Abstract

In this paper we introduce a new class of multifunction called Upper(lower) $g^{*}bp$-continuous multifunction, Upper(lower) almost $g^{*}bp$-continuous multifunction, Upper(lower) weakly $g^{*}bp$-continuous multifunction and Upper(lower) contrag$^{*}bp$-continuous multifunction in topological spaces, and study some of their basic properties and relations among them.

Keywords: $g^{*}b$-closed set, $g^{*}bp$-continuous, almost $g^{*}bp$-continuous, weakly $g^{*}bp$-continuous.

1. Introduction

Many mathematicians and they devote a great part of their research work on the study of generalised continuous multifunction. In 1999, Mahmoud introduced the concept of pre-irresolute multi-valued function while in 1996 Popa and Noiri and in 2001 Abd-El-Monsef and Nasef introduced other types of multifunctions. Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset $A$ of $X$, $\text{Cl}(A)$ and $\text{Int}(A)$ represents the closure of $A$ and Interior of $A$ respectively. A subset $A$ is said to be preopen [17] (resp., $\alpha$-open [19], semi open [12], regular open [25]) set if $A \subseteq \text{IntCl}(A)$ (resp., $A \subseteq \text{IntClInt}(A)$, $A \subseteq \text{ClInt}(A)$, $A = \text{IntCl}(A)$). The complement of a preopen set is called preclosed.

2. Preliminaries

We recall the following definition.

**Definition 2.1** A subset $A$ of a topological space $(X, \tau)$ is called

1. $b$-open set [3], if $A \subseteq \text{Cl}(@Int(A)) \cup \text{Int}(\text{Cl}(A))$ and $b$-closed set if $\text{Cl}(\text{Int}(A)) \cup \text{Int}(@\text{Cl}(A)) \subseteq A$.
2. generalized closed set (briefly $g$-closed) [11] ($g^{*}$-closed [23]), if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open($g$-open) in $X$.
3. $gb$-closed [20], and ($g^{*}b$-closed [24]) if $b\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open($g$-open) in $X$.
4. $p\delta$-open set [9], if for each $x \in A$, there exists a preopen set $U$ in $X$ such that $x \in U \subseteq p\text{IntpCl}(U) \subseteq A$.
5. regular preopen (resp., regular preclosed) set [6], if $A = p\text{IntpCl}(A)$ (resp. $A = p\text{ClpInt}(A)$).
\textbf{Definition 2.2} \cite{4} A space $X$ is said to be
\begin{enumerate}
\item Pre-$T_0$ if and only if to each pair of distinct points $x$, $y$ in $X$, there exists a preopen set containing one of the points but not the other.
\item Pre-$T_1$ if and only if to each pair of distinct points $x$, $y$ of $X$, there exists a pair of preopen sets one containing $x$ but not $y$ and other containing $y$ but not $x$.
\item Pre-$T_2$ if and only if to each pair of distinct points $x$, $y$ of $X$, there exists a pair of disjoint preopen sets one containing $x$ and the other containing $y$.
\end{enumerate}

\textbf{Definition 2.3} A topological space $(X, \tau)$ is said to be:
\begin{enumerate}
\item $g^b$-$T_0$ if for each pair of distinct points $x$, $y$ in $X$, there exists a $g^b$-open set $U$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
\item $g^b$-$T_1$ if for each pair of distinct points $x$, $y$ in $X$, there exist two $g^b$-open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.
\item $g^b$-$T_2$ if for each distinct points $x$, $y$ in $X$, there exist two disjoint $g^b$-open sets $U$ and $V$ containing $x$ and $y$ respectively.
\item $g^b$-$T_2$ if every $g^b$-closed set is $g$-closed.
\item $g^b$-space if every $g^b$-open set of $X$ is open in $X$.
\end{enumerate}

\textbf{Definition 2.4} A topological space $(X, \tau)$ is said to be:
\begin{enumerate}
\item submaximal \cite{7}, if the closure of every open set of $X$ is $X$.
\item extremally disconnected \cite{15}, if the closure of every open set of $X$ is open in $X$.
\item pre-$T_2$ \cite{16}, space if every $pg$-closed set is preclosed.
\item $r$-$T_2$ \cite{8}, if for each pair of distinct points $x$ and $y$ of $X$, there exists regular open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $y \notin U$ and $x \notin V$.
\end{enumerate}

\textbf{Theorem 2.5} \cite{7} A space $X$ is submaximal if and only if every preopen set is open.

\textbf{Theorem 2.6} \cite{2} Let $(Y, \tau_Y)$ be subspace of a space $(X, \tau)$. If $A \in PO(X, \tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$.

\textbf{Theorem 2.7} \cite{25} Let $A$ be a subset of a topological space $(X, \tau)$, if $A \in \tau$, then $Cl_0(A) = Cl(A)$.

\textbf{Theorem 2.8} \cite{24} Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $g^b$-closed in $X$, then $A$ is $g^b$-closed relative to $Y$.

\textbf{Definition 2.9} \cite{14} A multifunction $F : X \to Y$ is said to be:
\begin{enumerate}
\item Upper pre-irresolute at $x \in X$ if for each preopen set $A$ of $Y$ containing $F(x)$ ($F(x) \cap V \neq \emptyset$), there exists a preopen set $U$ of $X$ containing $x$ such that $F(U) \subseteq A$.
\item Lower pre-irresolute at $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \cap A \neq \emptyset$, there exists a preopen set $U$ of $X$ containing $x$ such that $F(u) \cap A \neq \emptyset$ for every $u \in U$.
\end{enumerate}

\textbf{Definition 2.10} \cite{1} For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set $A$ of $Y$ by $F^+(A)$ and $F^-(A)$, respectively, that is, $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

\textbf{Definition 2.11} A multifunction $F : X \to Y$ is said to be:
\begin{enumerate}
\item Upper $\alpha$-continuous \cite{21} at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq V$.
\item Lower $\alpha$-continuous \cite{21} at $x \in X$ if for each open set $A$ of $Y$ such that $F(x) \cap A \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap A \neq \emptyset$ for every $u \in U$.
\end{enumerate}
3. Upper (Lower) $\alpha$-continuous [18] if it has this property at each point of $X$.

Definition 2.12 [22] A multifunction $F : X \to Y$ is said to be;

1. Upper almost $\alpha$-continuous at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq \text{IntCl}(V)$.

2. Lower almost $\alpha$-continuous at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap A \neq \phi$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap \text{IntCl}(V) \neq \phi$ for every $u \in U$.

Definition 2.13 [13] A multifunction $F : X \to Y$ is said to be;

1. Upper $\delta$-continuous at $x \in X$ if for each regular open set $V$ of $Y$ containing $F(x)$, there exists a regular open set $U$ of $X$ such that $F(U) \subseteq V$.

2. Lower $\delta$-continuous at $x \in X$ if for each regular open set $V$ of $Y$ such that $F(x) \cap A \neq \phi$, there exists a regular open set $U$ of $X$ such that $F(u) \cap V \neq \phi$ for every $u \in U$.

Definition 2.14 [1] A multifunction $F : X \to Y$ is said to be;

1. Upper $b$-continuous at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists a $b$-open set $U$ of $X$ such that $F(U) \subseteq V$.

2. Lower $b$-continuous at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap A \neq \phi$, there exists a $b$-open set $U$ of $X$ such that $F(u) \cap V \neq \phi$ for every $u \in U$.

3. Upper and lower $g^*bp$-continuous multifunction

In this section, we introduce the concept of upper and lower $g^*bp$-continuous multifunctions in topological spaces.

Definition 3.1 A multifunction $F : X \to Y$ is said to be:

1. Upper $g^*bp$-continuous ($Ug^*bp.c.$) at $x \in X$ if for each preopen set $A$ of $Y$ containing $F(x)$, there exists a $g^*bp$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq A$.

2. Lower $g^*bp$-continuous ($Lg^*bp.c.$) at $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \cap A \neq \phi$, there exists a $g^*bp$-open set $U$ of $X$ containing $x$ such that $F(u) \cap A \neq \phi$ for every $u \in U$.

3. Upper (Lower) $g^*bp$-continuous if it has this property at each point of $X$.

Proposition 3.2 Let $X$ and $Y$ be topological spaces. For a multifunction $F : X \to Y$, the following statements are equivalents:

1. $F$ is $Ug^*bp.c.$ ($Lg^*bp.c.$),
2. For every preopen set $A$, $F^+(A)(F^-(A))$ is a $g^*bp$-open set in $X$,
3. For every preclosed set $K$, $F^-(K)(F^+(K))$ is a $g^*bp$-closed set in $X$.

Proof. $(1) \Rightarrow (2)$. If $A$ is preopen set of $Y$, then for each $x \in F^+(A), F(x) \subseteq A$. By $(1)$ there exists a $g^*bp$-open set $U$ of $x$ such that $F(U) \subseteq A$ which implies that $x \in U \subseteq F^+(A)$, therefore $F^+(A)$ is $g^*bp$-open in $X$.

$(2) \Rightarrow (3)$. Let $K$ be preclosed set of $Y$. Then $Y \setminus K$ is preopen set of $Y$. By $(2)$, $F^+(Y \setminus K) = X \setminus F^-(K)$ is $g^*bp$-open set in $X$ and hence $F^-(K)$ is $g^*bp$-closed in $X$.

$(3) \Rightarrow (1)$. Let $A$ be any preopen set of $Y$. Then $(Y \setminus A)$ is preclosed in $Y$. By $(3)$, $F^-(Y \setminus A)$ is $g^*bp$-closed set in $X$. But $F^-(Y \setminus A) = X \setminus F^+(A)$. Thus $X \setminus F^+(A)$ is $g^*bp$-closed in $X$ so $F^+(A)$ is $g^*bp$-open in $X$. Therefore, we obtain $F(F^+(A)) \subseteq A$, hence $F$ is $g^*bp$-continuous.

The proof for the case where $F$ is $Lg^*bp.c.$ is similarly proved.

Theorem 3.3 If a multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper $b$-continuous and $Y$ is submaximal, then $F$ is upper $g^*bp$-continuous.
proof. Let $A$ be preopen set in $Y$, since $Y$ is submaximal then $A$ is open set in $Y$. Since $F$ is upper $b$-continuous, then $F^+(A)$ is $b$-open in $X$ and by Theorem(3.4) [24], it is $g^b$-open in $X$. Hence $F$ is upper $g^b$-continuous.

Proposition 3.4 Let $X = R_1 \cup R_2$, where $R_1$ and $R_2$ are $g^b$-closed set in $X$. Let $F : R_1 \rightarrow Y$ and $G : R_2 \rightarrow Y$ be upper $g^b$-continuous. If $F(x) = G(x)$ for each $x \in R_1 \cap R_2$. Then $H : R_1 \cup R_2 \rightarrow Y$ such that

$$H(x) = \begin{cases} F(x) & \text{if } x \in R_1 \\ G(x) & \text{if } x \in R_2 \end{cases}$$

is upper $g^b$-continuous.

proof. Let $A$ be any preopen set in $Y$. Clearly $H^+(A) = F^+(A) \cup G^+(A)$. Since $F$ is upper $g^b$-continuous, then $F^+(A)$ is $g^b$-open in $R_1$. But $R_1$ is $g^b$-open in $X$. Then by Theorem (3.30) [24], $F^+(A)$ is $g^b$-open in $X$. Similarly, $G^+(A)$ is $g^b$-open in $R_2$ and hence a $g^b$-open in $X$. Since a union of two $g^b$-open sets is $g^b$-open. Therefore, $H^+(A) = F^+(A) \cup G^+(A)$ is $g^b$-open in $X$. Hence $H$ is upper $g^b$-continuous.

Theorem 3.5 For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ the following are equivalent.

1. $F$ is upper $g^b$-continuous.
2. $F(g^bCl(B)) \subseteq pCl(F(B))$, for every subset $B$ of $X$,
3. $g^bCl(F^+(A)) \subseteq F^+(pCl(A))$, for each subset $A$ of $Y$,
4. $F^-(pInt(A)) \subseteq g^bInt(F^-(A))$, for each subset $A$ of $Y$,
5. $pInt(F(B)) \subseteq F(g^bInt(B))$, for each subset $B$ of $X$.

proof. (1) $\Rightarrow$ (2). Let $B$ be any subset of $X$. Then $F(B) \subseteq pCl(F(B))$ and $pClF(B)$ is preclosed in $Y$. Hence $B \subseteq F^+(pCl(F(B)))$, since $F$ is $g^b$-continuous. By Proposition 3.2, $F^+(pCl(F(B)))$ is $g^b$-closed set in $X$. Therefore, $g^bCl(B) \subseteq F^+(pCl(F(B)))$. Hence $F(g^bCl(B)) \subseteq F^+(pCl(F(B)))$.

(2) $\Rightarrow$ (3). Let $A$ be any subset of $Y$, then $F^+(A)$ is a subset of $X$. By (2) we have $F(g^bCl(F^+(A)) \subseteq pCl(F(F^+(A)))) = pCl(A)$. It follow that $g^bCl(F^+(A)) \subseteq F^+(pCl(A))$.

(3) $\Rightarrow$ (4). Let $A$ be any subset of $Y$. Then apply(3) to $(Y, A)$ we obtain $g^bCl(F^+(Y \setminus A)) \subseteq F^+(pCl(Y \setminus A)) \Rightarrow g^bCl(X \setminus F^-(A)) \subseteq F^+(Y \setminus pInt(A)) \Rightarrow X \setminus g^bInt(F^-(A)) \subseteq X \setminus F^-(pInt(A)) \Rightarrow F^-(pInt(A)) \subseteq g^bInt(F^-(A))$.

(4) $\Rightarrow$ (5). Let $B$ be any subset of $X$, Then $F(B)$ is a subset of $Y$. By (4), we have $F^-(pInt(F(A))) \subseteq g^bInt(F^-(A))$. Therefore, $pInt(F(A)) \subseteq F(g^bInt(F(A)))$.

(5) $\Rightarrow$ (1). Let $x \in X$ and let $A$ be any preopen set of $Y$ containing $F(x)$. Then $x \in F^+(A)$ and $F^+(A)$ is a subset of $X$. By (5), we have $pInt(F(A)) \subseteq F(g^bInt(F^+(A)))$. Then $pInt(A) \subseteq F(g^bInt(F^+(A)))$, since $A$ is preopen, then $A \subseteq F(g^bInt(F^+(A)))$ implies that $F^+(A) \subseteq g^bInt(F^+(A))$. Therefore $F^+(A)$ is $g^b$-open in $X$ containing $x$ and clearly $F(F^+(A)) \subseteq A$. Hence $F$ is upper $g^b$-continuous.

Proposition 3.6 Let $F : X \rightarrow Y$ be upper $g^b$-continuous and $Y \subseteq Z$. If $Y$ is preclosed subset of a topological space $Z$ then $F : X \rightarrow Z$ is upper $g^b$-continuous.

proof. Let $K$ be any preclosed set in $Z$. Then $K \cap Y$ is preclosed in $Z$, by Theorem(2.22) [2], it is preclosed in $Y$. Since $F$ is upper $g^b$-continuous $F^+(K \cap Y)$ is $g^b$-closed in $X$ but $F(x) \in Y$ for each $x \in X$, and thus $F^+(K) = F^+(K \cap Y)$ is $g^b$-closed subset of $X$. Therefore, by Proposition 3.2 $F : X \rightarrow Z$ is upper $g^b$-continuous.

Theorem 3.7 If $F : X \rightarrow Y$ is upper $g^b$-continuous and $A$ is $g^b$-closed set in $X$ then $F[A] : A \rightarrow Y$ is upper $g^b$-continuous.

proof. Let $B$ be preclosed set in $Y$, since $F$ is upper $g^b$-continuous, then $F^+(B)$ is $g^b$-closed in $X$. If $F^+(B) \cap A = A_1$ then $A_1$ is $g^b$-closed in $X$, since intersection of two $g^b$-closed is $g^b$-closed. Since $(F[A])^+(A) = A_1$ by Theorem 2.8, $A_1$ is $g^b$-closed set in $A$. Therefore $F[A]$ is upper $g^b$-continuous.

Theorem 3.8 If $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ be any two multifunction, then $G \circ F : X \rightarrow Z$ is upper $g^b$-continuous if $G$ is preirresolute multifunction and $F$ is upper $g^b$-continuous.

proof. Let $A$ be any preclosed set in $Z$. Since $G$ is preirresolute multifunction then $G^+(A)$ is preclosed in $Y$, since $F$ is upper $g^b$-continuous then $F^+(G^+(A))$ is $g^b$-closed in $X$. Hence $G \circ F$ is upper $g^b$-continuous.
Theorem 3.9 If $F : X \to Y$ is a upper $g^*bp$-continuous injection and $Y$ is pre-$T_1$, then $X$ is $g^*bT_1$.

**proof.** Assume that $Y$ is pre-$T_1$. For any distinct points $x$ and $y$ in $X$, there exists preopen set $A$ and $W$ such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since $F$ is upper $g^*bp$-continuous, so there exists a $g^*b$-open sets $G$ and $H$ such that $x \in G$, $y \in H$, $F(G) \subseteq A$ and $F(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. This show that $X$ is $g^*bT_1$.

Theorem 3.10 If $F : X \to Y$ is upper $g^*bp$-continuous injection and $Y$ is pre-$T_2$ then $X$ is $g^*bT_2$.

**proof.** For any pair of distinct points $x$ and $y$ in $X$, there exists disjoint preopen sets $U$ and $V$ in $Y$ such that $F(x) \in U$ and $F(y) \in V$. Since $F$ is upper $g^*bp$-continuous, there exists $g^*b$-open sets $G$ and $H$ in $X$ containing $x$ and $y$, respectively, such that $F(G) \subseteq U$ and $F(H) \subseteq V$. Since $U$ and $V$ are disjoint, we have $U \cap V = \phi$, hence $G \cap H = \phi$. This shows that $X$ is $g^*bT_2$.

Theorem 3.11 An upper $g^*bp$-continuous image of a $g^*b$-connected space is $g^*b$-connected for a multifunction $F$.

**proof.** Let $F : X \to Y$ be an upper $g^*bp$-continuous multifunction from a $g^*b$-connected space $X$ onto a space $Y$. Suppose $Y$ is not connected and let $Y = A \cup B$ be a partition of $Y$. Then both $A$ and $B$ are preopen and preclosed subset of $Y$. Since $F$ is upper $g^*bp$-continuous, $F^+(A)$ and $F^+(B)$ are $g^*b$-open subset of $X$. In view of the fact that $F^+(A)$ and $F^+(B)$ are disjoint, $X = F^+(A) \cup F^+(B)$ is a partition of $X$. This is contrary to the connectedness of $X$.

**Definition 3.12** A multifunction $F : X \to Y$ is said to be;

1. Upper almost $g^*bp$-continuous at a point $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \in A$, there exists a $g^*b$-open set $U$ containing $x$ such that $F(U) \subseteq \text{IntCl}(A)$.
2. Lower almost $g^*bp$-continuous at a point $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \in A$, there exists a $g^*b$-open set $U$ of $X$ containing $x$ such that $F(U) \cap \text{IntCl}(A) \neq \phi$.
3. Upper (Lower) almost $g^*bp$-continuous if it has this property at each point of $X$.

Theorem 3.13 A multifunction $F : X \to Y$ is upper almost $g^*bp$-continuous if and only if for each $x \in X$ and each regular open set $A$ containing $F(x)$, there exists a $g^*b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq A$.

**proof.** For every $x \in X$ and let $A$ be any regular open set containing $F(x)$, then $A$ is preopen set containing $F(x)$. Since $F$ is upper almost $g^*bp$-continuous, then there exists a $g^*b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq \text{IntCl}(A)$. Conversely. Assume that for all regular open set $A$ containing $F(x)$, there exists a $g^*b$-open set $U$ in $X$ containing $x$ with $F(U) \subseteq A = \text{IntCl}(A)$ then $A$ is preopen set and hence $F$ is upper almost $g^*bp$-continuous.

Theorem 3.14 For a multifunction $F : X \to Y$, the following statements are equivalent:

1. $F$ upper almost $g^*bp$-continuous,
2. $F^+(\text{IntCl}(A))$ is $g^*b$-open set in $X$, for each preopen set $A$ in $Y$,
3. $F^-(\text{ClInt}(B))$ is $g^*b$-closed set in $X$, for each preclosed set $B$ in $Y$,
4. $F^-(B)$ is $g^*b$-closed set in $X$, for each regular closed set $B$ in $Y$,
5. $F^+(A)$ is $g^*b$-open set in $X$, for each regular open set $A$ in $Y$.

**proof.** (1) $\Rightarrow$ (2). Let $A$ be any preopen set in $Y$. We have to show that $F^+(\text{IntCl}(A))$ is $g^*b$-open set in $X$. Let $x \in F^+(\text{IntCl}(A))$. Then $F(x) \in \text{IntCl}(A)$ and $\text{IntCl}(A)$ is regular open set in $Y$. Since $F$ is upper almost $g^*bp$-continuous. By Theorem 3.13, there exists a $g^*b$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \text{IntCl}(A)$. Which implies that $x \in U \subseteq F^+(\text{IntCl}(A))$. Therefore, $F^+(\text{IntCl}(A))$ is $g^*b$-open set in $X$.

(2) $\Rightarrow$ (3). Let $B$ be any preclosed set of $Y$. Then $Y \setminus B$ is preopen set of $Y$. By (2), $F^+(\text{IntCl}(Y \setminus B))$ is $g^*b$-open set in $X$ and $F^+(\text{IntCl}(Y \setminus B)) = F^+(\text{Int}(Y \setminus \text{Int}(B))) = F^+(Y \setminus \text{ClInt}(B)) = X \setminus F^-(\text{ClInt}(B))$ is $g^*b$-open set in $X$ and hence $F^-(\text{ClInt}(B))$ is $g^*b$-closed set in $X$.

(3) $\Rightarrow$ (4). Let $B$ be any regular closed set of $Y$. Then $B$ is preclosed set of $Y$. By (3), $F^-(\text{ClInt}(B))$ is $g^*b$-closed set in $X$ since $B$ is regular closed set, then $F^-(\text{ClInt}(B))$ is $F^-(B)$. Therefore $F^-(B)$ is $g^*b$-closed set in $X$. 


Theorem 3.15 If a multifunction $F : X \rightarrow Y$ is upper $g^*\text{bp}$-continuous, then it is upper almost $g^*\text{bp}$-continuous but not conversely.

**proof.** Let $A$ be any regular open set in $Y$, so is preopen in $Y$. Since $F$ is upper $g^*\text{bp}$-continuous then $F^+(A)$ is $g^*\text{bp}$-open in $X$. Hence by Theorem 3.14, $F$ is upper almost $g^*\text{bp}$-continuous.

Remark 3.16 The converse of the theorem need not be true in general.

Example 3.17 Consider $X = Y = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, Y\}$ and with the identity multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, $F$ is upper almost $g^*\text{bp}$-continuous but not upper $g^*\text{bp}$-continuous since for preclosed set $B = \{b, c\}$ in $Y$, $F^+(B) = \{b, c\}$ is not $g^*\text{bp}$-closed in $X$.

Theorem 3.18 If a multifunction $F : X \rightarrow Y$ is upper almost $\alpha$-continuous then $F$ is upper almost $g^*\text{bp}$-continuous.

**proof.** Let $A$ be any regular open set in $Y$. Since $F$ is upper almost $\alpha$-continuous then $F^+(A)$ is semi open set in $X$, hence by Theorem (3.10) [24], is $g^*\text{bp}$-open in $X$. Therefore, $F$ is upper almost $g^*\text{bp}$-continuous.

Theorem 3.19 If a multifunction $F : X \rightarrow Y$ is upper $\delta$-continuous, then $F$ is upper almost $g^*\text{bp}$-continuous.

**proof.** Let $x \in X$ and let $A$ be any preopen set in $Y$, then $A \subseteq \text{IntCl}(A)$. Since $F$ is upper $\delta$-continuous, there exists an regular open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \text{IntCl}(\text{IntCl}(A))$, then $F(U) \subseteq \text{IntCl}(A)$. Since $U$ is regular open set, then it is preopen and by Theorem (3.12) [24], $U$ is $g^*\text{bp}$-open set of $X$. Therefore, $F$ is upper almost $g^*\text{bp}$-continuous.

Theorem 3.20 If $F : X \rightarrow Y$ is upper almost $g^*\text{bp}$-continuous function, then we have $F^-(A) \subseteq g^*\text{Int}(F^+(\text{IntCl}(A)))$ for every preopen set $A$ in $Y$.

**proof.** Let $A$ be any preopen set in $Y$, then $A \subseteq \text{IntCl}(A)$. Since $\text{IntCl}(A)$ is regular open set in $Y$, and Since $F$ is upper almost $g^*\text{bp}$-continuous multifunction, so by Theorem 3.14, $F^+(\text{IntCl}(A))$ is $g^*\text{bp}$-open set in $X$. So $F^+(A) \subseteq F^+(\text{IntCl}(A)) = g^*\text{Int}(F^+(\text{IntCl}(A)))$.

**Corollary 3.21** If $F : X \rightarrow Y$ is lower almost $g^*\text{bp}$-continuous function, then we have $g^*\text{bCl}(F^-(\text{ClInt}(E))) \subseteq F^-(E)$, for every preclosed set $E$ in $Y$.

**proof.** Let $E$ be any preclosed set in $Y$, so $Y \setminus E$ is preopen. By Theorem 3.20, $F^+(Y \setminus E) \subseteq g^*\text{Int}(F^+(Y \setminus \text{ClInt}(E)))$ this implies that $X \setminus F^-(E) \subseteq g^*\text{Int}(F^+(Y \setminus \text{ClInt}(E)))$, then $X \setminus F^-(E) \subseteq g^*\text{Int}(X \setminus F^-(\text{ClInt}(E)))$, it follow that $X \setminus F^-(E) \subseteq X \setminus g^*\text{bCl}(F^-(\text{ClInt}(E)))$. Hence $g^*\text{bCl}(F^-(\text{ClInt}(E))) \subseteq F^-(E)$.

Theorem 3.22 Let $F : X \rightarrow Y$ be an upper almost $g^*\text{bp}$-continuous. If $Y$ is preopen set in $Z$, then $F : X \rightarrow Z$ is upper almost $g^*\text{bp}$-continuous.

**proof.** Let $A$ be any regular open set of $Z$. Since $Y$ is preopen, then $A \cap Y$ is regular open set in $Y$ [see [10]]. Since $F$ is upper almost $g^*\text{bp}$-continuous then $F^+(A \cap Y)$ is $g^*\text{bp}$-open set in $X$. But $F(x) \in Y$ for each $x \in X$. Thus $F^+(A) = F^+(A \cap Y)$ is a $g^*\text{bp}$-open set in $X$. Therefore $F$ is upper almost $g^*\text{bp}$-continuous.

Theorem 3.23 If $F : X \rightarrow Y$ is an upper almost $g^*\text{bp}$-continuous multifunction and $A$ is $g^*\text{b}$-closed set of $X$, then the restriction function $F|A : A \rightarrow Y$ is almost $g^*\text{bp}$-continuous multifunction.

**proof.** Let $B$ be any regular closed set of $Y$. Since $F$ is upper almost $g^*\text{bp}$-continuous multifunction, then by Theorem 3.14, $F^+(B)$ is $g^*\text{b}$-closed set in $X$, and $(F|A)^+(B) = A \cap F^+(B)$. Since $A$ is $g^*\text{b}$-closed, so $A \cap F^+(B)$ is $g^*\text{b}$-closed set in $A$ (see Theorem 2.8). Hence $F|A$ is upper almost $g^*\text{bp}$-continuous multifunction.
Theorem 3.24 If $F: X \to Y$ is an upper almost $g^*bp$-continuous injection and $Y$ is r-$T_1$, then $X$ is $g^*b - T_1$.

proof. Assume that $Y$ is r-$T_1$. For any distinct points $x$ and $y$ in $X$, there exists regular open set $A$ and $W$ such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since $F$ is upper almost $g^*bp$-continuous there exists a $g^*b$-open set $G$ and $H$ such that $x \in G$, $y \in H$, $F(G) \subseteq A$ and $F(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. This show that $X$ is $g^*b - T_1$.

Theorem 3.25 If $F: X \to Y$ is upper almost $g^*bp$-continuous and $Y$ is pre-$T_2$ then $X$ is $g^*b - T_2$.

proof. For any pair of distinct points $x$ and $y$ in $X$, there exists disjoint preopen sets $U$ and $V$ in $Y$ such that $F(x) \in U$ and $F(y) \in V$. Since $F$ is upper almost $g^*bp$-continuous, there exists $g^*b$-open sets $G$ and $H$ in $X$ containing $x$ and $y$, respectively, such that $F(G) \subseteq IntCl(U)$ and $F(H) \subseteq IntCl(V)$. Since $U$ and $V$ are disjoint, we have $IntCl(U) \cap IntCl(V) = \phi$, hence $G \cap H = \phi$. This shows that $X$ is $g^*b - T_2$.

4. Weakly $g^*bp$-continuous multifunction

Definition 4.1 A multifunction $F: X \to Y$ is said to be:

1. Upper weakly $g^*bp$-continuous at a point $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \in A$, there exists a $g^*b$-open set $U$ containing $x$ such that $F(U) \subseteq Cl(A)$.

2. Lower weakly $g^*bp$-continuous at a point $x \in X$ if for each preopen set $A$ of $Y$ such that $F(x) \in A$, there exists a $g^*b$-open set $U$ of $X$ containing $x$ such that $F(U) \cap Cl(A) \neq \phi$.

3. Upper (Lower) almost $g^*bp$-continuous if it has this property at each point of $X$.

Theorem 4.2 Let $F: X \to Y$ be a multifunction. If $F^+(ClA)$ is $g^*b$-open set in $X$ for each preopen set $A$ in $Y$, then $F$ is upper weakly $g^*bp$-continuous.

proof. Let $x \in X$ and let $A$ be any preopen set of $Y$ containing $F(x)$. Then $x \in F^+(A) \subseteq F^+(ClA)$. By hypothesis, we have $F^+(ClA)$ is $g^*b$-open set in $X$ containing $x$. Therefore, we obtain $F(F^+(ClA)) \subseteq ClA$. Hence $F$ is upper weakly $g^*bp$-continuous.

It is obvious that upper almost $g^*bp$-continuous implies upper weakly $g^*bp$-continuous. However, the converse is not true in general as it shown in the following example.

Example 4.3 Consider $X = Y = \{a, b, c, d\}$ with the topology $\tau = \sigma = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, with identity multifunction $F: (X, \tau) \to (Y, \sigma)$ $F$ is upper weakly $g^*bp$-continuous but not upper almost $g^*bp$-continuous since for a preopen set $B = \{a, b\}$ in $Y$ $F^+(IntClB) = \{a, b\}$ which is not $g^*b$-open in $X$.

Theorem 4.4 If $F: X \to Y$ is upper weakly $g^*bp$-continuous multifunction and $Y$ is almost p-regular, then $F$ is upper almost $g^*bp$-continuous.

proof. Let $x \in X$ and let $A$ be preopen set of $Y$. By the almost p-regularity of $Y$ there exists a regular open set $G$ of $Y$ such that $F(x) \in G \subseteq Cl(G) \subseteq IntCl(A)$. Since $F$ is upper weakly $g^*bp$-continuous, there exists a $g^*b$-open set $U$ in $X$ such that $F(U) \subseteq Cl(G) \subseteq IntCl(A)$. Therefore $F$ is almost $g^*bp$-continuous.

Theorem 4.5 Let $F: X \to Y$ be a multifunction. If for each $x \in X$ and each regular closed set $R$ of $Y$ containing $F(x)$, there exists a $g^*b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq R$, then $F$ is upper weakly $g^*bp$-continuous.

proof. Let $x \in X$ and let $A$ be any preopen set of $Y$ containing $F(x)$. Then put $R = Cl(A)$ which is a regular closed set of $Y$ containing $F(x)$. By hypothesis, there exists a $g^*b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq R$. Hence $F$ is upper weakly $g^*bp$-continuous.

Theorem 4.6 Let $F: X \to Y$ be a multifunction. If the inverse image of each regular closed set of $Y$ is a $g^*b$-open set in $X$, then $F$ is upper weakly $g^*bp$-continuous.

proof. Let $A$ be any preopen set of $Y$. Then $Cl(A)$ is a regular closed set in $Y$. By hypothesis, we have $F^+(Cl(A))$ is a $g^*b$-open set in $X$. Therefore, by Theorem 4.2, $F$ is upper weakly $g^*bp$-continuous.

Corollary 4.7 Let $F: X \to Y$ be a multifunction. If the inverse image of each regular open set of $Y$ is a $g^*b$-closed set in $X$, then $F$ is upper weakly $g^*bp$-continuous.
Corollary 4.8 Let $F : X \rightarrow Y$ be a multifunction. If $F^+(\text{Int}F)$ is $g^*b$-closed set in $X$ for each preclosed set $F$ in $Y$, then $F$ is upper weakly $g^*b$-continuous.

Theorem 4.9 Let $F : X \rightarrow Y$ be upper weakly $g^*b$-continuous multifunction, if $A$ is $g^*b$-closed subset of $X$, then the restriction $F|A : A \rightarrow Y$ is upper weakly $g^*b$-continuous in the subspace $A$.

proof. Let $x \in A$ and let $B$ be a preclosed set of $Y$ containing $F(x)$. Since $F$ is upper weakly $g^*b$-continuous, by Corollary 4.8, $F^+(\text{Int}B)$ is $g^*b$-closed set in $X$, and $(F|A)^+(\text{Int}B) = A \cap F^+(\text{Int}B)$ is $g^*b$-closed in $X$, by Theorem (3.30)[24], it is $g^*b$-closed in $A$. Hence $F|A$ is upper weakly $g^*b$-continuous.

Theorem 4.10 Let $F : X \rightarrow Y$ be upper weakly $g^*b$-continuous multifunction and for each $x \in X$. If $Y$ is any subset of $Z$ containing $F(x)$, then $F : X \rightarrow Z$ is upper weakly $g^*b$-continuous.

proof. Let $x \in X$ and $A$ be any preclosed set of $Z$ containing $F(x)$. Then $A \cap Y$ is preclosed in $Y$ containing $F(x)$. Since $f : X \rightarrow Y$ is upper weakly $g^*b$-continuous, there exists a $g^*b$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \text{Cl}(A \cap Y)$ and hence $F(U) \subseteq \text{Cl}A$. Therefore, $F : X \rightarrow Z$ is upper weakly $g^*b$-continuous.

Theorem 4.11 For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

1. $f$ is upper weakly $g^*b$-continuous,
2. $g^*bClF^+(\text{IntpCl}B) \subseteq F^+(\text{pCl}B)$, for each $B \subseteq Y$,
3. $F^-(\text{pInt}B) \subseteq g^*b\text{Int}F^-(\text{ClpInt}B)$, for each $B \subseteq Y$,
4. $F^-(\text{pIntpCl}A) \subseteq g^*b\text{Int}F^-(\text{Clp}A)$, for each preclosed set $A$ of $Y$,
5. $F^-(A) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$, for each regular preclosed set $A$ of $Y$,
6. $g^*bClF^+(\text{IntF}) \subseteq F^+(\text{Cl}F)$, for each regular preclosed set $F$ of $Y$,
7. $g^*bClF^+(\text{IntF}) \subseteq F^+(\text{Cl}F)$, for each preclosed set $F$ of $Y$,
8. $g^*bClF^+(A) \subseteq F^+(\text{Cl}A)$, for each preclosed set $A$ of $Y$,
9. $F^-(\text{IntF}) \subseteq g^*b\text{Int}F^-(F)$, for each preclosed set $F$ of $Y$.

proof. (1) $\Rightarrow$ (2). Let $B$ be any subset of $Y$. Assume that $x \notin F^+(\text{pCl}B)$. Then $F(x) \notin \text{pCl}B$ and there exists a preclosed set $A$ containing $F(x)$ such that $A \cap B = \emptyset$, hence $A \cap \text{IntpCl}B = \emptyset$, then $A \subseteq Y \setminus (\text{IntpCl}B)$ and $\text{Cl}A \cap \text{IntpCl}B = \emptyset$. Hence, by (1), there exists a $g^*b$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \text{Cl}A$. Therefore, we have $f(U) \cap \text{IntpCl}B = \emptyset$ which implies $U \cap F^+(\text{IntpCl}B) = \emptyset$ and hence $x \notin g^*bClF^+(\text{IntpCl}B)$. Therefore, we obtain $g^*bClF^+(\text{IntpCl}B) \subseteq F^+(\text{pCl}B)$.

(2) $\Rightarrow$ (3). Let $B$ be any subset of $Y$. Then apply (2) to $Y \setminus B$ we obtain $g^*bClF^+(\text{IntpCl}(Y \setminus B)) \subseteq F^+(\text{pCl}(Y \setminus B)) \Rightarrow g^*bClF^+(\text{IntpCl}(Y \setminus pIntB)) \subseteq F^+(Y \setminus pIntB) \Rightarrow \text{ClpInt}F(Y \setminus pIntB) \subseteq \text{ClpInt}F(X \setminus F^+(\text{ClpIntB})) \subseteq X \setminus F^+(\text{ClpIntB}) \Rightarrow g^*bClF(X \setminus F^+(\text{ClpIntB})) \subseteq g^*bClF^+(\text{IntpCl}A)$, for each preclosed set $A$ of $Y$.

(3) $\Rightarrow$ (4). Let $A$ be any preclosed set of $Y$. Then apply (3) to $pClA$ we obtain $F^-(pClpIntClA) \subseteq g^*b\text{Int}F^-(\text{Cl}pClIntClA) = g^*b\text{Int}F^-(\text{Cl}A)$. Therefore we obtain $F^-(pClpIntClA) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$.

(4) $\Rightarrow$ (5). Let $A$ be any regular preclosed set of $Y$. Then $A$ is preclosed set of $Y$. By (4) we have $F^-(A) = F^-(pClpIntClA) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$. Therefore we obtain $F^-(A) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$.

(5) $\Rightarrow$ (6). Let $F$ be any regular preclosed set of $Y$. Then $Y \setminus F$ is a regular preclosed set of $Y$. By (5), we have $F^-(Y \setminus F) \subseteq g^*b\text{Int}F^-(\text{Cl}(Y \setminus F)) \Rightarrow X \setminus F^+(F) \subseteq g^*b\text{Int}F^-(Y \setminus \text{Int}F) \Rightarrow X \setminus F^+(F) \subseteq g^*b\text{Int}F^-(Y \setminus \text{Int}F) \Rightarrow F^-(Y \setminus F) \subseteq g^*b\text{Int}F^-(Y \setminus \text{Int}F) \Rightarrow F^-(F) \subseteq g^*b\text{Cl}F^+(\text{Int}F) \subseteq F^+(\text{Int}F) \subseteq F^+(F)$. Hence $g^*b\text{Cl}F^+(\text{Int}F) \subseteq F^+(F)$.

(6) $\Rightarrow$ (7). Let $F$ be any preclosed set of $Y$. Then $\text{ClpInt}F$ is regular preclosed set of $Y$. By (6) we have $g^*b\text{Cl}F^+(\text{IntpClpInt}F) = g^*b\text{Cl}F^+(\text{Int}F) \subseteq F^+(\text{pClpInt}F)$. Therefore we obtain $g^*b\text{Cl}F^+(\text{Int}F) \subseteq F^+(\text{pClpInt}F)$.

(7) $\Rightarrow$ (8). Let $A$ be any preclosed set of $Y$. Then by (7), we have $g^*b\text{Cl}F^+(A) \subseteq g^*b\text{Cl}F^+(\text{Int}ClA) \subseteq F^+(\text{pClpInt}ClA) \subseteq F^+(\text{Cl}A) \subseteq F^+(\text{Cl}A)$. Therefore we obtain $g^*b\text{Cl}F^+(\text{Int}F) \subseteq F^+(\text{Cl}F)$.

(8) $\Rightarrow$ (9). Let $F$ be any preclosed set of $Y$. Then $Y \setminus F$ is preclosed set of $Y$. By (8), we have $g^*b\text{Cl}F^+(Y \setminus F) \subseteq F^+(\text{Cl}F) \Rightarrow g^*b\text{Cl}(X \setminus F^-) \subseteq F^+(Y \setminus \text{Int}F) \Rightarrow X \setminus g^*b\text{Int}F^-(F) \subseteq X \setminus F^-(\text{Int}F) \subseteq F^-(\text{Int}F) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$. Therefore we obtain $F^-(\text{Int}F) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$.

(9) $\Rightarrow$ (1). Let $x \in X$ and let $A$ be any preclosed set in $Y$ containing $F(x)$. Then $x \in F^-(A)$ and $\text{Cl}A$ is a closed set, hence preclosed, in $Y$. By (9), we have $x \in F^-(A) \subseteq F^-(\text{Int}ClA) \subseteq g^*b\text{Int}F^-(\text{Cl}A)$. If we put $U = g^*b\text{Int}F^-(\text{Cl}A)$, then we obtain that $x \in U$ and $F(U) \subseteq \text{Cl}A$. Therefore, $F$ is weakly $g^*b$-continuous.
Theorem 4.12 The following are equivalent for a function $f: X \to Y$.

1. $F$ is upper weakly $g^{*}bp$-continuous,
2. $F(g^{*}bCl(A)) \subseteq Cl_{0}(F(A))$ for each subset $A$ of $X$,
3. $g^{*}bCl(F^{+}(B)) \subseteq F^{+}(Cl_{0}(B))$ for each subset $B$ of $Y$,
4. $g^{*}bCl(F^{+}(Int(Cl_{0}(B)))) \subseteq F^{+}(Cl_{0}(B))$ for every subset $B$ of $Y$.

Proof. (1) $\Rightarrow$ (2). Let $A$ be any subset of $X$. Suppose that $F(g^{*}bCl(A)) \not\subseteq Cl_{0}(F(A))$. Then there exists $y \in F(g^{*}bCl(A))$ such that $y \notin Cl_{0}(F(A))$, so there exists an open set $G$ in $Y$ containing $y$ such that $ClG \cap F(A) = \phi$. If $F^{+}(y) = \phi$, then there is nothing to prove. Suppose that $x$ be an arbitrary point of $F^{+}(y)$, so $F(x) \in G$. Since $G$ is open then it is preopen in $Y$ and by (1), there exists a $g^{*}b$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq Cl(G)$. Therefore, we have $F(U) \cap F(A) = \phi$, so $x \notin g^{*}bCl(A)$. Hence $y \notin F(g^{*}bCl(A))$ which is a contradiction. Therefore, $F(g^{*}bCl(A)) \subseteq Cl_{0}(F(A))$.

(2) $\Rightarrow$ (3). Let $B$ be any subset of $Y$. Set $A = F^{+}(B)$ in (2), then we have $f(g^{*}bCl(F^{+}(B))) \subseteq Cl_{0}(B)$ and $g^{*}bCl(F^{+}(B)) \subseteq F^{+}(Cl_{0}(B))$.

(3) $\Rightarrow$ (4). Let $B$ be any subset of $Y$. Since $Cl_{0}(B)$ is closed in $Y$ hence is preclosed in $Y$. We have $g^{*}bCl(F^{+}(Int(Cl_{0}(B)))) \subseteq F^{+}(Cl(Cl_{0}(B)))) \subseteq F^{+}(Cl(Cl_{0}(B)))) \subseteq F^{+}(Cl_{0}(B))$.

(4) $\Rightarrow$ (1). Let $G$ be any preopen set of $Y$, then $G \subseteq IntCl(G)$. Apply (4) to $IntCl(G)$, we get $g^{*}bClF^{+}(IntCl_{0}(IntCl(G))) \subseteq F^{+}(Cl[IntCl(G)]))$. By Theorem 2.7, we have $g^{*}ClF^{+}(IntCl_{0}(IntCl(G))) \subseteq F^{+}(Cl[IntCl(G)])$. So, we get, $g^{*}bCl(F^{+}(G)) \subseteq g^{*}bClF^{+}(IntCl(G)) \subseteq F^{+}(Cl[IntCl(G)]) \subseteq F^{+}(Cl(G))$. Hence, by Theorem 4.11, $F$ is upper weakly $g^{*}bp$-continuous.

Corollary 4.13 If a multifunction $F: X \to Y$ is upper weakly $g^{*}bp$-continuous, then $F^{+}(A)$ is $g^{*}b$-closed in $X$ for every $\theta$-closed set $A$ in $Y$.

Proof. If $A$ is $\theta$-closed, so by Theorem 4.12, we obtain that $g^{*}bCl(F^{+}(A)) \subseteq F^{+}(Cl_{0}(A)) = F^{+}(A)$. Therefore, $F^{+}(A)$ is $g^{*}b$-closed.

Corollary 4.14 Let $F: X \to Y$ be any multifunction. If $F^{+}(Cl_{0}(B))$ is $g^{*}b$-closed in $X$ for each subset $B$ of $Y$, then $F: X \to Y$ is upper weakly $g^{*}bp$-continuous.

Proof. Since $F^{+}(Cl_{0}(B))$ is $g^{*}b$-closed in $X$, we have $g^{*}bCl(F^{+}(B)) \subseteq g^{*}bClF^{+}(Cl_{0}(B)) = F^{+}(Cl_{0}(B))$. Therefore, by Theorem 4.12, $F$ is upper weakly $g^{*}bp$-continuous.

Theorem 4.15 A multifunction $F: X \to Y$ is upper weakly $g^{*}bp$-continuous if and only if $F^{+}(A) \subseteq g^{*}bIntF^{+}(Cl(A))$ for each preopen set $A$ in $Y$.

Proof. Necessity. Let $F$ be upper weakly $g^{*}bp$-continuous and let $A$ be any preopen set of $Y$, then $A \subseteq IntCl(A)$. Therefore, by Theorem 4.11, we get $F^{+}(A) \subseteq F^{+}(IntCl(A)) \subseteq g^{*}bIntF^{+}(Cl(A))$. Hence, $F^{+}(A) \subseteq g^{*}bIntF^{+}(Cl(A))$.

Sufficiency. Let $A$ be any regular preopen set of $Y$, then $A$ is preopen set in $Y$. By hypothesis, we have $F^{+}(A) \subseteq g^{*}bIntF^{+}(Cl(A))$. Therefore, by Theorem 4.11, $F$ is upper weakly $g^{*}bp$-continuous.

Corollary 4.16 A multifunction $F: X \to Y$ is upper weakly $g^{*}bp$-continuous if and only if $g^{*}bClF^{+}(Int(F)) \subseteq F^{+}(F)$ for each preopen set $F$ in $Y$.

Theorem 4.17 If $F: X \to Y$ is a upper weakly $g^{*}bp$-continuous function and $Y$ is extremally disconnected space, then $F$ is upper almost $g^{*}bp$-continuous.

Proof. Let $x \in X$ and let $A$ be any preopen set of $Y$ containing $F(x)$. Since $F$ is upper weakly $g^{*}bp$-continuous, there exists a $g^{*}b$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq Cl(A)$. Since $Y$ is extremally disconnected, then $F(U) \subseteq IntCl(A)$. Therefore, $F$ is upper almost $g^{*}bp$-continuous.

Theorem 4.18 If $F: X \to Y$ is upper weakly $g^{*}bp$-continuous injection and $Y$ is pre-$T_{1}$ then $X$ is $g^{*}b-T_{1}$.

Proof. Assume that $Y$ is pre-$T_{1}$. For any distinct points $x$ and $y$ in $X$, there exist preopen sets $A$ and $W$ such that $F(x) \in A$, $F(y) \notin A$, $F(x) \notin W$ and $F(y) \in W$. Since $F$ is upper weakly $g^{*}bp$-continuous, there exists a $g^{*}b$-open sets $G$ and $H$ in $X$ containing $x$ and $y$ respectively, such that $F(G) \subseteq Cl(U)$, $F(H) \subseteq Cl(A)$, $F(H) \subseteq Cl(W)$ since $A$ and $W$ are disjoint then $Cl(A)$ and $Cl(W)$ are disjoint. Thus we obtain $y \notin G$, $x \notin H$. This show that $X$ is $g^{*}b-T_{1}$.
Theorem 4.19 If $F : X \rightarrow Y$ is upper weakly $g^*bp$-continuous and $Y$ is pre-$T_2$ then $X$ is $g^*b - T_2$.

**proof.** For any pair of distinct points $x$ and $y$ in $X$, there exist disjoint preopen sets $U$ and $V$ in $Y$ such that $F(x) \in U$ and $F(y) \in V$. Since $F$ is upper weakly $g^*bp$-continuous, there exist $g^*b$-open sets $G$ and $H$ in $X$ containing $x$ and $y$, respectively, such that $F(G) \subseteq Cl(U)$ and $F(H) \subseteq Cl(V)$. Since $U$ and $V$ are disjoint, we have $Cl(U) \cap Cl(V) = \phi$, hence $G \cap H = \phi$. This shows that $X$ is $g^*b - T_2$.

5. **Contra $g^*bp$-continuous function**

**Definition 5.1** A multifunction $F : X \rightarrow Y$ is called:

1. Upper contra $g^*bp$-continuous at $x \in X$ if for each preclosed set $A$ such that $x \in F^+(A)$, there exists a $g^*b$-open set $U$ containing $x$ such that $U \subseteq F^+(A)$.

2. Lower contra $g^*bp$-continuous at $x \in X$ if for each preclosed set $A$ such that $x \in F^-(A)$, there exists a $g^*b$-open set $U$ containing $x$ such that $U \subseteq F^-(A)$.

3. Lower (upper) contra $g^*bp$-continuous if $F$ has this property at each point of $X$.

**Theorem 5.2** The following are equivalent for a multifunction $F : X \rightarrow Y$.

1. $F$ is upper contra $g^*bp$-continuous.
2. $F^+(A)$ is $g^*b$-open set for any preclosed set $A \subseteq Y$.
3. $F^-(U)$ is $g^*b$-closed set for any preopen set $U \subseteq Y$.
4. For each $x \in X$ and each preclosed set $A$ containing $F(x)$, there exists a $g^*b$-open set $U$ containing $x$ such that if $y \in U$, then $F(y) \subseteq A$.

**proof.** (1) $\Rightarrow$ (2). Let $A$ be a preclosed set in $Y$ and $x \in F^+(A)$. Since $F$ is upper contra $g^*bp$-continuous, there exists a $g^*b$-open set $U$ containing $x$ such that $U \subseteq F^+(A)$. Thus, $F^+(A)$ is $g^*b$-open. The converse of the proof is similar. (2) $\Rightarrow$ (3). This follows from the fact that $F^+(Y \setminus A) = X \setminus F^-(A)$ for every subset $A$ of $Y$. (1) $\Leftrightarrow$ (4). Obvious.

**Theorem 5.3** The following are equivalent for a multifunction $F : X \rightarrow Y$.

1. $F$ is upper contra $g^*bp$-continuous.
2. $F^+(A)$ is $g^*b$-open set for any preclosed set $A \subseteq Y$.
3. $F^+(U)$ is $g^*b$-closed set for any preopen set $U \subseteq Y$.
4. For each $x \in X$ and each preclosed set $A$ such that $F(x) \cap A \neq \phi$, if $y \in U$, then $F(y) \subseteq A$, there exists a $g^*b$-open set $U$ containing $x$ such that if $y \in U$, then $F(y) \cap A \neq \phi$.

**proof.** The proof is similar to the proof of Theorem 5.2.

**Theorem 5.4** If a multifunction $F : X \rightarrow Y$ is upper contra $g^*bp$-continuous and $Y$ is preregular, then $F$ is upper $g^*bp$-continuous.

**proof.** Let $x \in X$ and $A$ be preopen set of $Y$ containing $F(x)$. Since $Y$ is preregular, then there exists a preopen set $G$ in $Y$ containing $F(x)$ such that $pCl(G) \subseteq A$. Since $F$ is upper contra $g^*bp$-continuous, then by Theorem 5.2, there exists a $g^*b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq pCl(G)$. Then $F(U) \subseteq pCl(G) \subseteq A$. Hence $F$ is upper $g^*bp$-continuous.

**Theorem 5.5** If a multifunction $F : X \rightarrow Y$ is upper contra $g^*bp$-continuous, then $F$ is upper weakly $g^*bp$-continuous.
proof. Let $A$ be any preopen set in $Y$. Since $F$ is upper contra $g^*bp$-continuous, then $F^+(A)$ is $g^*b$-closed set of $X$. Hence, by Theorem 4.2, we obtain that $F$ is upper weakly $g^*bp$-continuous.

The converse of Theorem 5.5 is not true in general as it is shown in the following example.

Example 5.6 Consider $X = Y = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{b\}, \{a, b\}, Y\}$ and a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $F(a) = c$, $F(b) = b$ and $F(c) = a$, $F$ is upper weakly $g^*bp$-continuous but not upper contra $g^*bp$-continuous since for preopen set $B = \{a\}$ in $Y$ and $F^{-1}(B) = \{a, b\}$ is not $g^*b$-closed in $X$.

Theorem 5.7 If a multifunction $F : X \rightarrow Y$ is upper contra $g^*bp$-continuous and $X$ is $g^*b$-space, then $F$ is upper contra continuous.

proof. Let $A$ be an open set in $Y$, then $i$ is preopen. Since $F$ is upper contra $g^*bp$-continuous, so $F^+(A)$ is $g^*b$-closed in $X$. Since $X$ is $g^*b$-space, hence, $F^+(A)$ is closed in $X$. Thus $F$ is upper contra continuous.

6. Multifunctions with $g^*bp$-closed graphs

Definition 6.1 Let $F : X \rightarrow Y$ be any multifunction, the graph of the function $F$ is denoted by $G(F)$ and is said to be $g^*bp$-closed if for each $(x, y) \notin G(F)$, there exists a $g^*b$-open set $U$ in $X$ containing $x$, and a preopen set $V$ of $Y$ containing $y$ such that $(U \times V) \cap G(F) = \phi$.

Lemma 6.2 The multifunction $F : X \rightarrow Y$ has a $g^*bp$-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $y \neq F(x)$, there exists a $g^*b$-open set $U$ and a preopen set $V$ containing $x$ and $y$ respectively, such that $F(U) \cap V = \phi$.

proof. Follows from Definition 6.1.

Proposition 6.3 If $F : X \rightarrow Y$ is upper weakly $g^*bp$-continuous, and $Y$ is pre-$T_2$ space, then $G(F)$ is a $g^*bp$-closed graph.

proof. Suppose that $(x, y) \notin G(F)$, then $F(x) \neq y$. By the fact that $Y$ is pre-$T_2$, there exist preopen sets $W$ and $V$ such that $F(x) \in W$, $y \in V$ and $W \cap V = \phi$. It follow from above that $F(U) \subseteq CIW$. Hence, we have $F(U) \cap V = \phi$. This means that $G(F)$ is $g^*bp$-closed graph.

Theorem 6.4 Let $F : X \rightarrow Y$ be a preirresolute multifunction where $X$ is an arbitrary topological space and $Y$ is pre-$T_2$. Then $G(f)$ is $g^*bp$-closed.

proof. Let $(x, y) \notin G(F)$. Then $F(x) \neq y$. Since $Y$ is pre-$T_2$, there exists $U \in PO(Y, F(x))$, $V \in PO(Y, y)$ such that $U \cap V = \phi$. It is upper preirresolute, this implies that $F^+(U) = W \subseteq PO(X, x)$, $W \subseteq g^*bO(X, x)$. Hence $F(U) = F(F^+(U)) \subseteq U$. It follow from above that $F(U) \cap V = \phi$. Therefore, by the Lemma 6.2, we obtain that $G(F)$ is $g^*bp$-closed.

Definition 6.5 The graph $G(F)$ of a multifunction $F : X \rightarrow Y$ is called contra $g^*bp$-closed if for each $(x, y) \notin G(F)$, there exist $U \in g^*bO(X, x)$, $V \in PC(Y, y)$ such that $(U \times V) \cap G(F) = \phi$.

Lemma 6.6 The graph $G(F)$ of a multifunction $F : X \rightarrow Y$ is contra $g^*bp$-closed if and only if for each $(x, y) \notin G(F)$, there exist $U \in g^*bO(X, x)$, $V \in PC(Y, y)$ such that $F(U) \cap V = \phi$.

Theorem 6.7 If a multifunction $F : X \rightarrow Y$ is upper contra $g^*bp$-continuous and $Y$ is pre-Urysohn, then $G(F)$ is contra $g^*bp$-closed.

proof. Let $(x, y) \notin G(F)$. Then $y \neq F(x)$ and there exists preopen sets $H_1, H_2$ such that $F(x) \in H_1$, $y \in H_2$ and $pCl(H_1) \cap pCl(H_2) = \phi$. From hypothesis, there exists $V \in g^*bO(X, x)$ such that $F(V) \subseteq pCl(H_1)$. Therefore, we obtain $F(V) \cap pCl(H_2) = \phi$. This shows that $G(F)$ is contra $g^*bp$-closed.

Theorem 6.8 If a multifunction $F : X \rightarrow Y$ is upper $g^*bp$-continuous and $Y$ is pre-$T_1$, then $G(F)$ is upper contra $g^*bp$-closed.
It follows that $U \in F$. Hence $F$. References

Since upper $g^*bp$-continuous, there exists $g^b$-open set $U$ in $X$ containing $x$ such that $F(U) \subseteq H$. Therefore we obtain $F(U) \cap (Y - H) = \phi$ and $(Y - H) \in PC(Y, y)$. This show that $G(F)$ is contra $g^*bp$-closed.

**Theorem 6.9** Let $F : X \to Y$ be a multifunction and $G : X \to X \times Y$ the graph function of $F$, defined by $G(x) = (x, F(x))$ for every $x \in X$. If $G$ is upper contra $g^*bp$-continuous, then $F$ is upper contra $g^*bp$-continuous.

**proof.** Let $U$ be any preopen set in $Y$, then $X \times U$ is preopen set in $X \times Y$. Since $G$ is upper contra $g^*bp$-continuous. It follows that $F^+(U) = G^+(X \times U)$ is an $g^b$-closed in $X$. Thus $F$ is upper contra $g^*bp$-continuous.

**Definition 6.10** Let $X$ and $Y$ be topological spaces. A multifunction $F : X \to Y$ is said to have strongly $g^*bp$-closed graph if for each $(x, y) \notin G(F)$, there exists $U \in g^bO(X, x)$, $V \in PO(Y, y)$ such that $(U \times Cl(V)) \cap G(F) = \phi$.

**Lemma 6.11** A multifunction $F : X \to Y$ has strongly $g^*bp$-closed graph if for each $(x, y) \notin G(F)$, there exists $U \in g^bO(X, x)$, $V \in PO(Y, y)$ such that $F(U) \cap Cl(V) = \phi$.

**Remark 6.12** Evidently every multifunction has a strongly $g^*bp$-closed graph it has a $g^*bp$-closed graph but the converse is not true as it is shown by the following example.

**Example 6.13** Let $X = Y = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$, $\sigma = \{\phi, \{b\}, Y\}$, then the identity multifunction $I : (X, \tau) \to (Y, \sigma)$ has a $g^*bp$-closed graph but it has not strongly $g^*bp$-closed graph.

**Theorem 6.14** If $F : X \to Y$ is upper almost $g^*bp$-continuous and $Y$ is pre-$T_2$, then $G(F)$ is strongly $g^*bp$-closed graph.

**proof.** Let $(x, y) \notin G(F)$. Since $Y$ is pre-$T_2$, then there exists preopen set $V$ of $Y$ containing $y$ such that $F(x) \notin Cl(V)$. Now $Cl(V)$ is regular closed set in $Y$. So, $Y \setminus Cl(V)$ is regular open in $Y$ containing $F(x)$. Therefore, by the upper almost $g^*bp$-continuous of $F$ there exists $U \in g^bO(X, x)$ such that $F(U) \subseteq Y \setminus Cl(V)$. Hence $F(U) \cap Cl(V) = \phi$.

**Corollary 6.15** If $F : X \to Y$ is upper $g^*bp$-continuous and $Y$ is pre-$T_2$ then $G(F)$ is strongly $g^*bp$-closed.

**proof.** Since upper $g^*bp$-continuous implies upper almost $g^*bp$-continuous, the result follows.

**References**


