^-)=D(z^++z^-)=D(|z|)$.

Theorem 3.2 [4, Theorem 10(b)(ii)] *Suppose that $D\in\mathcal{D}(A)$, $x\in A$, and n is a positive integer. If A is reduced, then $x^n\in\ker D$ implies that $x\in\ker D$.*

Proof. Suppose that $x^n\in\ker D$. By [1, 9.1.10(iii)], since A is an f-ring, $|x^n|=|x|^n$, and thus by Lemma 3.1, since $D(x^n)=0$,

$$0=|D(x^n)|=D(|x^n|)=D(|x|^n)=|x|^{n-1}D(|x|)+D(|x^{n-1}|)|x|.$$

Therefore, since D is positive, $|x|^{n-1}D(|x|)=0$. Then

$$0=D(|x|^{n-1}D(|x|))=|x|^{n-1}D(D(|x|))+D(|x|^{n-1})D(|x|),$$

and thus

$$0=D(|x|^{n-1})D(|x|)=|x|^{n-2}D(|x|)D(|x|)+D(|x|^{n-2})|x|D(|x|)$$

so that $|x|^{n-2}D(|x|)^2=0$. Applying this argument sufficiently many times, we can conclude that $D(|x|)^n=0$, and hence since A is reduced, that $D(|x|)=0$, i.e., that $|x|\in\ker D$. But since D is positive, $\ker D$ is convex, and thus $x\in\ker D$.

Theorem 3.3 *Suppose that R is a reduced ℓ-ring, D is a positive derivation on R, and n is a positive integer. Then $D^n=0$ implies that $D=0$.*

Proof. Let $x \in R^+$. By hypothesis,

$$0 = D^n(x^n) = D^{n-1}(xD(x^{n-1}) + D(x)x^{n-1}) = D^{n-1}(xD(x^{n-1})) + D^{n-1}(D(x)x^{n-1}),$$

and thus since D is positive, $D^{n-1}(D(x)x^{n-1}) = 0$. Then

$$0 = D^{n-2}(D(x)D(x^{n-1}) + D^2(x)x^{n-1}) = D^{n-2}(D(x)D(x^{n-1})) + D^{n-2}(D^2(x)x^{n-1}),$$

and hence $D^{n-2}(D(x)D(x^{n-1})) = 0$. So

$$0 = D^{n-2}(D(x)(xD(x^{n-2}) + D(x)x^{n-2}) = D^{n-2}(D(x)xD(x^{n-2})) + D^{n-2}(D(x)^2x^{n-2}),$$

and thus $D^{n-2}(D(x)^2x^{n-2}) = 0$. Applying this argument sufficiently many times, we conclude that $D(x)^n = 0$, and thus since R is reduced, that $D(x) = 0$. Since this is true for all $x \in R^+$, $D = 0$.

Corollary 3.4 [**4, Theorem 10(b)(iv)**] *Suppose that $D \in \mathcal{D}(A)$, $x \in A$, and n is a positive integer. If A is reduced, then $D^n = 0$ implies that $D = 0$.*

The other results in [4] whose proofs use Theorem 10(b)(i) are Theorem 10(b)(iii) and the first part of Theorem 10(c) (that $U(A) \subset \ker D$). We conclude by showing that these assertions are indeed true for totally ordered rings (Theorem 3.6) and for reduced f-rings that possess a separating set of minimal ℓ-prime ℓ-ideals whose projections preserve infinite sups (Theorem 3.7).

Lemma 3.5 *Suppose that A has a unit element and that $D \in \mathcal{D}(A)$. If u is a unit of A, then $u \in \ker D$; if $\ker D$ is a band, then $U(A) \subseteq \ker D$.*

Proof. By [1, 9.1.10(vi)], $1 > 0$ and by [1, 9.1.10(iii)], $|uu^{-1}| = |u||u^{-1}|$. So $0 = D(1) = |u|D(|u^{-1}|) + D(|u|)|u^{-1}|$, and thus $D(|u|)|u^{-1}| = 0$. But then $D(|u|) = D(|u|)|u^{-1}u| = 0|u| = 0$, i.e., $|u| \in \ker D$. Since D is positive, $\ker D$ is convex so that $u \in \ker D$. It follows that if $\ker D$ is a band, then $U(A) \subseteq \ker D$.

Theorem 3.6 *Let T be a totally ordered ring and let D be a positive derivation on T. Then $\ker D$ is a band, and if T has an identity element, $U(T) \subseteq \ker D$.*

Proof. Suppose that $X \subseteq \ker D$ and that $x = \bigvee X$ in T. For any $a \in X$, $x \geq a$, and since D is a positive derivation on T, $D(x) \geq D(a) = 0$. Suppose that $0 \neq z \in X$. Then $x - |z| < x$ and thus $x - |z|$ is not an upper bound for X. So since T is totally ordered, there exists an element $w \in X$ such that $x - |z| \leq w$. It follows that $D(x) - D(|z|) \leq D(w) = 0$. But since T is totally ordered, either $D(|z|) = D(z) = 0$ or $D(|z|) = D(-z) = 0$, and hence $D(x) \leq 0$. So $0 \leq D(x) \leq 0$, i.e., $x \in \ker D$, and thus $\ker D$ is a band. If T has an identity element, then $U(T) \subseteq \ker D$ by Lemma 3.5.

Theorem 3.7 *Suppose $D \in \mathcal{D}(A)$.*

1. *Then $\ker D$ is an intersection of quasi-bands of A.*
2. *If A has a separating set of minimal ℓ-prime ℓ-ideals whose projections preserve infinite sups, then $\ker D$ is a band.*
3. *If $\ker D$ is a band and A has an identity element, then $U(A) \subseteq \ker D$.*

Proof. Note that since A is reduced, there are minimal ℓ-prime ℓ-ideals P_i of A such that $\bigcap P_i = \{0\}$ (see [3, Section 10] and [1, 9.3.2]), and by [1, 9.2.5(iii)], each A/P_i is a totally ordered domain. For each P_i, take $0 \leq x_i \in P_i$. By [5, Theorem 3.2.22], there exists $0 \leq y_i \notin P_i$ such that $x_iy_i = 0$. Then $0 = D(x_iy_i) = x_iD(y_i) + D(x_i)y_i$ and hence $D(x_i)y_i = 0$. Since $y_i \notin P_i$, $D(x_i) \in P_i$ by [1, 9.3.1(iv)]. That is, $D(P_i) \subseteq P_i$. Now let $\pi_i : A \to A/P_i$ be the usual projection $\pi_i(a) = a + P_i$. Define $D_i : A/P_i \to A/P_i$ by $D_i(a + P_i) = D(a) + P_i$. Since $D(P_i) \subseteq P_i$, D_i is well-defined, and it is straightforward to verify that D_i is a positive derivation on A/P_i.

(1) By Theorem 3.6, $\ker D_i$ is a band in A/P_i, and it is not difficult to check that $\ker D = \bigcap_i \pi_i^{-1}(\ker D_i)$.

(2) We will show that each $\pi_i^{-1}(\ker D_i)$ is a band of A. Let $X \subseteq \pi_i^{-1}(\ker D_i)$ and suppose that $x = \bigvee X$ exists in A. Then by hypothesis, $\pi_i(x) = \bigvee \pi_i(X)$ exists in A/P_i, and since $\pi_i(X) \subseteq \ker D_i$, $\pi_i(x) \in \ker D_i$ by Theorem 3.6. That is, $x \in \pi_i^{-1}(\ker D_i)$. So each $\pi_i^{-1}(\ker D_i)$ is a band of A, and since $\ker D = \bigcap_i \pi_i^{-1}(\ker D_i)$ by (1), $\ker D$ is also a band [3, Proposition 21.4].

(3) If A has an identity element, then $U(A) \subseteq \ker D$ by Lemma 3.5.

A collection of reduced f-rings that satisfy the conditions of Theorem 3.7(2) is the following.

Corollary 3.8 *Suppose that $\{T_\alpha\}$ is a collection of reduced totally ordered rings and recall that $\prod_\alpha T_\alpha$ is a reduced f-ring with respect to the coordinatewise operations and order. If R is an ℓ-subring of $\prod_\alpha T_\alpha$ that contains $\sum_\alpha T_\alpha$ and $D \in \mathcal{D}(R)$, then* ker D *is a band, and if as well R has an identity element, then $U(R) \subseteq$* ker D.

Proof. We will show that each projection $\pi_\alpha : R \to T_\alpha$ preserves infinite sups. Suppose by way of contradiction that the projection π_γ does not preserve infinite sups. Then there exists $\{x_i\} \subseteq R$ such that $\bigvee_i x_i$ exists in R but $\pi_\gamma(\bigvee_i x_i) \neq \bigvee_i \pi_\gamma(x_i)$. Then since T_γ is totally ordered and $\pi_\gamma(\bigvee_i x_i) > \pi_\gamma(x_i)$ for all i, there exists $0 < t \in T_\gamma$ such that $\pi_\gamma(\bigvee_i x_i) - t \geq \pi_\gamma(x_i)$ for all i. Since $\sum_\alpha T_\alpha \subseteq R$, the following element $\bar{t}$ exists in R:

$$\bar{t}_\alpha = \begin{cases} t & \text{if } \alpha = \gamma \\ 0 & \text{if } \alpha \neq \gamma \end{cases}.$$

Then $\pi_\gamma(\bigvee_i x_i - \bar{t}) = \pi_\gamma(\bigvee_i x_i) - t \geq \pi_\gamma(x_i)$ for all i and $\pi_\alpha(\bigvee_i x_i - \bar{t}) = \pi_\alpha(\bigvee_i x_i) \geq \pi_\alpha(x_i)$ for all i if $\alpha \neq \gamma$. So $\bigvee_i x_i > \bigvee_i x_i - \bar{t} \geq x_i$ for all i, a contradiction. It follows that each π_α preserves infinite sups and hence by Theorem 3.7 that ker D is a band and if R has an identity element as well, that $U(R) \subseteq$ ker D.

Example 2.2 shows that Corollary 3.8 may fail if R is not reduced.

Acknowledgement

Jingjing Ma made many excellent suggestions that improved the paper enormously. The author thanks Professor Ma for his help.

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