S_s-Open sets and S_s-Continuous Functions

Alias B. Khalaf 1*, Abdulrahman H. Majeed 2, Jamil M. Jamil 2

1 University of Duhok, Faculty of Science, Department of Mathematics
2 University of Baghdad, College of Science, Department of Mathematics
*Corresponding author E-mail: aliasbkhalaf@gmail.com

Abstract

In this paper we introduce and study the concept of S_s-open sets also, a study new class of functions called S_s-continuous functions, the relationships between S_s-continuity and other types of continuity are investigated.

Keywords: S_s-open set, S_s-continuous function, semi-open set, semi-continuous functions.

1. Introduction

In 1963, Levine [16], introduced the concept of semi-open set and semi continuity and gave several properties about these functions. Njastad [18] introduced the concepts of α-sets and Abd-El-Monsef et al [1] defined β-open sets and β-continuous functions. Khalaf and Ameen in [14], defined the concept of S-s-open sets and in 2012, Khalaf and Ahmed [15], introduced another type of semi-open sets called S_β-open sets. Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represents non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, Cl(A) and Int(A) represents the closure and Interior of A respectively. A subset A is said to be pre-open [17] (resp., α-open [18], semi-open [16], regular open [21], β-open [1]) set if A ⊆ IntCl(A) (resp. A ⊆ IntClInt(A), A ⊆ ClInt(A), A = IntCl(A), A ⊆ ClIntCl(A)). The complement of a pre-open (resp., α-open, semi-open , regular open, β-open) set is called pre-closed (resp., α-closed, semi-closed, regular closed, β-closed) set. The intersection of all semi-closed sets containing A is called the semi-closure [5] of A and it is denoted by sClA. The semi-interior of a set A is the union of all semi-open sets contained in A and is denoted by sIntA. A subset A of a topological space (X, τ) is said to be θ-open [23] (resp., θ-semi-open [13], semi-θ-open [6]) set if for each x ∈ A, there is an open (resp., semi-open, semi-open) set U such that x ∈ U ⊆ Cl(U) ⊆ A (resp., x ∈ U ⊆ Cl(U) ⊆ A, x ∈ U ⊆ sCl(U) ⊆ A). For more properties of semi-θ-open sets (see [24]) also. A subset A of a topological space X is said to be regular-semi-open [4] if there exists a regular-open set U such that U ⊆ A ⊆ ClU equivalently A is regular-semi-open [22] if and only if A = sIntsClA. A set A is called semi-regular [12], if it is both semi-open and semi-closed. The family of all regular-semi-open (resp., θ-open, θ-semi-open, semi-θ-open, semi-regular) sets of X is denoted by RSO(X) (resp., θO(X), θSO(X), SθO(X), SR(X)). The aim of the present paper is to define a new type of sets, we call it S_s-open set. Since the families SO(X) and PO(X) are incomparable [17], so the it is obvious that the concept of S_s-open sets incomparable with S_β-open sets but it is strictly weaker than S_α-open sets and stronger than S_β-open sets.
2. Preliminaries

In this section, we recall the following definitions and results:

Lemma 2.1 Let \((Y, \tau_Y)\) be a subspace of a space \((X, \tau)\).

1. If \(A \in SO(X, \tau)\) and \(A \subseteq Y\), then \(A \in SO(Y, \tau_Y)\).[16]
2. If \(A \in SO(Y, \tau_Y)\) and \(Y \in SO(X, \tau)\), then \(A \in SO(X, \tau)\).[8]

Lemma 2.2 Let \(A\) be a subset of a space \(X\), then the following properties hold.

1. If \(A \in SO(X)\), then \(sCl(A) \in RSO(X)\).[22]
2. If \(A \in SO(X)\), then \(sCl(A) = sCl_0(A)\).[3]
3. If \(A\) is open subset of \(X\), then \(sCl(A) = \text{IntCl}(A)\).[12]

Lemma 2.3 [9] For any topological space \(X\). If \(A \in \alpha O(X)\) and \(B \in SO(X)\), then \(A \cap B \in SO(X)\).

Definition 2.4 A semi-open subset \(A\) of a space \(X\) is called \(S_c\)-open [14] (resp., \(S_{\beta}\)-open [15], \(S_p\)-open [20]) set if for each \(x \in A\), there exists a closed set (resp., \(\beta\)-closed, pre-closed set) \(F\) such that \(x \in F \subseteq A\).

Definition 2.5 A topological space \((X, \tau)\) is called:

1. semi-\(T_1\) [2], if for every two distinct points \(x, y\) in \(X\), there exist two semi-open sets, one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).
2. semi-regular [11], if for each \(x \in X\) and each \(H \in SO(X)\) containing \(x\), there exists \(G \in SO(X)\) such that \(x \in G \subseteq sCl(G) \subseteq H\).

Lemma 2.6 [2] A space \(X\) is semi-\(T_1\), if and only if, the singleton set \(\{x\}\) is semi closed for any point \(x \in X\).

Lemma 2.7 The following properties hold:

1. If a space \(X\) is semi-regular, then each \(SO(X) = S\theta O(X)\).
2. If a space \(X\) is semi-regular, then \(sCl(A) = sCl_0(A)\) for each subset \(A\) of \(X\).

Proof. It is clear that each semi-\(\theta\)-open is semi-open. If \(X\) is semi-regular space and if \(G\) is a non-empty semi-open set in \(X\), the by Definition 2.5, there exists a semi-open set \(U\) such that \(x \in U \subseteq sCl(U) \subseteq G\), this implies that \(G\) is semi-\(\theta\)-open. Therefore, \(SO(X) = S\theta O(X)\).

Part (2). Follows from part (1).

Definition 2.8 A space \(X\) is locally indiscrete [9], if every open set is closed.

Lemma 2.9 [9] A space \(X\) is locally indiscrete if and only if every semi open set in \(X\) is closed.

Definition 2.10 [19] A function \(f : X \to Y\) is said to be strongly \(\theta\)-semi-continuous at a point \(x \in X\), if for each open set \(V\) containing \(f(x)\), there exists a semi-open set \(U\) containing \(x\) such that \(f(sCl(U)) \subseteq V\).

The function \(f\) is said to be strongly \(\theta\)-semi-continuous on \(X\) if it is strongly \(\theta\)-semi-continuous at every point of \(X\), we shall denote by \(f\) is st.sc on \(X\).

Definition 2.11 [10] A function \(f : X \to Y\) is said to be semi-continuous (resp., contra-semi-continuous) if the inverse image of every open set in \(Y\) is semi-open (resp., semi-closed) in \(X\).

Theorem 2.12 [2] For any spaces \(X\) and \(Y\). If \(A \subseteq X\) and \(B \subseteq Y\) then,

1. \(sInt_{X \times Y}(A \times B) = sInt_X(A) \times sInt_Y(B)\).
2. \(sCl_{X \times Y}(A \times B) = sCl_X(A) \times sCl_Y(B)\).
3. $S_\gamma$-Open Sets

In this section, we introduce the concept of $S_\gamma$-open sets in topological spaces.

**Definition 3.1** A semi-open subset $A$ of a space $X$ is called $S_\gamma$-open if for each $x \in A$, there exists a semi-closed set $F$ such that $x \in F \subseteq A$.

The family of all $S_\gamma$-open subsets of a topological space $(X, \tau)$ is denoted by $S_\gamma O(X, \tau)$ or $S_\gamma O(X)$.

**Proposition 3.2** A subset $A$ of a space $X$ is $S_\gamma$-open if and only if $A = \cup F_\gamma$ where $A$ is semi-open set and $F_\gamma$ semi-closed set for each $\gamma$.

**Proof.** Obvious.

**Remark 3.3** It is clear from the definition that every $S_\gamma$-open subset of a space $X$ is semi-open, but the converse is not true in general as it is shown in Example 3.11.

**Proposition 3.4** If a space $X$ is semi-$T_1$, then $S_\gamma O(X) = SO(X)$.

**Proof.** Follows from the fact that in a semi-$T_1$ space, every singleton set is semi-closed (Lemma 2.6).

**Remark 3.5** Since any union of semi-open sets is semi-open [16], so any union of $S_\gamma$-open sets in a topological space $(X, \tau)$ is also $S_\gamma$-open. The intersection of two $S_\gamma$-open sets need not be $S_\gamma$-open in general as it is shown by the following example:

**Example 3.6** Consider the intervals $[0, 1]$ and $[1, 2]$ in $R$ with the usual topology. Since $R$ is $T_1$ space and hence it is semi-$T_1$, so by Proposition 3.4, both the intervals are $S_\gamma$-open sets and we have $[0, 1] \cap [1, 2] = \{1\}$ which is not $S_\gamma$-open.

**Proposition 3.7** Every semi-$\theta$-open subset of a space $X$ is $S_\gamma$-open.

**Proof.** Suppose that the subset $A$ of $X$ is semi-$\theta$-open, then clearly it is semi-open and by definition, for each $x \in A$, there exists a semi-open set $U$ such that $x \in U \subseteq sClU \subseteq A$. Hence, $sClU$ is the semi-closed set containing $x$ contained in $A$, so $A$ is $S_\gamma$-open.

The relation of $S_\gamma$-open sets to some other types of sets is illustrated in the following remark:

**Remark 3.8** If $X$ is any topological space, then the following properties hold:

1. Since every $\theta$-semi-open subset of $X$ is semi-$\theta$-open, so, from Proposition 3.7, we obtain that every $\theta$-semi-open set is $S_\gamma$-open.
2. It is obvious that every semi-regular subset of $X$ is $S_\gamma$-open.
3. It is obvious that every $s_\gamma$-open set is $S_\gamma$-open.
4. Every $s_\gamma$-open set is $s\beta$-open, because every semi open set is $\beta$-open.

Although not every open set is an $S_\gamma$-open set as we can see in Example 3.11 but we have the following results:

**Proposition 3.9** Let $(X, \tau)$ be a semi regular space, then $\tau \subseteq S_\gamma O(X)$.

**Proof.** Let $A$ be any non-empty open subset of $X$, then for each $x \in A$, there is a semi-open set $G$ such that $x \in G \subseteq sCl(G) \subseteq A$ implies that $x \in sCl(G) \subseteq A$. Hence $A$ is $S_\gamma$-open.

**Proposition 3.10** The following properties hold.

1. If $A$ is a semi-open subset of a space $X$, then $sClA$ is $S_\gamma$-open.
2. If $A$ is a semi-closed subset of a space $X$, then $sIntA$ is $S_\gamma$-open.
3. $sClIntA$ is $S_\gamma$-open subset, for every subset $A$ of $X$.
4. $sIntsClA$ is $S_\gamma$-open subset, for every subset $A$ of $X$. 
5. Every regular semi-open subset of $X$ is $S_s$-open.

**Proof.** (1) For any subset $A$ of $X$, we have $sCl A = A \cup IntCl A$. Hence $sCl A$ is both semi-open and semi-closed, so it is $S_s$-open.

The proof of parts (2), (3), (4) and (5) are similar.

We get the following diagram of implications:

$$
\begin{array}{cccc}
\text{\theta-open set} & \downarrow & \text{\delta-open set} & \longrightarrow \text{semi-\theta-open set} & \leftarrow \text{\theta-semi-open set} \\
\downarrow & & \downarrow & & \\
\text{open set} & & \text{semi-open set} & \leftarrow \text{$S_s$-open set} & \leftarrow \text{sc-open set}
\end{array}
$$

The following examples show that the above implications are not reversible.

**Example 3.11** Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then we have: $SO(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, and hence $SC(X) = \{\phi, \{c\}, \{b\}, \{b, c\}, X\}$. So, $S_s O(X) = \{\phi, X\}$ implies that the set $\{a\} \in SO(X)$, but $\{a\} \notin S_s O(X)$.

**Example 3.12** Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c, d\}, X\}$. Then : $SO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c, d\}, \{a, d\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, c\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \{d\}, X\}$. Hence, the set $\{a, d\}$ is an $S_s$-open set which is not $S_c$-open.

**Example 3.13** Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, X\}$. Then we can easily find the following families of sets:

$SO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c, d\}, \{a, d\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, c\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \{d\}, X\}$, also $SC(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{b, c, d\}, \{a, b, c, d\}, \{a, b, c\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \{d\}, X\}$. Hence, the set $\{a, c\}$ is an $S_s$-open set which is not $\theta$-semi-open set also it is not semi-regular set.

**Proposition 3.14** For any space $X$, $sCl(sInt(\{x\})) = \{x\}$ if and only if $\{x\}$ is $S_s$-open.

**proof.** Let $sCl(sInt(\{x\})) = \{x\}$, this implies that $sInt(\{x\}) = \{x\}$ and so, $\{x\}$ is both semi-open and semi-closed, then $\{x\}$ is semi regular open. Hence, $\{x\} \in S_s O(X)$.

Conversely. Let $\{x\}$ be an $S_s$-open set in $X$, then there exists a semi-closed $F$ such that $x \in F \subseteq \{x\}$, this implies that $x \in \{x\} \subseteq \{x\}$, so, $\{x\}$ is semi-open and semi-closed. Therefore, $sCl(sInt(\{x\})) = \{x\}$.

**Proposition 3.15** Let $(x, \tau)$ be a topological space. Then $\{x\} \in S_s O(X)$ if and only if it is semi-regular.

**proof.** If $\{x\}$ is semi-regular, then, by Remark 3.8, $\{x\} \in S_s O(X)$.

Conversely. Suppose $\{x\}$ is $S_s O(X)$, then $\{x\}$ is semi-open and by definition it is semi-closed. Hence, $\{x\}$ is semi-regular.

**Proposition 3.16** A subset $A$ of a space $(X, \tau)$ is $S_s$-open if and only if for each $x \in A$, there exists an $S_s$-open set $B$ such that $x \in B \subseteq A$.

**proof.** If $A$ is an $S_s$-open subset in the space $(X, \tau)$, then for each $x \in A$, putting $A = B$, which is $S_s$-open containing $x$ such that $x \in B \subseteq A$.

Conversely. Suppose that for each $x \in A$, there exists a $S_s$-open set $B$ such that $x \in B \subseteq A$. So, $A = \cup B_{\gamma}$ where $B_{\gamma} \in S_s O(X)$ for each $\gamma$. Therefore, by Remark 3.5, $A$ is $S_s$-open.

**Proposition 3.17** Let $X$ be a topological space, and $A, B \subseteq X$. If $A \in S_s O(X)$ and $B$ is both $\alpha$-open and semi-closed, then $A \cap B \in S_s O(X)$.
proof. Let \( A \in S_s O(X) \) and \( B \) be \( \alpha \)-open, then \( A \) is semi open set, so by Lemma 2.3, we have \( A \cap B \in SO(X) \). Now let \( x \in A \cap B \), then \( x \in A \) and therefore, there exists a semi-closed set \( F \) such that \( x \in F \subseteq A \). Since \( B \) is semi-closed, so \( F \cap B \) is semi-closed set. Hence, \( x \in F \cap B \subseteq A \cap B \). Thus \( A \cap B \) is \( S_s \)-open set in \( X \).

**Proposition 3.18** Let \((Y, \tau_Y)\) be an \( \alpha \)-open subspace of a space \((X, \tau)\). If \( A \in S_s O(X, \tau) \) and \( A \subseteq Y \), then \( A \in S_s O(Y, \tau_Y) \).

**proof.** Let \( A \in S_s O(X, \tau) \), then \( A \in SO(X, \tau) \) and for each \( x \in A \), there exists a semi-closed set \( F \) in \( X \) such that \( x \in F \subseteq A \). Since \( A \in SO(X, \tau) \) and \( A \subseteq Y \), so, by Lemma 2.1, \( A \in SO(Y, \tau_Y) \). Since \( F \) semi-closed set in \( X \), then \( X \setminus F \) is semi-open and hence, by Lemma 2.3, \( Y \setminus (X \setminus F) \) is semi-open in \( Y \). So, by Lemma 2.1, \( Y \setminus (X \setminus F) \) is semi-open in \( Y \). Therefore, \( F = Y \setminus (Y \setminus X \setminus F) \) is semi-closed set in \( Y \). Hence \( A \in S_s O(Y, \tau_Y) \).

**Proposition 3.19** Let \( Y \) be a semi-regular set in a space \((X, \tau)\). If \( A \in S_s O(Y, \tau_Y) \), then \( A \in S_s O(X, \tau) \).

**proof.** Let \( A \in S_s O(Y, \tau_Y) \), then \( A \in SO(Y, \tau_Y) \) and for each \( x \in A \), there exists a semi-closed set \( F \) in \( Y \) such that \( x \in F \subseteq A \). Since \( A \in SO(Y, \tau_Y) \) and \( Y \) is semi-regular. So, by Lemma 2.1, \( A \in SO(X, \tau) \). Since \( F \) semi-closed set in \( Y \), then \( Y \setminus F \) is semi-open in \( Y \) and also, by Lemma 2.1, \( Y \setminus F \) is semi-open in \( X \). Again \( Y \) is semi-regular in \( X \) implies that \( X \setminus Y \) is semi-open. Hence, \( Y \setminus F \cup X \setminus Y = X \setminus F \) is semi-open in \( X \). So, \( F \) is semi-closed in \( X \). Therefore, \( A \in S_s O(X, \tau) \).

**Definition 3.20** Let \( A \) be a subset of a topological space \((X, \tau)\).

1. The union of all \( S_s \)-open sets which are contained in \( A \) is called the \( S_s \)-interior of \( A \) and is denoted by \( S_s \text{Int}(A) \).
2. The intersection of all \( S_s \)-closed sets containing \( A \) is called the \( S_s \)-closure of \( A \) and we denote it by \( S_s \text{Cl}(A) \).
3. The \( S_s \)-boundary of \( A \) is \( S_s \text{Cl}(A) \setminus S_s \text{Int}(A) \) and is denoted by \( S_s \text{Bd}(A) \).

**Proposition 3.21** Let \( A \) be any subset of a space \( X \). If a point \( x \) is in the \( S_s \)-interior of \( A \), then there exists a semi closed set \( F \) of \( X \) containing \( x \) such that \( F \subseteq A \).

**proof.** Suppose that \( x \in S_s \text{Int}(A) \), then there exists a \( S_s \)-open set \( U \) of \( X \) containing \( x \) such that \( U \subseteq A \). Since \( U \) is \( S_s \)-open set, so there exists a semi closed set \( F \) containing \( x \) such that \( F \subseteq U \subseteq A \). Hence \( x \in F \subseteq A \).

**Proposition 3.22** For any subset \( A \) of a topological space \( X \). The following statements are true:

1. \( S_s \text{Int}(A) \) is the largest \( S_s \)-open set contained in \( A \).
2. \( A \) is \( S_s \)-open if and only if \( A = S_s \text{Int}(A) \).
3. \( S_s \text{Cl}(A) \) is the smallest \( S_s \)-Closed set in \( X \) containing \( A \).
4. \( A \) is \( S_s \)-closed set if and only if \( A = S_s \text{Cl}(A) \).

Some other properties of \( S_s \)-interior of a set \( A \) are in the following result:

**Theorem 3.23** If \( A \) and \( B \) are any subsets of a topological space \((X, \tau)\), then the following properties hold:

1. if \( A \subseteq B \), then \( S_s \text{Int}(A) \subseteq S_s \text{Int}(B) \) and \( S_s \text{Cl}(A) \subseteq S_s \text{Cl}(B) \).
2. \( S_s \text{Int}(A) \cup S_s \text{Int}(B) \subseteq S_s \text{Int}(A \cup B) \).
3. \( S_s \text{Int}(A) \cap S_s \text{Int}(B) \subseteq S_s \text{Int}(A \cap B) \).
4. \( S_s \text{Cl}(A) \cup S_s \text{Cl}(B) \subseteq S_s \text{Cl}(A \cup B) \).
5. \( S_s \text{Cl}(A \cap B) \subseteq S_s \text{Cl}(A) \cap S_s \text{Cl}(B) \).

**proof.** Obvious.

In general, \( S_s \text{Int}(A) \cup S_s \text{Int}(B) \neq S_s \text{Int}(A \cup B) \). and \( S_s \text{Int}(A) \cap S_s \text{Int}(B) \neq S_s \text{Int}(A \cap B) \). Also, the equalities in (4) and (5) does not hold as shown in the following example:
Example 3.24 Consider the space $(X, \tau)$ defined in Example 3.13, then we have the following cases:

1. if $A = \{a, c\}$ and $B = \{a, d\}$, then $S_s\text{Int}(A) = \{a\}$, $S_s\text{Int}(B) = \{a\}$. Hence, $S_s\text{Int}(A) \cup S_s\text{Int}(B) = \{a\}$ and $S_s\text{Int}(A \cup B) = S_s\text{Int}(\{a, b, c\}) = \{a, b, c\}$. It follows that $S_s\text{Int}(A) \cup S_s\text{Int}(B) \neq S_s\text{Int}(A \cup B)$.

2. If $A = \{a, b\}$ and $B = \{b, c, d\}$, then $S_s\text{Int}(A) = \{a, b\}$, $S_s\text{Int}(B) = \{b, c, d\}$, so $S_s\text{Int}(A) \cap S_s\text{Int}(B) = \{b\}$ and $S_s\text{Int}(A \cap B) = S_s\text{Int}(\{b\}) = \phi$. It follows that $S_s\text{Int}(A) \cap S_s\text{Int}(B) \neq S_s\text{Int}(A \cap B)$.

3. If $A = \{a\}$ and $B = \{c, d\}$, then $S_s\text{Cl}(A) = F$, $S_s\text{Cl}(B) = B$. Hence, $S_s\text{Cl}(A) \cup S_s\text{Cl}(B) = \{a, c, d\}$, and $S_s\text{Cl}(A \cup B) = S_s\text{Cl}(\{a, c, d\}) = X$. It follows that $S_s\text{Cl}(A) \cup S_s\text{Cl}(B) \neq S_s\text{Cl}(A \cup B)$.

4. If $A = \{a, c, d\}$ and $B = \{b, c, d\}$, then $S_s\text{Cl}(A) = X$ and $S_s\text{Cl}(B) = B$, so $S_s\text{Cl}(A) \cap S_s\text{Cl}(B) = B$, and $S_s\text{Cl}(A \cap B) = S_s\text{Cl}(\{c, d\}) = \{c, d\}$. It follows that $S_s\text{Cl}(A \cap B) \neq S_s\text{Cl}(A) \cap S_s\text{Cl}(B)$.

Proposition 3.25 Let $A$ be a subset of a topological space $X$. Then $x \in S_s\text{Cl}(A)$ if and only if for any $S_s$-open set $U$ containing $x$, $U \cap A \neq \phi$.

proof. Let $x \in S_s\text{Cl}(A)$ and suppose that $U \cap A = \phi$ for some $S_s$-open set $U$ which contains $x$. Then $(X \setminus U)$ is $S_s$-closed set and $A \subseteq (X \setminus U)$, thus $S_s\text{Cl}(A) \subseteq (X \setminus U)$. But this implies that $x \in (X \setminus U)$, which is contradiction. Therefore $U \cap A \neq \phi$.

Conversely, suppose that there exists an $S_s$-open set containing $x$ with $A \cap U = \phi$, then $A \subseteq X \setminus U$ and $X \setminus U$ is an $S_s$-closed with $x \notin X \setminus U$. Hence, $x \notin S_s\text{Cl}(A)$.

Proposition 3.26 For any subset $A$ of a topological space $X$. The following statements are true.

1. $X \setminus S_s\text{Cl}(A) = S_s\text{Int}(X \setminus A)$.

2. $S_s\text{Cl}(A) = X \setminus S_s\text{Int}(X \setminus A)$.

3. $X \setminus S_s\text{Int}(A) = S_s\text{Cl}(X \setminus A)$.

4. $S_s\text{Int}(A) = X \setminus S_s\text{Cl}(X \setminus A)$.

proof. We only prove (1), and the other parts can be proved similarly. (1) For any point $x \in X$, if $x \in X \setminus S_s\text{Cl}(A) \Leftrightarrow x \notin S_s\text{Cl}(A)$ for each $B \in S_sO(X)$ containing $x$, we have $A \cap B = \phi \Leftrightarrow x \in B \subseteq X \setminus A \Leftrightarrow x \in S_s\text{Int}(X \setminus A)$.

Theorem 3.27 If $A$ is a subset of a topological space $X$. Then $\text{Int}_\theta(A) \subseteq S_s\text{Int}(A) \subseteq S_s\text{Cl}(A) \subseteq S_s\text{Cl}(A) \subseteq \text{Cl}_\theta(A)$.

proof. Obvious.

Proposition 3.28 Let $A$ be any subset of a space $X$. If $A \in S_s\text{O}(X)$, then $S_s\text{Cl}(A) \subseteq S_s\text{Cl}(A)$.

proof. Assume that $x \notin S_s\text{Cl}(A)$, then there exists an $S_s$-open set $U$ containing $x$ such that $A \cap U = \phi$ and $A \cap S_s\text{Cl}(U) = \phi$ since $A \in S_s\text{O}(X)$, but $S_s\text{Cl}(U) \subseteq S_s\text{Cl}(U)$ implies that $A \cap S_s\text{Cl}(U) = \phi$ and hence $x \notin S_s\text{Cl}(A)$.

Proposition 3.29 Let $(X, \tau)$ be a semi regular space and $A$ be any subset of $X$. Then, $S_s\text{Cl}(A) = S_s\text{Cl}(A) = S_s\text{Cl}(A)$.

proof. From Theorem 2.6, we have $S_s\text{Cl}(A) = S_s\text{Cl}(A)$, so we get that $S_s\text{Cl}(A) = S_s\text{Cl}(A) = S_s\text{Cl}(A)$.

4. $S_s$-Continuous Functions

Definition 4.1 A function $f : X \rightarrow Y$ is called $S_s$-continuous at a point $x \in X$, if for each open set $V$ of $Y$ containing $f(x)$, there exists an $S_s$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

If $f$ is $S_s$-continuous at every point $x \in X$, then it is called $S_s$-continuous.

Proposition 4.2 A function $f : X \rightarrow Y$ is $S_s$-continuous if and only if the inverse image of every open set in $Y$ is an $S_s$-open in $X$. 
proof. Let \( f \) be \( S_s \)-continuous, and \( V \) be any open set in \( Y \). If \( f^{-1}(V) \neq \emptyset \), then there exists \( x \in f^{-1}(V) \) which implies \( f(x) \in V \). Since, \( f \) is \( S_s \)-continuous, there exists an \( S_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). This implies that \( x \in U \subseteq f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \( S_s \)-open.

Conversely, let \( V \) be any open set in \( Y \), \( f(x) \in V \), then \( x \in f^{-1}(V) \). By hypothesis, \( f^{-1}(V) \) is an \( S_s \)-open set in \( X \) containing \( x \), thus \( f(f^{-1}(V)) \subseteq V \). Therefore, \( f \) is \( S_s \)-continuous.

Proposition 4.3 If a function \( f : X \rightarrow Y \) is strongly \( \theta \)-semi-continuous, then \( f \) is \( S_s \)-continuous.

proof. Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). Since, \( f \) is strongly \( \theta \)-semi-continuous, then, there exists a semi-open set \( G \) in \( X \) containing \( x \) such that \( f(sCl(G)) \subseteq V \). Hence, by Proposition 3.10(1), \( sCl(G) \) is an \( S_s \)-open set. Therefore, \( f \) is \( S_s \)-continuous.

Corollary 4.4 If a function \( f : X \rightarrow Y \) is strongly \( \theta \)-continuous, then \( f \) is \( S_s \)-continuous.

proof. Follows from Remark 3.4 of [19] and Proposition 4.3.

The following example shows that the converse of Corollary 4.4 is not true in general.

Example 4.5 Let \( X = \{a, b, c, d\} \) equipped with the two topologies \( \tau = \sigma = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, X\} \).

If \( f : (X, \tau) \rightarrow (X, \sigma) \) is the identity function, then \( f \) is \( S_s \)-continuous, but it is not strongly \( \theta \)-continuous because \( f^{-1}(\{a, c, d\}) = \{a, c, d\} \) which is not \( \theta \)-open.

The proof of the following result follows directly from their definitions.

Corollary 4.6 Every \( S_s \)-continuous function is semi-continuous.

proof. Obvious.

Example 4.7 Let \( X = \{a, b, c\} \) with the topology \( \tau = \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). Let \( f : (X, \tau) \rightarrow (X, \sigma) \) be the identity function. Then, \( f \) is semi-continuous, but it is not \( S_s \)-continuous, because \( \{a\} \) is an open set in \( (X, \sigma) \) and \( f^{-1}(\{a\}) \) is not \( S_s \)-open.

Corollary 4.8 If a function \( f : X \rightarrow Y \) is both semi-continuous and contra-semi-continuous, then it is \( S_s \)-continuous.

proof. Follows from Definition 2.11 and Proposition 4.2.

Remark 4.9 The function \( f \) in Example 4.5 is not contra-semi-continuous.

Proposition 4.10 A function \( f : X \rightarrow Y \) is \( S_s \)-continuous if and only if \( f \) is semi-continuous and for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi-closed set \( G \) of \( X \) containing \( x \) such that \( f(G) \subseteq V \).

proof. Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). By hypothesis, there exists an \( S_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \). Since \( U \) is \( S_s \)-open, then for each \( x \in U \), there exists a semi-closed set \( G \) of \( X \) such that \( x \in G \subseteq U \). Therefore, we have \( f(G) \subseteq V \).

Conversely, let \( V \) be any open set of \( Y \). It should be shown that \( f^{-1}(V) \) is \( S_s \)-open set in \( X \). Since, \( f \) is semi-continuous, then \( f^{-1}(V) \) is semi-open set in \( X \). Let \( x \in f^{-1}(V) \), then \( f(x) \in V \). By hypothesis, there exists a semi-closed set \( G \) of \( X \) containing \( x \) such that \( f(G) \subseteq V \), which implies that \( x \in G \subseteq f^{-1}(V) \). Therefore, \( f^{-1}(V) \) is \( S_s \)-open in \( X \). Hence, by Proposition 4.2, \( f \) is \( S_s \)-continuous.

5. Characterizations and Properties

In this section, we give some characterizations and properties of \( S_s \)-continuous functions and we start with the following result.

Proposition 5.1 For a function \( f : X \rightarrow Y \), the following statements are equivalent:
1. \( f \) is \( S_\rho \)-continuous.

2. \( f^{-1}(V) \) is an \( S_\rho \)-open set in \( X \), for each open set \( V \) of \( Y \).

3. \( f^{-1}(F) \) is an \( S_\rho \)-closed set in \( X \), for each closed set \( F \) of \( Y \).

4. \( f(S_\rho Cl(A)) \subseteq Cl(f(A)) \), for each subset \( A \) of \( X \).

5. \( S_\rho Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B)) \), for each subset \( B \) of \( Y \).

6. \( f^{-1}(Int(B)) \subseteq S_\rho Int(f^{-1}(B)) \), for each subset \( B \) of \( Y \).

7. \( Int(f(A)) \subseteq f(S_\rho Int(A)) \), for each subset \( A \) of \( X \).

**proof.** (1) \( \Rightarrow \) (2): Follows from Proposition 4.2.

(2) \( \Rightarrow \) (3): Let \( F \) be any closed set of \( Y \). Then, \( Y \setminus F \) is an open set of \( Y \). By (2), \( f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \) is an \( S_\rho \)-open set in \( X \) and hence \( f^{-1}(F) \) is \( S_\rho \)-closed in \( X \).

(3) \( \Rightarrow \) (4): Let \( A \) be any subset of \( X \). Then, \( f(A) \subseteq Cl(f(A)) \) and \( Cl(f(A)) \) is a closed set in \( Y \). By (3), we have \( f^{-1}(Cl(f(A))) \subseteq S_\rho \)-closed in \( X \). Therefore, \( S_\rho Cl(A) \subseteq f^{-1}(Cl(f(A))) \). Hence, \( f(S_\rho Cl(A)) \subseteq Cl(f(A)) \).

(4) \( \Rightarrow \) (5): Let \( B \) be any subset of \( Y \), so \( f^{-1}(B) \) is a subset of \( X \). By (4), we have \( f(S_\rho Cl(f^{-1}(B))) \subseteq Cl(f^{-1}(B)) \subseteq Cl(B) \). Hence, \( S_\rho Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B)) \).

(5) \( \Leftrightarrow \) (6): Let \( B \) be any subset of \( Y \). Then apply (5) to \( Y \setminus B \), we obtain \( S_\rho Cl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B)) \Leftrightarrow S_\rho Cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B)) \Leftrightarrow X \setminus S_\rho Int(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq S_\rho Int(f^{-1}(B)) \).

Therefore, \( f^{-1}(Int(B)) \subseteq S_\rho Int(f^{-1}(B)) \).

(6) \( \Rightarrow \) (7): Let \( A \) be any subset of \( X \). Then, \( f(A) \) is a subset of \( Y \). By (6), we have \( f^{-1}(Int(f(A))) \subseteq S_\rho Int(f^{-1}(f(A))) \subseteq S_\rho Int(A) \). Therefore, \( Int(f(A)) \subseteq f(S_\rho Int(A)) \).

(7) \( \Rightarrow \) (1): Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). Then, \( x \in f^{-1}(V) \) and \( f^{-1}(V) \) is a subset of \( X \). By (7), we have \( Int(f^{-1}(V)) \subseteq f(S_\rho Int(f^{-1}(V))) \). Hence, \( Int(V) \subseteq f(S_\rho Int(f^{-1}(V))) \). Since, \( V \) is an open set, so \( V \subseteq f(S_\rho Int(f^{-1}(V))) \) implies \( f^{-1}(V) \subseteq S_\rho Int(f^{-1}(V)) \). Therefore, \( f^{-1}(V) \) is an \( S_\rho \)-open set in \( X \) which contains \( x \) and clearly \( f(f^{-1}(V)) \subseteq V \). Hence, \( f \) is \( S_\rho \)-continuous.

**Proposition 5.2** For a function \( f : X \to Y \), the following statements are equivalent:

1. \( f \) is \( S_\rho \)-continuous.

2. \( S_\rho Cl(f^{-1}(V)) \subseteq f^{-1}(Cl_\vartheta(V)) \), for each open set \( V \) of \( Y \).

3. \( f^{-1}(Int_\vartheta(V)) \subseteq S_\rho Int(f^{-1}(V)) \), for each closed \( V \) of \( Y \).

**proof.** (1) \( \Rightarrow \) (2): Let \( V \) be any open set in \( Y \). Suppose that \( x \notin f^{-1}(Cl_\vartheta(V)) \), then \( f(x) \notin Cl_\vartheta(V) \) and there exists an open set \( G \) containing \( f(x) \), such that \( Cl_\vartheta(G) \cap V = \emptyset \) implies \( G \cap V = \emptyset \). Since, \( f \) is \( S_\rho \)-continuous, there exists a \( S_\rho \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq G \). Therefore, we have \( f(U) \cap V = \emptyset \) and \( U \cap f^{-1}(V) = \emptyset \). This shows that \( x \notin S_\rho Cl(f^{-1}(V)) \). Thus, we obtain \( S_\rho Cl(f^{-1}(V)) \subseteq f^{-1}(Cl_\vartheta(V)) \).

(2) \( \Rightarrow \) (3). It is quite similar to part (5) \( \Rightarrow \) (6) in Proposition 5.1.

(3) \( \Rightarrow \) (1). From the Proposition 5.1 (6) and the fact that \( Int(V) = Int_\vartheta(V) \) for each closed set \( V \).

**Proposition 5.3** A \( f : X \to Y \) is \( S_\rho \)-continuous if and only if \( S_\rho Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B)) \), for each subset \( B \) in \( Y \).

**proof.** Let \( B \) be any subset of \( Y \), then we have \( f^{-1}(Bd(B)) = f^{-1}(Cl(B) \setminus Int(B)) = f^{-1}(Cl(B)) \setminus f^{-1}(Int(B)) \). Hence, by Proposition 5.1 (5) and (6), we have \( f^{-1}(Cl(B)) \setminus f^{-1}(Int(B)) \supseteq f^{-1}(S_\rho Cl(B) \setminus S_\rho Int(B)) \). Hence, \( S_\rho Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B)) \).

Conversely, let \( V \) be any open set in \( Y \) and \( F = Y \setminus V \). Then, by hypothesis, we have \( S_\rho Bd(f^{-1}(F)) \subseteq f^{-1}(Bd(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \) and hence \( S_\rho Cl(f^{-1}(F)) = S_\rho Int(f^{-1}(F)) \cup S_\rho Bd(f^{-1}(F)) \subseteq f^{-1}(F) \). Thus, \( f^{-1}(F) \) is \( S_\rho \)-closed and hence \( f^{-1}(V) \) is \( S_\rho \)-open in \( X \).

**Theorem 5.4** Let \( f : X \to Y \) be a function. Let \( B \) be any basis for \( \tau \) in \( Y \). Then, \( f \) is \( S_\rho \)-continuous if and only if for each \( B \in B \), \( f^{-1}(B) \) is a \( S_\rho \)-open subset of \( X \).
proof. Necessity. Suppose that \( f \) is \( S_s \)-continuous. Then, since each \( B \in \mathcal{B} \) is an open subset of \( Y \). Therefore, by Theorem 5.1, \( f^{-1}(B) \) is a \( S_s \)-open subset of \( X \).

Sufficiency. Let \( V \) be any open subset of \( Y \). Then, \( V = \bigcup \{ B_i : i \in I \} \) where every \( B_i \) is a member of \( \mathcal{B} \) and \( I \) is a suitable index set. It follows that \( f^{-1}(V) = f^{-1}(\bigcup \{ B_i : i \in I \}) = \bigcup f^{-1}(\{ B_i : i \in I \}) \). Since, \( f^{-1}(B_i) \) is a \( S_s \)-open subset of \( X \) for each \( i \in I \). Hence, \( f^{-1}(V) \) is the union of a family of \( S_s \)-open sets of \( X \) and hence is \( S_s \)-open set of \( X \). Therefore, by Proposition 5.1, \( f \) is \( S_s \)-continuous.

**Proposition 5.5** Let \( f : X \to Y \) be a \( S_s \)-continuous function. If \( Y \) is any subset of a topological space \( Z \), then \( f : X \to Z \) is \( S_s \)-continuous.

proof. Let \( x \in X \) and \( V \) be any open set of \( Z \) containing \( f(x) \), then \( V \cap Y \) is open in \( Y \). But, \( f(x) \in Y \) for each \( x \in X \), then \( f(x) \in V \cap Y \). Since, \( f : X \to Y \) is \( S_s \)-continuous, then there exists a \( S_s \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \cap Y \subseteq V \). Therefore, \( f : X \to Z \) is \( S_s \)-continuous.

**Proposition 5.6** Let \( f : X \to Y \) be a function and \( X \) is locally indiscrete space. Then, \( f \) is \( S_s \)-continuous if and only if \( f \) is semi-continuous.

proof. Follows from Lemma 2.9.

**Proposition 5.7** Let \( f : X \to Y \) be a function and \( X \) is semi-\( T_1 \) space. Then, \( f \) is \( S_s \)-continuous if and only if \( f \) is semi-continuous.

proof. Follows from Proposition 3.4.

**Proposition 5.8** Let \( f : X \to Y \) be an \( S_s \)-continuous function. If \( A \) is \( \alpha \)-open and semi-closed subset of \( X \), then \( f|A : A \to Y \) is \( S_s \)-continuous in the subspace \( A \).

proof. Let \( V \) be any open set of \( Y \). Since, \( f \) is \( S_s \)-continuous. Then, by Proposition 4.2, \( f^{-1}(V) \) is \( S_s \)-open set in \( X \). Since, \( A \) is \( \alpha \)-open and semi-closed subset of \( X \). By Proposition 3.17, \( (f|A)^{-1}(V) = f^{-1}(V) \cap A \) is an \( S_s \)-open subset of \( A \). This shows that \( f|A : A \to Y \) is \( S_s \)-continuous.

**Proposition 5.9** A function \( f : X \to Y \) is \( S_s \)-continuous, if for each \( x \in X \), there exists a semi-regular set \( A \) of \( X \) containing \( x \) such that \( f|A : A \to Y \) is \( S_s \)-continuous.

proof. Let \( x \in X \), then by hypothesis, there exists a semi-regular set \( A \) containing \( x \) such that \( f|A : A \to Y \) is \( S_s \)-continuous. Let \( V \) be any open set of \( Y \) containing \( f(x) \), then there exists an \( S_s \)-open set \( U \) in \( A \) containing \( x \) such that \( (f|A)(U) \subseteq V \). Since, \( A \) is semi-regular set, by Remark 3.8, \( U \) is \( S_s \)-open set in \( X \) and hence \( f(U) \subseteq V \). This shows that \( f \) is \( S_s \)-continuous.

**Proposition 5.10** Let \( f : X_1 \to Y \) and \( g : X_2 \to Y \) be two \( S_s \)-continuous functions. If \( Y \) is Hausdorff, then the set \( E = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = g(x_2) \} \) is \( S_s \)-closed in the product space \( X_1 \times X_2 \).

proof. Let \( (x_1, x_2) \notin E \). Then, \( f(x_1) \neq g(x_2) \). Since, \( Y \) is Hausdorff, there exist open sets \( V_1 \) and \( V_2 \) of \( Y \) such that \( f(x_1) \subseteq V_1 \), \( g(x_2) \subseteq V_2 \) and \( V_1 \cap V_2 = \emptyset \). Since, \( f \) and \( g \) are \( S_s \)-continuous, then there exist \( S_s \)-open sets \( U_1 \) and \( U_2 \) of \( X_1 \) and \( X_2 \) containing \( x_1 \) and \( x_2 \) such that \( f(U_1) \subseteq (V_1) \) and \( g(U_2) \subseteq (V_2) \), respectively. Put \( U = U_1 \times U_2 \), then \( (x_1, x_2) \in U \) and by Proposition 2.12, \( U \) is an \( S_s \)-open set in \( X_1 \times X_2 \) and \( U \cap E = \emptyset \). Therefore, we obtain \( (x_1, x_2) \notin S_s\text{Cl}(E) \). Hence, \( E \) is \( S_s \)-closed in the product space \( X_1 \times X_2 \).

**Proposition 5.11** Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. If \( f \) is \( S_s \)-continuous and \( g \) is continuous. Then, the composition function \( g \circ f : X \to Z \) is \( S_s \)-continuous.

proof. Let \( V \) be any open subset of \( Z \). Since, \( g \) is continuous, \( g^{-1}(V) \) is open subset of \( Y \). Since, \( f \) is \( S_s \)-continuous, then by Proposition 4.2, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is \( S_s \)-open subset in \( X \). Therefore, \( g \circ f \) is \( S_s \)-continuous.

**Proposition 5.12** Let \( f : (X, \tau) \to (Y, \rho) \) be a surjection function such that \( f(U) \) is \( S_s \)-open in \( Y \), for any \( S_s \)-open set \( U \) in \( X \) and let \( g : (Y, \rho) \to (Z, \sigma) \) be any function. If \( g \circ f \) is \( S_s \)-continuous then \( g \) is \( S_s \)-continuous.

proof. Let \( y \in Y \). Since, \( f \) is surjection, there exists \( x \in X \) such that \( f(x) = y \). Let \( V \in \sigma \) with \( g(y) \in V \), then \( (g \circ f)(x) \in V \). Since, \( g \circ f \) is \( S_s \)-continuous, there exists an \( S_s \)-open set \( U \) in \( X \) containing \( x \) such that \( (g \circ f)(U) \subseteq V \). By assumption \( H = f(U) \) is an \( S_s \)-open set in \( Y \) and contains \( f(x) = y \). Thus, \( g(H) \subseteq V \). Hence, \( g \) is \( S_s \)-continuous.
Proposition 5.13 If $f_i : X_i \to Y_i$ is $S_s$-continuous functions for $i = 1, 2$. Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be a function defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then, $f$ is $S_s$-continuous.

**proof.** Let $R_1 \times R_2 \subseteq Y_1 \times Y_2$, where $R_i$ is open set in $Y_i$ for $i = 1, 2$. Then, $f^{-1}(R_1 \times R_2) = f_1^{-1}(R_1) \times f_2^{-1}(R_2)$. Since, $f_i$ is $S_s$-continuous for $i = 1, 2$. By Proposition 4.2, $f^{-1}(R_1 \times R_2)$ is $S_s$-open set in $X_1 \times X_2$.

Proposition 5.14 Let $f : X \to Y$ be any function. If the function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$, is an $S_s$-continuous function, then $f$ is $S_s$-continuous.

**proof.** Let $H$ be an open subset of $Y$, then $X \times H$ is an open subset of $X \times Y$. Since $g$ is $S_s$-continuous, then $g^{-1}(X \times H) = f^{-1}(H)$ is an $S_s$-open subset of $X$. Hence $f$ is $S_s$-continuous.

References


