

Further Geometric Properties of a Subclass of Univalent Functions

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Abstract

This present paper aims to investigate further, certain characterization properties for a subclass of univalent function defined by a generalized differential operator. In particular, necessary and sufficient conditions for the function $f(z)$ to belong to the subclass $\varphi_{\mu}^n(\beta, \alpha)$ is established. Additionally, we provide the δ -neighborhood properties for the function $[f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0] \in \varphi_{\mu}^n(\beta, \alpha)$ by making use of the necessary and sufficient conditions. The results obtained are new geometric properties for the subclass $\varphi_{\mu}^n(\beta, \alpha)$.

Keywords: Analytic Functions; Univalent Functions; Differential Operator; Neighborhood.

1. Introduction

Let A denotes the class of functions $f(z)$ which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let the class of all functions in A which are univalent in U be denoted by the symbol S and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

It is well known that any function $f \in S$ has the Taylor series expansion of the form (1), for details (see Duren [1] and Pommerenke [2]). The form (1) is the normalized form of functions $f(z) \in A$ for which the normalization condition is given by

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Thus,

$$S = \{f \in A : f(0) = f'(0) - 1 = 0\}.$$

Some well-known properties of functions in the class S can be found elsewhere (see [1], [3] and [4]), while some special classes of univalent functions have also been investigated by various authors (see [5], [6], [7], [8], [9], [10] and [11]).

Furthermore, we denote by T the subclass of A consisting of functions $f(z) \in A$ which are analytic and univalent in U and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0 \quad (2)$$

The class $\varphi_{\mu}^n(\beta, \alpha)$, a subclass of univalent functions was introduced and studied by Oyekan [10]. For this class, the author established both convolution and inclusion properties for the class. Other subsequent work on the class can be found in Oyekan and Kehinde [12].

Definition 1: [10] A function $f(z) \in A$ is in the class $\varphi_{\mu}^n(\beta, \alpha)$ of provided $D_{\mu,p}^n[f(z)]' \in p(\alpha)$. That is, if

$$\operatorname{Re} \left[D_{\mu,p}^n(f(z))' \right] > \alpha, z \in U, \text{ for } 0 \leq \alpha < 1, 1 \leq \mu \leq \beta, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We note that $p(\alpha) \in P$ which is the class of the Caratheodory functions.

In the sequel, we shall state and prove our new results for the class $\varphi_{\mu}^n(\beta, \alpha)$. These new results presented in section 2, are motivated by the results in Opoola [9].

2. Results and discussion

2.1. Necessary and sufficient conditions

Theorem 2.1: Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$.

If

$$z + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k| < 1 - \alpha,$$

then $f(z) \in \varphi_{\mu}^n(\beta, \alpha)$.

Proof: It suffices to show that

$$\left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| < 1 - \alpha, 0 \leq \alpha < 1$$

Now,

$$\begin{aligned} \left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| &= \left| 1 + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n a_k z^{k-1} - 1 \right| \\ &= \left| \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n a_k z^{k-1} \right| = \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k| |z|^{k-1} \\ &\leq \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k|. \end{aligned}$$

Thus, by the condition of the theorem, we have that

$$\left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| < 1 - \alpha.$$

Hence, the proof is complete.

Theorem 2.2: A function $f(z)$ of the form given by (2) belongs to the class $\varphi_{\mu}^n(\beta, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k| < 1 - \alpha, 0 \leq \alpha < 1.$$

Proof: Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \varphi_{\mu}^n(\beta, \alpha)$, $a_k \geq 0$.

Then

$$\operatorname{Re} \left(D_{\mu, \beta}^n f(z) \right)' > \alpha, \quad (3)$$

Which implies

$$\left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| < 1 - \alpha \quad (4)$$

$$\left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| = \left| 1 + \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n a_k z^{k-1} - 1 \right|$$

$$\operatorname{Re} \left(\sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n a_k z^{k-1} \right) < 1 - \alpha. \quad (5)$$

Taking values of z on real axis and letting $z \rightarrow -1$ through real values we have
From (5) that

$$\sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k| < 1 - \alpha.$$

Conversely,

$$\begin{aligned} \left| \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n a_k z^{k-1} \right| &\leq \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k| \\ &= \sum_{k=2}^{\infty} k[k(1 + \beta - \mu)]^n |a_k|. \end{aligned}$$

Hence, by the condition of the theorem we have that

$$\left| \left(D_{\mu, \beta}^n f(z) \right)' - 1 \right| < 1 - \alpha.$$

Consequently

$$\operatorname{Re}\left(D_{\mu,\beta}^n f(z)\right)' > \alpha,$$

And hence

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \varphi_{\mu}^n(\beta, \alpha).$$

2.2 Neighborhoods for $\varphi_{\mu}^n(\beta, \alpha)$

Let $f(z) \in \varphi_{\mu}^n(\beta, \alpha)$ and $\delta \geq 0$, we define the δ – neighborhood of $f(z)$ as

$$N_{\delta}(f) := \{g \in A : g(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \varphi_{\mu}^n(\beta, \alpha) \text{ and } \sum_{k=2}^{\infty} b_k |a_k - b_k| \leq \delta\} \quad (6)$$

In particular, for the identity function $e(z) = z$, we immediately have

$$N_{\delta}(e) := \{g \in A : g(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \varphi_{\mu}^n(\beta, \alpha) \text{ and } \sum_{k=2}^{\infty} k |b_k| \leq \delta\}. \quad (7)$$

The concept of neighborhood of analytic functions above was sequel to the works of Goodman [13] and Ruscheweyh [14]. The main goal in this subsection is to investigate the δ – neighborhood of $f(z) \in \varphi_{\mu}^n(\beta, \alpha)$ with negative coefficients.

Theorem 2.3: *If*

$$\delta = \frac{1-\alpha}{[2(1+\beta-\mu)]^n}, \quad (8)$$

Then $\varphi_{\mu}^n(\beta, \alpha) \subset N_{\delta}(e)$.

Proof: Let $f(z) \in \varphi_{\mu}^n(\beta, \alpha)$.

Then from Theorem 2.1, we have that

$$\sum_{k=2}^{\infty} k [k(1+\beta-\mu)]^n |a_k| < 1-\alpha,$$

Which implies that

$$[2(1+\beta-\mu)]^n \sum_{k=2}^{\infty} k |a_k| < 1-\alpha.$$

That is,

$$\sum_{k=2}^{\infty} k |a_k| < \frac{1-\alpha}{[2(1+\beta-\mu)]^n},$$

Which by (7) gives that $f(z) \in N_{\delta}(e)$.

Hence,

$$\varphi_{\mu}^n(\beta, \alpha) \subset N_{\delta}(e).$$

3. Conclusion

For the class $\varphi_{\mu}^n(\beta, \alpha)$, various results have been obtained and can be found in [10, 12]. Whereas, the results presented in this present work are new geometric properties for the class.

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