On a Subclass of Multivalent Functions with Bounded Positive Real Part

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Abstract

In the present paper, by introducing a new subclass of multivalent functions with respect to \((j, k)\) - symmetric points, we have obtained the integral representations and conditions for starlikeness using differential subordination.

Keywords: multivalent functions; \((j, k)\) - symmetric points; Differential subordination.

1. Introduction, Definitions And Preliminaries

Let \(\mathcal{H}\) be the class of functions analytic in the open unit disc \(U = \{z \in \mathbb{C} : |z| < 1\}\). Let \(\mathcal{H}(a, m)\) be the subclass of \(\mathcal{H}\) consisting of functions of the form \(f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \ldots\). Let \(\mathcal{A}_k\) be the class of functions \(f(z)\), of the form

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n
\]

(1)

which are analytic in the unit disc \(U = \{z \in \mathbb{C} : |z| < 1\}\). And let \(\mathcal{A} = \mathcal{A}_1\).

We denote by \(\mathcal{S}, \mathcal{C}, \mathcal{H}\) and \(\mathcal{E}\) the familiar subclasses of \(\mathcal{A}\) consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in \(U\).

Let \(\mathcal{P}\) be the subclass of \(\mathcal{A}\) consisting of all functions which are univalent in \(U\). Also, let \(\mathcal{P}\) denote the class of functions of the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n
\]

which are analytic and convex in \(U\) and satisfy the condition

\[
\Re(p(z)) > 0, \quad (z \in U).
\]

Let \(f(z)\) and \(g(z)\) be analytic in \(U\). Then we say that the function \(f(z)\) is subordinate to \(g(z)\) in \(U\), if there exists an analytic function \(w(z)\) in \(U\) such that \([w(z)] < |z|\) and \(f(z) = g(w(z))\), denoted by \(f(z) \prec g(z)\). If \(g(z)\) is univalent in \(U\), then the subordination is equivalent to \(f(0) = g(0)\) and \(f(U) \subset g(U)\).

Motivated by the concept introduced by Sakaguchi in [8], recently several subclasses of analytic functions with respect to \(k\)-symmetric points were introduced and studied by various authors (see [1], [2], [9], [10] and [12]). Parvatham in ([7]) introduced and investigated \(K_{\alpha}(\alpha, h)\) - so called class of \(\alpha\) starlike functions with respect to \(n\) symmetric points.

Let \(k\) be a positive integer and \(j = 0, 1, 2, \ldots (k - 1)\). A domain \(D\) is said to be \((j, k)\)-fold symmetric if a rotation of \(D\) about the origin through an angle \(2\pi j/k\) carries \(D\) onto itself. A function \(f \in \mathcal{A}\) is said to be \((j, k)\)-symmetrical if for each \(z \in U\)

\[
f(\varepsilon z) = \varepsilon^j f(z),
\]

(2)

where \(\varepsilon = \exp(2\pi i/k)\). The family of \((j, k)\)-symmetrical functions will be denoted by \(\mathcal{S}_k^j\). For every function \(f\) defined on a symmetrical subset \(U\) of \(\mathbb{C}\), there exists a unique sequence of \((j, k)\)-symmetrical functions \(f_{j,k}(z), j = 0, 1, \ldots, k - 1\) such that

\[
f = \sum_{j=0}^{k-1} f_{j,k}.
\]
Also let \( f_{j,k}(z) \) be defined by the following equality
\[
f_{j,k}(z) = \frac{1}{k} \sum_{v=j}^{k-1} f(e^{v}z), \quad (f \in \mathcal{A}_p; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1)).
\] (3)

where, \( v \) is an integer.

This decomposition is a generalization of the well known fact that each function defined on a symmetrical subset \( U \) of \( \mathbb{C} \) can be uniquely represented as the sum of an even function and an odd functions (see Theorem 1 of [5]). It is obvious that \( f_{j,k}(z) \) is a linear operator from \( U \) into \( \mathbb{U} \). The notion of \((j, k)\)-symmetrical functions was first introduced and studied by P. Liczberski and J. Polubiński in [5].

The following identities directly follow from (3):
\[
\begin{align*}
f_{j,k}(e^vz) &= e^{vp}f_{j,k}(z) \\
f_{j,k}'(e^vz) &= e^{vp}f_{j,k}'(z) \\
f_{j,k}''(e^vz) &= e^{vp-2v}f_{j,k}''(z)
\end{align*}
\] (4)

In [4], Kirthikeyan et al., investigated the class
\[
\mathcal{S}_p^{s,t} (b, \alpha, \beta) = \left\{ f \in \mathcal{A}_p : \alpha < \Re \left( 1 + \frac{1}{b} \left( (1-\lambda)f_j(z) + \lambda zf_j'(z) \right) - m \right) \right\}, \quad (z \in \mathbb{U}),
\]
where, \( 0 \leq \alpha < 1 < \beta, 0 \leq \lambda \leq 1 \) and \( f_{j,k}(z) \neq 0 \) is defined by the equality (3). Similarly, we say that a function \( f \in \mathcal{A}_p \) is in the class \( \mathcal{S}_p^{s,t} (\gamma, \lambda, \alpha, \beta) \) if and only if
\[
z f'(z) \in \mathcal{S}_p^{s,t} (\gamma, \lambda, \alpha, \beta).
\]

Remark 1.1. If \( \lambda = 0, j = k = p = 1 \) and \( \alpha \geq 0 \), then \( f(z) \) reduces to the well-known class of starlike functions of complex order. Similarly, if we let \( \lambda = 1, j = k = p = 1 \) and \( \alpha \geq 0 \), then \( f(z) \) reduces to the well-known class of convex functions of complex order.

We observe that for a given \( \alpha \) and \( \beta (0 \leq \alpha < 1 < \beta) \), \( f \in \mathcal{S}_p^{s,t} (\gamma, \lambda, \alpha, \beta) \) satisfies each of the following subordination equations
\[
1 + \frac{1}{t} \left( (1-\lambda)f_j(z) + \lambda zf_j'(z) \right) - \frac{1 + (1 - 2\alpha)z}{1 - z} \leq 0
\]
and
\[
1 + \frac{1}{t} \left( (1-\lambda)f_j(z) + \lambda zf_j'(z) \right) - \frac{1 + (1 - 2\beta)z}{1 - z} \leq 0.
\]

Both superordinate functions in the above expressions maps the unit disc onto right half plane, so it is obvious that the above expression is mapped on to a plane having real part greater than \( \alpha \) but less than \( \beta \).

Kuroki and Owa [3], defined an analytic function \( p : \mathbb{U} \to \mathbb{C} \) by
\[
p(z) = 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - 2\pi e^{(1-m)z}}{1 - z} \right).
\]

The above function \( p \) maps \( \mathbb{U} \) onto a convex domain \( \Lambda = \{ w : \alpha < \Re \{ w \} < \beta \} \), conformally. Using this fact and the definition of subordination, we can obtain the following:

Let \( f \in \mathcal{A}_p \) and \( 0 \leq \alpha < 1 < \beta \). Then \( f \in \mathcal{S}_p^{s,t} (\gamma, \lambda, \alpha, \beta) \) if and only if
\[
1 + \frac{1}{t} \left( (1-\lambda)f_j(z) + \lambda zf_j'(z) \right) - \frac{1 + (1 - 2\alpha)z}{1 - z} \leq p(z),
\]
and \( p(z) \) is of the form
\[
p(z) = 1 + \sum_{n=1}^{\infty} c_nz^n,
\]
and
\[
c_n = \left( \frac{\beta - \alpha}{n\pi} \right) i \left( 1 - e^{2n\pi i (1-m)} \right).
\]
Lemma 1.1. [6] Let the functions $q$ be univalent in the open unit disc $U$ and $\Theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $K(z) = \Theta(q(z)) + Q(z)$. Suppose that

1. $Q$ is starlike univalent in $U$ and
2. $\Re\left(\frac{\Theta(z)}{\Theta_1(z)}\right) > 0$ for $z \in U$.

If

$$\Theta(p(z)) + zp'(z)\phi(p(z)) < \Theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) < q(z)$ and $q$ is the best dominant.

2. **Main Results**

In this section, we prove the integral representation of the function class $\mathcal{H}_{\beta}^{j,k}(\gamma, \lambda, \alpha, \beta)$.

**Theorem 2.1.** Let $f \in \mathcal{H}_{\beta}^{j,k}(\gamma, \lambda, \alpha, \beta)$ with $0 \leq \alpha < 1 < \beta$ and $0 < \lambda \leq 1$. Then we have

$$f_{j,k}(z) = \frac{1}{\lambda} \mathcal{L}^{-j-\frac{1}{2}} \int_0^z \exp \left\{ \frac{\gamma}{k} \sum_{v=0}^{k-1} \left[ \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i (1-v)}{\gamma}} w(e^{\nu} \zeta) \right) \right] \right\} z^{1+\nu-2} du$$

(6)

where $f_{j,k}(z)$ defined by (3), $w(z)$ is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$.

**Proof.** Let $f \in \mathcal{H}_{\beta}^{j,k}(\gamma, \lambda, \alpha, \beta)$ with $0 \leq \alpha < 1 < \beta$ and $0 < \lambda \leq 1$. Then we have

$$1 + \frac{1}{\gamma} \left( \frac{1 - \lambda}{1 - \lambda} z f'(z) + \lambda z f'(z) \right) = 1 + \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(z) \right)$$

(7)

where $w(z)$ is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$. Substituting $z$ by $e^{\nu}z$ in (7), we have

$$1 + \frac{1}{\gamma} \left( \frac{1 - \lambda}{1 - \lambda} z f'(z) + \lambda z f'(z) \right) = 1 + \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right)$$

(8)

Using the identities (4), we have

$$1 + \frac{1}{\gamma} \left( \frac{1 - \lambda}{1 - \lambda} z f'(z) + \lambda z f'(z) \right) = 1 + \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right)$$

(9)

On simplifying, we get

$$1 + \frac{1}{\gamma} \left( \frac{1 - \lambda}{1 - \lambda} z f'(z) + \lambda z f'(z) \right) = 1 + \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right)$$

(10)

Let $\nu = 0, 1, 2, \ldots, (k-1)$ in (10) respectively and summing them, we get

$$1 + \frac{1}{\gamma} \left( \frac{1 - \lambda}{1 - \lambda} z f'(z) + \lambda z f'(z) \right) = 1 + \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right)$$

(11)

From this equality, we get

$$\frac{(1 - \lambda) z f'(z) + \lambda z f'(z)}{(1 - \lambda) f_{j,k}(z) + \lambda f_{j,k}(z)} = \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right)$$

(12)

Integrating, we get

$$\log \left( \frac{(1 - \lambda) f_{j,k}(z) + \lambda f_{j,k}(z)}{z^\nu} \right) = \frac{\gamma}{k} \sum_{v=0}^{k-1} \left\{ \frac{1}{\nu} \int \left( \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right) \right) \right\} dt$$

(13)

Simplifying (13), we have

$$\left(1 - \lambda\right) f_{j,k}(z) + \lambda z f_{j,k}(z) = z^\nu \exp \left\{ \frac{\gamma}{k} \sum_{v=0}^{k-1} \left\{ \frac{1}{\nu} \int \left( \frac{\beta - \alpha}{\pi} \log \left( 1 - e^{\frac{2\pi i}{\gamma}} w(e^{\nu} z) \right) \right) \right\} dt \right\}$$

(14)

A simple computation in (14), gives the required conclusion of this theorem.
Theorem 2.2. Let $f \in \mathcal{S}_p^{{\lambda},k}(\gamma, \lambda, \alpha, \beta)$ with $0 \leq \alpha < 1 < \beta$ and $0 < \lambda \leq 1$. Then we have

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^1 \left( \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right) d\zeta$$

where $w(z)$ is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$.

Proof. From (7), we have

$$(1 - \lambda) zf'(z) + \lambda z (zf'(z))' = (1 - \lambda) f_{j,k}(z) + \lambda zf_{j,k}'(z)$$

$$\times \left[ p + \frac{\gamma(\beta - \alpha)}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right].$$

From (14) and (16), we have

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^1 \left( \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right) d\zeta$$

On simplifying and integrating the above equality (17), we get (15).

If we put $\lambda = 1, j = k = 1$ in Definition 1.1 and Theorem 2.1, we get the following corollary:

Corollary 2.3. If $f \in \mathcal{S}_p$ satisfies the analytic condition

$$\alpha < \Re \left\{ 1 + \frac{1}{\gamma} \left( 1 + \frac{z f''(z)}{f'(z)} - p \right) \right\} < \beta,$$

then the integral representation of $f(z)$ is given by

$$f(z) = \int_0^1 t^{p-1} \exp \left\{ \gamma \sum_{k=0}^{k-1} \int_0^t 1 \left( \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right) d\zeta \right\} dt.$$  

Remark 2.1. If we put $\lambda = 1, j = k = 1$ in (5) then this result is reduced into the Corollary 2.5 in [4].

Remark 2.2. If we put $\lambda = 0, j = k = 1$ in (14), then

$$f(z) = z^p \exp \left\{ \gamma \sum_{k=0}^{k-1} \int_0^t 1 \left( \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right) d\zeta \right\}.$$

Take $p = 1$, this result was proved by K.Kuroki and S.Owa [3].

Theorem 2.4. Let the function $h(z)$ analytic in $U$ be defined by

$$h(z) = \delta + (\delta + \kappa) \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - z} \right) + \kappa \left( \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - w(z)} \right) \right)$$

$$\kappa \left( \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - z} \right) \right)^2 - \kappa \left( \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{(1-z)}{1-w(z)}}}{1 - z} \right) \right)^2$$

where $\kappa > 0, \kappa + \delta > 0$. If $f \in \mathcal{S}$ with $F_{j,k}(z) \neq 0$ satisfies the condition

$$\delta + \frac{\delta + \kappa}{\gamma} \left[ \frac{F(z)}{F_{j,k}(z)} - 1 \right] + \kappa \left[ \frac{F(z)}{F_{j,k}(z)} - 1 \right]^2 + \kappa \left[ \frac{\zeta F(z)}{F_{j,k}(z)} - \frac{\zeta F(z) F_{j,k}(z)}{(F_{j,k}(z))^2} \right] \prec h(z),$$

where

$$F(z) = (1 - \lambda) zf'(z) + \lambda z (zf'(z))' \quad \text{and} \quad F_{j,k}(z) = (1 - \lambda) f_{j,k}(z) + \lambda zf_{j,k}'(z)$$

then $f \in \mathcal{S}_p^{{\lambda},k}(\gamma, \lambda, \alpha, \beta)$. 

Proof. Let the function \( p(z) \) be defined by

\[
p(z) = 1 - \frac{1}{\gamma} \left( \frac{F(z)}{F_{j,k}(z)} - 1 \right) \quad (z \in \mathbb{U}; z \neq 0; f \in \mathcal{A}),
\]

where \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}, F(z) \) and \( F_{j,k}(z) \) defined by (20). On simplification, we get

\[
z p'(z) = 1 - \frac{z F'(z) - z F(z) F'_{j,k}(z)}{F_{j,k}(z) - F_{j,k}(z)^2}.
\]

Thus by (19), we have

\[
\kappa z p'(z) + \kappa p^2(z) + (\delta - \kappa) p(z) \prec h(z).
\]  \hspace{1cm} (22)

Let

\[
g(z) = 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{2\pi i (\frac{1-\alpha}{\gamma})} w(z)}{1 - w(z)} \right).
\]  \hspace{1cm} (23)

Set

\[
\theta(w) = \kappa w^2 + (\delta - \kappa) w \quad \text{and} \quad \phi(w) = \kappa,
\]

it can be easily verified that \( \theta \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C} \) with \( \phi(0) \neq 0 \) in the \( w \)-plane. Also, let

\[
Q(z) = zg'(z)\phi(g(z)) = \kappa zg'(z)
\]

and

\[
h(z) = \theta(g(z)) + Q(z) = \kappa (g(z))^2 + (\delta - \kappa) g(z) + \kappa zg'(z).
\]

Since \( g(z) \) is convex univalent in \( \mathbb{U} \) provided \( \alpha \geq 0 \), it gives that \( Q(z) \) is starlike univalent in \( \mathbb{U} \). In view of the result proved in [3], \( g(z) \) given by (23) is starlike for \( \alpha \geq 0 \), we have

\[
\Re \left( \frac{z h'(z)}{Q(z)} \right) = \Re \left\{ \kappa \left( \frac{g(z)}{z g'(z)} (g(z) - 1) + 1 \right) + \delta \frac{\theta(z)}{z g'(z)} \right\} > 0.
\]

By the application of Lemma 1.1, we get the required assertion of this theorem. \( \square \)

If we put \( \lambda = 0, \gamma = 1 \) in Theorem 2.4, we get the following corollary:

**Corollary 2.5.** Let the function \( h(z) \) be defined as in (18). If \( f \in \mathcal{A} \) with \( \frac{f(z)}{z} \neq 0 \) satisfies the condition

\[
\kappa \left\{ \frac{zf''(z)}{f_{j,k}(z)} - \frac{zf'(z)f'_{j,k}(z)}{(f_{j,k}(z))^2} + \frac{zf'(z)^2}{(f_{j,k}(z))^2} \right\} + \delta \frac{zf'(z)}{f_{j,k}(z)} \prec h(z),
\]

then

\[
\frac{zf'(z)}{f_{j,k}(z)} < 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{2\pi i (\frac{1-\alpha}{\gamma})} z}{1 - z} \right).
\]

**Remark 2.3.** If we take \( j = k = 1 \) in the corollary 2.5, then this result was analogous to the result obtained by Xu et al. in [11].

**References**


