Research Paper

Construction of generalized atomic decompositions in Banach spaces

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Abstract

G-atomic decompositions for Banach spaces with respect to a model space of sequences have been introduced and studied as a generalization of atomic decompositions. Examples and counter example have been provided to show its existence. It has been proved that an associated Banach space for G-atomic decomposition always has a complemented subspace. The notion of a representation system is introduced and exhibits its relation with G-atomic decomposition. Also It has been observed that G-atomic decompositions are exactly compressions of Schauder decompositions for a larger Banach space. We give a characterization for finite G-atomic decomposition in terms of finite-dimensional expansion of identity.

Keywords: complemented coefficient spaces, finite-dimensional expansion of identity, G-atomic decomposition, representation system.

1. Introduction

Frames are main tools for use in signal and image processing, compression, sampling theory, optics, filter banks, signal detection etc. In order to have many more uses of frames, several notions generalizing the concept of frames have been introduced and studied, namely; Banach frames [13], pseudo frames [15], oblique frames [7], frames of subspaces (fusion frames) [4, 5], G-frames [20] etc.

Coifman and Weiss [9] introduced a concept, similar to that of frames, called atomic decompositions for function spaces. Later, the concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Grochenig [11] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Grochenig [13] who introduced the notion of Banach frames for Banach spaces. Frazier and Jawerth [12] had constructed wavelet atomic decompositions for Besov spaces, which they called “ϕ-transform”. Feichtinger [10] constructed Gabor atomic decomposition for the modulation spaces. Christensen [6] in 1996, studied atomic decomposition via group representation, while Christensen and Heil [8], in 1997, discussed stability of atomic decompositions for Banach spaces under small perturbations. Casazza, Han and Larson [3] relate an atomic decomposition with several forms of the approximation property in Banach space theory and with Banach frames. Banach frames and atomic decompositions were further studied in [1, 2].

In this article we generalize the classical construction of Pelczynski [16]. In fact, we introduce the notion of G-atomic decomposition for Banach spaces. It has been proved that an associated Banach space, for G-atomic decomposition always has a complemented subspace. We define representation system and exhibits its relation with
G-atomic decomposition. Also, it has been observed that G-atomic decompositions are exactly compressions of Schauder decompositions for a larger Banach space. We give a characterization for finite G-atomic decomposition in terms of finite-dimensional expansion of identity.

2. Preliminaries

Throughout this paper, $E$ will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $E^*$ the conjugate space of $E$, and $L(E, F)$ will denote the Banach space of all continuous linear operators from $E$ into $F$.

A sequence $\{x_n\}$ in $E$ is said to be complete if $[x_n] = E$ and a sequence $\{f_n\}$ in $E^*$ is said to be total over $E$ if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. In the case where $F = E$, we write $L(E) = L(E, E)$. A sequence $\{v_n\} \subset L(E)$ is said to be total on $E$ if $v_n(x) = 0$, for all $n \in \mathbb{N}$, implies $x = 0$.

By a Banach sequence space (often called a BK-space) we mean a Banach space of scalar sequences, indexed by $\mathbb{N}$, for which the coordinate functionals are continuous. We say that the space is a Schauder sequence space if, in addition, the unit vectors $(e_i)$ given by $(e_i)_j = \delta_{ij}$ (where $\delta_{ij}$ is the Kronecker delta) form a basis for it.

Definition 2.1 ([11]). Let $E$ be a Banach space and let $E_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in $E$ and let $\{f_n\}$ be a sequence in $E^*$. Then, the pair $(\{f_n\}, \{x_n\})$ is called an atomic decomposition for $E$ with respect to $E_d$, if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$

(b) there exist constants $A, B$ with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$$

(c) $x = \sum_{n=1}^{\infty} f_n(x) n, \text{ for all } x \in E$.

The positive constants $A, B$ are called atomic bounds for the atomic decomposition $(\{f_n\}, \{x_n\})$.

Definition 2.2 ([18]). A sequence $\{G_n\}$ of subspaces of $E$ is a decomposition of $E$ if for each $x \in E$ there exists a unique sequence $\{x_n\}$ in $E$ such that

$$x = \sum_{n=1}^{\infty} x_n, \quad x_n \in G_n \text{ for each } n,$$

the convergence being in the norm topology of $E$. Uniqueness implies the existence of projections (not necessarily continuous) $v_n$ from $E$ onto $G_n$ such that $v_n v_j = \delta_{ij} v_j$, where $\delta_{ij}$ is the Kronecker delta. If each $v_n$ is continuous, then decomposition is called a Schauder decomposition.

Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for $E$ with respect to a Banach sequence space $Z$. There is a natural procedure that allows us to replace $Z$ by a Schauder sequence space so that $(\{f_n\}, \{x_n\})$ is also, an atomic decomposition of $E$ with respect to $E_d$ (see [3, Theorem 2.6]).

3. Main results

The theory of spaces of sequences of scalars admits a natural generalization to a vector sequence spaces. If $\Phi = \{G_n\}$ is a sequence of Banach spaces, a sequence space $X_\Phi$ associated with $\{G_n\}$ is a linear subspace of $\prod_{n=1}^{\infty} G_n$ (the collection of all sequences $\{y_n\}$ with $y_n \in G_n$, $n = 1, 2, \ldots$, endowed with product topology). The coordinate operators $P_n : X_\Phi \to G_n$ are defined by $P_n(\{y_n\}) = y_n$, $n = 1, 2, \ldots$. Then $X_\Phi$ is called a generalized BK-space induced by $\{G_n\}$ if $X_\Phi$ is a Banach space and $P_n$ is a continuous operator on $X_\Phi$, for every $n \in \mathbb{N}$. The scalar BK-spaces containing all unit vectors $e_n$ are generalized by the spaces $X_\Phi$ containing all canonical subspaces

$$F_n = \{0\} \times \ldots \times \{0\} \times G_n \times \{0\} \times \ldots \quad (G_n \neq \{0\}, n = 1, 2 \ldots).$$

These $F_n$’s are closed linear subspaces of $X_\Phi$. We refer to the space $X_\Phi$ as a model space.
The following is the example of such type of a model space.

Let \( \Phi = \{G_n\} \) be a sequence of closed linear subspaces of a Banach space \( E \). Consider the linear space \( X_\Phi \), that is, the space of all element sequences \( y = \{y_n\}_{n=1}^\infty \) for which the series \( \sum_{n=1}^\infty y_n \) is convergent equipped with the norm

\[
\|y\|_{X_\Phi} = \sup_{n \geq 1} \left\| \sum_{k=1}^n y_k \right\|_E, \quad y_n \in G_n \ (n = 1, 2, \ldots).
\]

The space \( X_\Phi \) is complete with respect to this norm and the system \( \{F_n\} \) defined by above is a Schauder decomposition of \( X_\Phi \). Clearly, any model space \( X_\Phi \) can be obtained by the method described above, indeed, if \( X \) is a model space of the sequence of subspaces \( \Phi = \{F_n\} \) then \( X_\Phi = X \).

We begin with the following generalization of Atomic decomposition.

**Definition 3.1.** Let \( \Phi = \{G_n\} \) be a sequence of non-trivial subspaces of a Banach space \( E \) and \( \{v_n : v_n \in L(E, G_n)\} \) be a sequence of linear operators (not necessarily projections). Let \( X_\Phi \) be a model space associated with \( E \). Then we say \( \{G_n\}, \{v_n\} \) is \( G \)-atomic decomposition for \( E \) with respect to \( X_\Phi \) if

(a) \( \{v_n(x)\} \in X_\Phi \), for all \( x \in E \)

(b) there exist constants \( A, B \) with \( 0 < A \leq B < \infty \) such that

\[
A\|x\|_E \leq \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \quad x \in E
\]

(c) \( x = \sum_{n=1}^\infty v_n(x) \), for all \( x \in E \).

The positive constants \( A \) and \( B \), respectively, are called lower and upper atomic bounds for the \( G \)-atomic decomposition \( \{G_n\}, \{v_n\} \).

Next, we have following lemma on the line of [19, p. 189], which used in the subsequent work.

**Lemma 3.2.** Let \( \{G_n\} \) be a sequence of subspaces of \( E \) and \( \{v_n\} \subset L(E, G_n) \) be a sequence of operators, \( \forall \ n \in \mathbb{N} \). If \( \{v_n\} \) is total over \( E \), then \( X = \{\{v_n(x)\} : x \in E\} \) is a Banach space with norm \( \|\{v_n(x)\}\|_X = \|x\|_E, x \in E \).

Regarding existence of \( G \)-atomic decompositions, we have the following examples. The modified sequence \( \{G_n\} \) used below was constructed in [14].

**Example 3.3.** Consider the Banach space

\[
E = \ell^\infty(\chi) = \{\{x_n\} : x_n \in \chi; \sup_{1 \leq n < \infty} \|x_n\|_\chi < \infty\}
\]

equipped with the norm \( ||\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_\chi \), \( \{x_n\} \in E \), where \( (\chi, \|\cdot\|) \) is a Banach space.

Define a sequence \( \{G_n\} \) of subspaces of \( E \) by

\[
G_{2n-1} = \{\delta_{2n-1}^{x} + 2^n \delta_{2n}^{x} : x \in \chi\}
\]

\[
G_{2n} = \{\delta_{2n}^{x} : x \in \chi\}
\]

where

\[
\delta_n^x = (0, 0, \ldots, 0, x, 0, \ldots) \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \chi.
\]

Define operators \( v_n : \ell^\infty(\chi) \to \ell^\infty(\chi) \) by

\[
v_{2n-1}(x) = \delta_{2n-1}^{x} + 2^n \delta_{2n}^{x}
\]

\[
v_{2n}(x) = 2^n \delta_{2n}^{x} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}.
\]

Then, by Lemma 3.2, there exists an associated model space \( X = \{v_n(x) : x \in E\} \) with norm given by

\[
\|\{v_n(x)\}\|_X = \|x\|_E, \quad x \in E.
\]
Also
\[ \sum_{n=1}^{\infty} v_n(x) = x, \quad x \in E. \]

Therefore \( \{\{G_n\}, \{v_n\}\} \) is \( G \)-atomic decomposition of \( E \) with respect to model space \( X \).

**Example 3.4.** Let \( E = c_0 \) and \( \{e_n\} \) be the unit vector basis in \( c_0 \). Write
\[ G_n = [x_n] \quad \text{and} \quad v_n(x) = f_n(x)x_n, \quad n \in \mathbb{N}, \]
where \( \{x_n\} \subset E \) and \( \{f_n\} \subset E^* \) are given by
\[ x_{2n-1} = 2^{1-n}e_{2n-1} - e_{2n}, \quad x_{2n} = e_{2n} \quad (n \in \mathbb{N}) \]
\[ f_{2n-1} = 2^{n-1}h_{2n-1}, \quad f_{2n} = 2^{n-1}h_{2n-1} + h_{2n} \quad (n \in \mathbb{N}), \]
\( \{h_n\} \) being the sequence of coordinate functionals to \( \{e_n\} \). Then, it can be easily prove that there exist an associated model space \( X = \{v_n(x) : x \in E\} \), such that \( \{\{G_n\}, \{v_n\}\} \) is \( G \)-atomic decomposition for \( E \) with respect to \( X \).

**Example 3.5.** Let \( E \) be a Banach space defined as
\[ E = \ell^2(\chi) = \left\{ \{x_n\} : x_n \in \chi; \sum_{n=1}^{\infty} \|x_n\|_\chi^2 < \infty \right\}, \]
where \( (\chi, \|\cdot\|) \) is a Banach space, equipped with the norm given by
\[ \|\{x_n\}\|_E = \left( \sum_{n=1}^{\infty} \|x_n\|_\chi^2 \right)^{\frac{1}{2}}. \]

Define for \( n \in \mathbb{N}, G_n = \{\delta^n x : x \in \chi\} \) and \( v_n(x) = \delta^n x + \delta^{n+1} x, x = \{x_n\} \in E, \) where \( \delta^n x = (0, 0, \ldots, 0, x, 0, \ldots) \) for all \( n \in \mathbb{N} \) and \( x \in \chi \). But, since for any \( 0 \neq x \in \chi, \delta^n = (x, 0, 0, \ldots) \in E \) is such that \( v_n(\delta^n) = 0, \) for all \( n \in \mathbb{N}, \) there exist no associated model space \( X \) such that \( \{\{G_n\}, \{v_n\}\} \) is a \( G \)-atomic decomposition for \( E \) with respect to \( X \).

**Remark 3.6.** Any Banach space \( E \) admits the trivial \( G \)-atomic decomposition \( \{G_n\} \), where \( G_1 = E \) and \( G_n \neq \{0\} \) \( (n = 2, 3, \ldots) \) are arbitrary with operators \( v_1 = I_E, v_n = 0 \) \( (n = 2, 3, \ldots) \).

**Theorem 3.7.** If \( \{\{G_n\}, \{v_n\}\} \) is a \( G \)-atomic decomposition for \( E \) with respect to \( X_\Phi \), then there exist a complemented coefficientsubspace \( G \) of \( X_\Phi \) and an isomorphism \( T \) from \( E \) into \( X_\Phi \) such that \( X_\Phi = T(E) \oplus G \).

**Proof.** Let \( \{\{G_n\}, \{v_n\}\} \) be a \( G \)-atomic decomposition of \( E \) with respect to \( X_\Phi \) where
\[ X_\Phi = \left\{ \{y_n\} \subset E; \sum_{n=1}^{\infty} y_n \text{ converges; } y_n \in G_n \ (n = 1, 2, \ldots) \right\} \]
eq (1)
eq equipped with norm \( \|\{y_n\}\|_{X_\Phi} = \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} y_i \right\| \).

Then the mapping \( T : E \to X_\Phi \) defined by
\[ T(x) = \{v_n(x)\}, \quad x \in E \]
is an isomorphism from \( E \) into \( X_\Phi \). Since \( \sum_{n=1}^{\infty} v_n(x) \) converges to \( x \) by (1) and
\[ \|x\|_E = \left\| \sum_{i=1}^{\infty} v_i(x) \right\| \leq \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i(x) \right\| \]
\[ = \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \quad x \in E, \]
where
\[ B = \sup_{1 \leq n < \infty} \left( \sum_{i=1}^{n} \|v_i\| \right). \]
where
\[ B = \sup_{1 \leq n < \infty} \|S_n\| < \infty \quad \text{and} \quad S_n(x) = \sum_{i=1}^{n} v_i(x). \]

Now, define \( S : X_\Phi \to E \) by
\[ S(\{x_n\}) = \sum_{i=1}^{\infty} x_i, \quad \{x_n\} \in X_\Phi, \quad n \in \mathbb{N}. \]

Then \( S \) is a bounded linear operator from \( X_\Phi \) to \( E \). Put \( G = \ker S \). Then
\[ G = \left\{ \{x_n\} \subseteq E \mid x_n \in G_n(n = 1, 2, \ldots) \sum_{i=1}^{\infty} x_i = 0 \right\}, \]
is a closed subspace of \( X_\Phi \).

Furthermore, if \( \{v_n(x)\} \in G \) for some \( x \in E \), then
\[ 0 = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x) = x. \]

So
\[ T(E) \cap G = \{0\}. \]

Now, let \( \{x_n\} \in X_\Phi \) be arbitrary such that \( x = \sum_{i=1}^{\infty} x_i \). Then \( \{v_n(x)\} \in T(E) \) such that
\[ \sum_{i=1}^{\infty} (x_i - v_i(x)) = \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{\infty} v_i(x) = x - x = 0. \]

Therefore \( \{x_n - v_n(x)\} \in G \) such that
\[ \{x_n\} = \{v_n(x)\} + \{x_n^{(0)}\}, \quad \text{where} \quad \{v_n(x)\} \in T(E) \quad \text{and} \quad \{x_n^{(0)}\} = \{x_n - v_n(x)\} \in G. \]

Hence, we have \( X_\Phi = T(E) \oplus G \).

**Definition 3.8.** A system \( \Phi = \{G_n\} \) of closed linear subspaces of a Banach space \( E \), with \( G_n \neq \{0\} \) \((n = 1, 2, \ldots)\) is called a representation system of \( E \) with respect to model space \( X_\Phi \) if for every \( x \in E \), there exists a sequence \( \{x_n\} \subseteq E \) with \( x_n \in G_n \) \((n = 1, 2, \ldots)\) such that \( x = \sum_{n=1}^{\infty} x_n \) and \( G = \left\{ \{x_n\} \subseteq E \mid \sum_{n=1}^{\infty} x_n = 0 \right\} \) is a complemented coefficient subspace of \( X_\Phi \).

**Theorem 3.9.** Let \( E \) be a Banach space and \( X_\Phi \) be an associated Banach space indexed by \( \mathbb{N} \). Then \( \Phi = \{G_n\} \) is a representation system if and only if \( \{G_n\}, \{v_n\} \) is a \( G \)-atomic decomposition with respect to \( X_\Phi \).

**Proof.** Necessity. Let \( \Phi = \{G_n\} \) be a representation system of \( E \) then for \( \{x_n\} \subseteq E \) with \( x_n \in G_n \), we have \( x = \sum_{n=1}^{\infty} x_n, \quad n \in \mathbb{N} \). Let \( G = \left\{ \{x_n\} \subseteq E \mid \sum_{n=1}^{\infty} x_n = 0 \right\} \) be a complemented coefficient subspace of \( X_\Phi \) then \( X_\Phi = G \oplus F \) and \( F \) is complemented to \( G \). Define \( S : X_\Phi \to E \) by
\[ S(\{x_n\}) = \sum_{n=1}^{\infty} x_n, \quad \{x_n\} \in X_\Phi, \quad n \in \mathbb{N}. \]

As in Theorem 3.7, \( T \) is an isomorphism from \( E \) into \( X_\Phi \), then \( S|_F \) is an isomorphism from \( F \) onto \( E \). Indeed, if \( S(\{x_n\}) = 0 \) for some \( \{x_n\} \in F \), then \( \sum_{n=1}^{\infty} x_n = 0 \). Hence \( \{x_n\} \in G \cap F = \{0\} \), which proves that \( S|_F \) is one
to one. Also, if \( y \in E \) then, since \( \Phi \) is a representation system, there exists a sequence \( \{y_n\} \in X_\Phi \) such that

\[
y = \sum_{n=1}^{\infty} y_n = S(\{y_n\}),
\]

write

\[
\{y_n\} = \{x_n^{(0)}\} + \{x_n\}, \quad \text{where} \quad \{x_n^{(0)}\} \in G, \quad \{x_n\} \in F.
\]

Then \( y = S(\{x_n^{(0)}\}) + S(\{x_n\}) = S(\{x_n\}) \), which proves that \( S|_F \) is onto.

Now let \( x \in E \) be an arbitrary element and let \( \{v_n(x)\} = (S|_F)^{-1}(x) \in F \) then

\[
\{x_n\} = \{v_n(x)\} + \{x_n^{(0)}\}, \quad \{x_n^{(0)}\} \in G.
\]

So, we have

\[
S(\{x_n\}) = S(\{v_n(x)\}).
\]

Therefore

\[
x = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x).
\]

Since, \( F \subset X_\Phi \), we have \( v_n(x) \in G_n, \ x \in E, \ n \in \mathbb{N} \).

\[
x = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x), \quad x \in E,
\]

and each \( v_n \) is linear on \( E \) and satisfies

\[
\|v_n(x)\| \leq 2 \sup_{1\leq k<\infty} \left\| \sum_{i=1}^{k} v_i(x) \right\|
\]

\[
= 2\|\{v_n(x)\}\|
\]

\[
\leq 2\|(S|_F)^{-1}\||\|x\|\quad (x \in E, n = 1, 2, \ldots).
\]

Also, by the principle of uniform boundedness,

\[
\|x\|_E = \left\| \sum_{n=1}^{\infty} v_n(x) \right\|
\]

\[
\leq \sup_{1\leq n<\infty} \left\| \sum_{i=1}^{n} v_i(x) \right\|
\]

\[
= \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E,
\]

where \( B = \sup_{1\leq n<\infty} \|S_n\| < \infty \) and \( S_n(x) = \sum_{i=1}^{n} v_i(x) \). Therefore \( \{\{G_n\}, \{v_n\}\} \) is a \( G \)-atomic decomposition of \( E \) with respect to \( X_\Phi \).

**Sufficiency**, follows with the argument of the proof of Theorem 3.7.

In the following result, we show that an \( G \)-atomic decomposition for a Banach space produces another \( G \)-atomic decomposition for the space.

**Theorem 3.10.** If \( \{\{G_n\}, \{v_n\}\} \) is a \( G \)-atomic decomposition for \( E \) with respect to \( X_\Phi \), then there exists a projection \( P \) of \( X_\Phi \) onto \( T(E) \) along \( G \) such that \( \{(T^{-1}P(F_n))\}, \{v_n\}\) is an \( G \)-atomic decomposition for \( E \) with respect to \( X_\Phi \), where \( \{F_n\} \) is the Schauder decomposition of \( X_\Phi \).

**Proof.** Let \( P \) be a projection of \( X_\Phi \) onto \( T(E) \) along \( G \). Then

\[
P(\{x_n\}) = \left\{ v_n \left( \sum_{i=1}^{\infty} x_i \right) \right\}, \quad \{x_n\} \in X_\Phi.
\]

(2)
Since for every \( \{v_n(x)\} \in T(E) \), we have
\[
P(\{v_n(x)\}) = \{v_n(x)\} = \left\{ v_n \left( \sum_{i=1}^{\infty} v_i(x) \right) \right\}
\]
and since for every \( \{x_n\} \in G \) we have
\[
P(\{x_n\}) = 0 = \{v_n(0)\} = \left\{ v_n \left( \sum_{i=1}^{\infty} x_i \right) \right\}.
\]
By (2) we have, in particular, for any \( \{\delta_{nk}x_n\} \in F_k, k = 1, 2, \ldots \)
\[
P(\{\delta_{nk}x_n\}) = \left\{ v_n \left( \sum_{\substack{i=1 \\ i \neq k}}^{\infty} \delta_{ik}x_i \right) \right\} = \{v_n(x_k)\} = T(x_k)
\]
where
\[
\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}
\]
Since \( T \) is invertible, then \( x_k = T^{-1}(P(\{\delta_{nk}x_n\})) \), \( x_k \in G_k, k = 1, 2, \ldots \). Therefore \( G_k = T^{-1}P(F_k), k = 1, 2, \ldots . \)
Hence \( (\{T^{-1}P(F_n)\}, \{v_n\}) \) is a G-atomic decomposition for \( E \) with respect to model space \( X_\Phi \).

In next theorem we want to classify G-atomic decomposition in terms of bases of subspaces (Schauder decomposition) for Banach space.

**Theorem 3.11.** If \( D \) is any Banach space with Schauder decomposition \( \{F_n\} \) and an isomorphism \( T \) from \( E \) into \( D \) and a projection \( P \) from \( D \) onto \( T(E) \) such that \( G_n = T^{-1}P(F_n) \). Then, there exists an associated sequence of operators \( \{v_n\} \) such that \( (\{G_n\}, \{v_n\}) \) is a G-atomic decomposition for \( E \).

**Proof.** Since \( \{F_n\} \) is a Schauder decomposition for \( D \). Assume \( \{u_n\} \subset L(D,D) \) is an associated sequence of coordinate projection to \( \{F_n\} \). Then, for \( y \in P(D) \)
\[
y = P(y) = P\left( \sum_{n=1}^{\infty} u_n(y) \right) = \sum_{n=1}^{\infty} P(u_n(y)).
\]
Since \( Pu_n|_{P(D)} \in L(P(D), P(F_n)), n = 1, 2, \ldots \) Therefore \( (\{P(F_n)\}, \{Pu_n\}) \) is G-atomic decomposition of \( P(D) = T(E) \) by (3).
Now, \( T \) is an isomorphism from \( E \) onto \( T(E) \) and \( G_n = T^{-1}P(F_n) \). Put \( Pu_n = v_n, n = 1, 2, \ldots \)
It follows that \( (\{G_n\}, \{v_n\}) \) is G-atomic decomposition for \( E \).

In the next theorem we generalize Theorem 2.6 [3], which is a classical construction of Pelczynski [16].

**Theorem 3.12.** Let \( E \) be a Banach space. Then the following are equivalent:

(i) There is a Banach space of scalar valued sequences \( X_\Phi \), so that \( (\{G_n\}, \{u_n\}) \) satisfies Definition 3.1 (i.e. is an G-atomic decomposition for \( E \)).

(ii) There is a Banach space \( D \) with a Schauder decomposition \( \psi = \{A_n\} \) so that \( E \subset D \) and there is a bounded linear projection \( P : D \to E \) with \( P\{A_n\} = G_n \), for all \( n \in \mathbb{N} \).

**Proof.** (i)⇒(ii) This implication is obvious with \( D = E \) and \( \{A_n\} = \{G_n\} \) and \( P = I_E \) in Theorem 3.10.
(ii)⇒(i) This follows with the argument of the above proof of Theorem 3.11.

Now we proceed to examine the general relationship between finite-dimensional G-atomic decompositions and approximation property in Banach space theory.
We recall the following definition:
Definition 3.13 ([17]). A sequence of non zero finite rank operators \( \{v_i\} \) from a Banach space \( E \) into itself is called a finite dimensional expansion of the identity of \( E \), if

\[
x = \sum_{i=1}^{\infty} v_i(x), \quad x \in E.
\]

In view of above definition, we prove the following result.

**Theorem 3.14.** A Banach space \( E \) has a finite-dimensional \( G \)-atomic decomposition \( (\{G_n\}, \{v_n\}) \) (i.e., such that \( \dim G_n < \infty \) for all \( n = 1, 2, \ldots \)) if and only if \( E \) admits a finite-dimensional expansion \( \{v_n\} \) of the identity of \( E \).

**Proof.** Let \( (\{G_n\}, \{v_n\}) \) be finite dimensional \( G \)-atomic decomposition for \( E \). Then \( \dim G_n < \infty \) for all \( n = 1, 2, \ldots \) and \( x = \sum_{n=1}^{\infty} v_n(x), \; x \in E \).

Therefore, an associated sequence of operators \( \{v_n\} \) for \( \{G_n\} \) is a finite-dimension expansion of \( E \).

Conversely, if \( \{v_n\} \) is a finite-dimensional expansion of the identity of \( E \), then

\[
\{G_n\} = \{v_n(E)\}
\]

i.e. \( (\{G_n\}, \{v_n\}) \) is a finite-dimensional \( G \)-atomic atomic decomposition of \( E \). \( \square \)

**Remark 3.15.** With the help of above result we can classified finite \( G \)-atomic decompositions in terms of several forms of the approximation property for Banach spaces.

**References**


