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Research paper

Further Geometric Properties of a Subclass of Univalent Functions

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Abstract

This present paper aims to investigate further, certain characterization properties for a subclass of univalent function defined by a generalized differential operator. In particular, necessary and sufficient conditions for the function $f(z)$ to belong to the subclass $\varphi_\mu^n(\beta,\alpha)$ is established. Additionally, we provide the **δ**-neighborhood properties for the function $[f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0] \in \varphi_\mu^n(\beta,\alpha)$ by making use of the necessary and sufficient conditions. The results obtained are new geometric properties for the subclass $\varphi_\mu^n(\beta,\alpha)$.

Keywords: *Analytic Functions; Univalent Functions; Differential Operator; Neighborhood.*

1. Introduction

Let A denotes the class of functions f(z) which are analytic in the unit disk $U = \{z \in C: |z| < 1\}$. Also, let the class of all functions in A which are univalent in U be denoted by the symbol S and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n \tag{1}$$

It is well known that any function $f \in S$ has the Taylor series expansion of the form (1), for details (see Duren [1] and Pommerenke [2]). The form (1) is the normalized form of functions $f(z) \in A$ for which the normalization condition is given by

$f(0) = 0$ and $f'(0) = 1$.

Thus,

$S = \{f \in A: f(0) = f'(0) - 1 = 0\}$.

Some well-known properties of functions in the class S can be found elsewhere (see [1], [3] and [4]), while some special classes of univalent functions have also been investigated by various authors (see [5], [6], [7], [8], [9], [10] and [11]).
Furthermore, we denote by T the subclass of A consisting of functions $f(z) \in A$ which are analytic and univalent in U and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0 \tag{2}$$

The class $\varphi_\mu^n(\beta,\alpha)$, a subclass of univalent functions was introduced and studied by Oyekan [10]. For this class, the author established both convolution and inclusion properties for the class. Other subsequent work on the class can be found in Oyekan and Kehinde [12].

Definition 1: *[10] A function $f(z) \in A$ is in the class $\varphi_\mu^n(\beta,\alpha)$ of provided $D_{\mu,p}^n[f(z)]' \in p(\alpha)$. That is, if*

$$\mathrm{Re}\left[D_{\mu,p}^n\left(f(z)\right)'\right] > \alpha, z \in U, \text{for } 0 \leq \alpha < 1, 1 \leq \mu \leq \beta, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We note that $p(\alpha) \in P$ which is the class of the Caratheodory functions.
In the sequel, we shall state and prove our new results for the class $\varphi_\mu^n(\beta,\alpha)$. These new results presented in section 2, are motivated by the results in Opoola [9].

2. Results and discussion

2.1. Necessary and sufficient conditions

Theorem 2.1: *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$.

If

$z + \sum_{k=2}^{\infty} k[k(1+\beta-\mu)]^n |a_k| < 1-\alpha$,

then $f(z) \in \varphi_\mu^n(\beta,\alpha)$.

Proof: It suffices to show that

$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| < 1-\alpha, 0 \le \alpha < 1$

Now,

$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| = \left|1 + \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k z^{k-1} - 1\right|$

$= \left|\sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k z^{k-1}\right| = \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k |z|^{k-1}$

$\le \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]|a_k|$.

Thus, by the condition of the theorem, we have that

$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| < 1-\alpha$.

Hence, the proof is complete.

Theorem 2.2: *A function* $f(z)$ *of the form given by* (2) *belongs to the class* $\varphi_\mu^n(\beta,\alpha)$ *if and only if*

$\sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k < 1-\alpha, 0 \le \alpha < 1$.

Proof: Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \varphi_\mu^n(\beta,\alpha), a_k \ge 0$.
Then

$$\mathrm{Re}\left(D_{\mu,\beta}^n f(z)\right)' > \alpha, \tag{3}$$

Which implies

$$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| < 1-\alpha \tag{4}$$

$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| = \left|1 + \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k z^{k-1} - 1\right|$

$$\mathrm{Re}\left(\sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k z^{k-1}\right) < 1-\alpha. \tag{5}$$

Taking values of z on real axis and letting $z \to -1$ through real values we have
From (5) that

$\sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k < 1-\alpha$.

Conversely,

$\left|\sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, a_k z^{k-1}\right| \le \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n]\, |a_k|$

$= \sum_{k=2}^{\infty} k[k(1+\beta-\mu)^n] a_k$.

Hence, by the condition of the theorem we have that

$\left|\left(D_{\mu,\beta}^n f(z)\right)' - 1\right| < 1-\alpha$.

Consequently

$Re\left(D^n_{\mu,\beta}f(z)\right)' > \alpha,$

And hence

$f(z) = z - \sum_{k=2}^{\infty} a_k\, z^k \in \varphi^n_\mu(\beta,\alpha).$

2.2 Neighborhoods for $\varphi^n_\mu(\beta,\alpha)$

Let $f(z) \in \varphi^n_\mu(\beta,\alpha)$ and $\delta \geq 0$, we define the δ – neighborhood of $f(z)$ as

$$N_\delta(f) := \left\{g \in A: g(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \in \varphi^n_\mu(\beta,\alpha)\ and\ \sum_{k=2}^{\infty} b_k|a_k - b_k| \leq \delta\right\} \tag{6}$$

In particular, for the identity function $\mathrm{e(z) = z}$, we immediately have

$$\mathrm{N_\delta(e)} := \left\{\mathrm{g \in A: g(z) = z} + \sum_{k=2}^{\infty} \mathrm{a_k\, z^k} \in \varphi^n_\mu(\beta,\alpha) \text{ and } \sum_{k=2}^{\infty} \mathrm{k|b_k|} \leq \delta\right\}. \tag{7}$$

The concept of neighborhood of analytic functions above was sequel to the works of Goodman [13] and Ruscheweyh [14].
The main goal in this subsection is to investigate the δ – neighborhood of $\mathrm{f(z)} \in \varphi^n_\mu(\beta,\alpha)$ with negative coefficients.

Theorem 2.3: *If*

$$\delta\ = \frac{1-\alpha}{[2(1+\beta-\mu)]^n}, \tag{8}$$

Then $\varphi^n_\mu(\beta,\alpha) \subset \mathrm{N}\delta(\mathrm{e})$.

Proof: Let $\mathrm{f(z)} \in \varphi^n_\mu(\beta,\alpha)$.
Then from Theorem 2.1, we have that

$\sum_{k=2}^{\infty} \mathrm{k}\,[\mathrm{k}(1 + \beta - \mu)]^n\ |\mathrm{ak}| <\ 1 - \alpha,$

Which implies that

$[2(1 + \beta - \alpha)]^n \sum_{k=2}^{\infty} \mathrm{k|a_k|} < 1 - \alpha.$

That is,

$\sum_{k=2}^{\infty} \mathrm{k|a_k|} < \frac{1 - \alpha}{[2(1+\beta-\mu)]^n},$

Which by (7) gives that $\mathrm{f(z)} \in \mathrm{N_\delta(e)}$.
Hence,

$\varphi^n_\mu(\beta,\alpha) \subset \mathrm{N_\delta(e)}.$

3. Conclusion

For the class $\varphi^n_\mu(\beta,\alpha)$, various results have been obtained and can be found in [10, 12]. Whereas, the results presented in this present work are new geometric properties for the class.

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