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Application of the SBA method to solving optimal control problem governed by systems of nonlinear partial differential equations

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Abstract

In this article, we are interested in solving optimal control problems governed by a system of nonlinear partial differential equations by the iterative method SBA(combination of the Adomian method, successive approximations method and the principle of Picard) [17]. This technique allows reducing the problem of optimal control of the error function

$$J = \int_0^T \left(u(l,t) - \varepsilon\right)^2 dt$$

in a classical minimization problem.

Keywords: *Adomian method, successive approximations method, Picard principle, SBA method*.

1 Introduction

We are interested in the resolution of optimal control problems governed by systems of nonlinear partial differential equations of Cauchy type:

$$\text{(S):} \begin{cases} u_t = F(t)L(u,v) + F(t)G(t)N(u,v) + \varphi(u,v)\ ;\ 0 < t < T,\ 0 < x < L \\ v_t = G(t)\tilde{L}(u,v) + F(t)G(t)\tilde{N}(u,v) + \psi(u,v)\ ;\ 0 < t < T,\ 0 < x < L \\ u(0) = f \\ v(0) = \tilde{f} \end{cases}$$

Modeling many phenomena [13, 16] in biomathematics. This is for instance the case of two substances of respective concentrations $u(t,x)$ and $v(t,x)$ in interaction; and ψ can represent different types of nonlinearity.

The problem is not the classical resolution of the system (S), it comes to determine the control functions

$$F, G : [0;T] \to \mathbb{R}$$

Minimizing the error functional

$$J = \int_0^T \left(u(l,t) - \varepsilon\right)^2 dt \quad (P)$$

Where T represents the observation duration of the phenomenon and $\varepsilon > 0$.

Several numerical resolution methods of such systems have been developed, among others, the spectral method, the finite difference these conventional methods of resolution usually use the technique of discretization or linearization. The new technique SBA [1-5], does not discretize the system of equations and retains the biological properties of the model, and this is important for the applications [13-16].

2 Description of the approach SBA for the system (S)

The algorithm SBA (Some Blaise - Abbo) [1-5] is applied to the following system

$$(S):\begin{cases} u_t = F(t)L(u,v)+F(t)G(t)N(u,v)+\varphi(u,v)\ ;\ 0<t<T,\ 0<x<L \\ v_t = G(t)\tilde{L}(u,v)+F(t)G(t)\tilde{N}(u,v)+\psi(u,v)\ ;\ 0<t<T,\ 0<x<L \\ u(0)=f \\ v(0)=\tilde{f} \end{cases}$$

In a functional space with $(L,\tilde{L})$ a pair of linear differential operator, $(N,\tilde{N})$ the nonlinear term and u,v unknown functions such as

$$u_t=\frac{\partial u}{\partial t}, v_t=\frac{\partial v}{\partial t}.$$

Using the idea of the method of successive approximations, the above problem can be approximated by the following equivalent problem:

$$(S_{appr}):\begin{cases} u_t = F(t)L(u^k,v^k)+F(t)G(t)N(u^{k-1},v^{k-1})+\varphi(u^{k-1},v^{k-1})\ ;\ 0<t<T \\ v_t = G(t)\tilde{L}(u^k,v^k)+F(t)G(t)\tilde{N}(u^{k-1},v^{k-1})+\psi(u^{k-1},v^{k-1})\ ;\ 0<t<T \\ u^k(0)=f, \qquad k=1,2,... \\ v^k(0)=\tilde{f}, \qquad k=1,2,... \end{cases}$$

Solving the problem (S_{appr}) by the method of successive approximations, then we determine at each iteration $(k=1,2,...)$ the following approximate solutions: $u^1,u^2,...,u^n,...,v^1,v^2,...,v^n,...$

But this requires first the choice of the initial condition (u^0,v^0). the resulting solution (u,v) of the problem (S) sought is obtained by:

$$(u,v)=\lim_{k\to\infty}(u^k,v^k)$$

If (u^k,v^k) is convergent.

Note that various successive approximate solutions are usually obtained by using the method of Picard [17]. In the specific case, we use the decompositional method of Adomian at each iteration step in choosing a hypothesis (u^0,v^0).

Reconsidering the following problem:

$$(S_{appr}):\begin{cases} u_t = F(t)L(u^k,v^k)+F(t)G(t)N(u^{k-1},v^{k-1})+\varphi(u^{k-1},v^{k-1})\ ;\ 0<t<T \\ v_t = G(t)\tilde{L}(u^k,v^k)+F(t)G(t)\tilde{N}(u^{k-1},v^{k-1})+\psi(u^{k-1},v^{k-1})\ ;\ 0<t<T \\ u^k(0)=f, \qquad k=1,2,... \\ v^k(0)=\tilde{f}, \qquad k=1,2,... \end{cases}$$

We use the Adomian method to solve the problem at each iteration step. But above all, we note that after the choice of the couple (u^0,v^0),

And $(g_k,\tilde{g}_k)$ is a pair of functions depending only on t (in the case of ODE) or on x and t (in the PDE).

As a result the problem (S_{appr}) becomes

$$\begin{cases} u_t^k = L(u^k,v^k)+g_k, 0<t<T \\ v_t^k = \tilde{L}(u^k,v^k)+\tilde{g}_k, 0<t<T \\ u^k(0)=f \qquad k=1,2,... \\ v^k(0)=\tilde{f} \qquad k=1,2,... \end{cases}$$

Which is a problem of semi-linear easily solved by the Adomian method EDO.

Indeed, witch the choice of operators

$$L_1 = \frac{\partial}{\partial t} \text{ And } L_1^{-1} = \int_0^t (.)\, ds$$

The system of equations

$$\begin{cases} u_t^k = L\left(u^k, v^k\right) + N\left(u^{k-1}, v^{k-1}\right) + \varphi\left(u^{k-1}, v^{k-1}\right) \\ v_t^k = \tilde{L}\left(u^k, v^k\right) + \tilde{N}\left(u^{k-1}, v^{k-1}\right) + \psi\left(u^{k-1}, v^{k-1}\right) \end{cases}$$

Can be rewritten as follows:

$$\begin{cases} u^k = u^k(0) + L_1^{-1}\left(L\left(u^k, v^k\right)\right) + L_1^{-1}\left(N\left(u^{k-1}, v^{k-1}\right)\right) + L_1^{-1}\left(\varphi\left(u^{k-1}, v^{k-1}\right)\right) \\ v^k = v^k(0) + L_1^{-1}\left(\tilde{L}\left(u^k, v^k\right)\right) + L_1^{-1}\left(\tilde{N}\left(u^{k-1}, v^{k-1}\right)\right) + L_1^{-1}\left(\psi\left(u^{k-1}, v^{k-1}\right)\right) \end{cases}$$

Which is a canonical form of Adomian. And we can deduce the following Adomian algorithm:

$$\begin{cases} u_0^k = u^k(0) + L_1^{-1} g_k, & k = 1,2,... \\ u_n^k = L_1^{-1}\left(L\left(u_{n-1}^k, v_{n-1}^k\right)\right) + L_1^{-1}\left(\varphi\left(u^{k-1}, v^{k-1}\right)\right), & k = 1,2,... \\ v_0^k = v^k(0) + L_1^{-1} \tilde{g}_k, & k = 1,2,... \\ v_n^k = L_1^{-1}\left(L\left(u_{n-1}^k, v_{n-1}^k\right)\right) + L_1^{-1}\left(\psi\left(u^{k-1}, v^{k-1}\right)\right), & k = 1,2,... \end{cases}$$

Then the solution at each step is given by:

$$\left(u^k, v^k\right) = \left(\sum_{n=0}^{\infty} u_n^k, \sum_{n=0}^{\infty} v_n^k\right), \quad k = 1,2,...$$

And the solution (u,v) of (S) is obtained by: $(u,v) = \lim_{k \to \infty}\left(u^k, v^k\right) = \Lambda\left(F(t); G(t)\right)$. This solution typically solve the problem (P). Indeed, with the analytical solution obtained by the SBA method, the functional J can be rewritten as follows: $J = \int_0^T \left(\Lambda_1\left(F(t), G(t)\right) - \varepsilon\right)^2 dt \quad (P)$

Where we replace by its equivalent: $u(l,t) = \Lambda_1\left(F(t), G(t)\right)$ Let's put $\Lambda_2(m,t) = \left(F(t), G(t)\right)$ and J becomes

$$J = \int_0^T \left(\Lambda_1 \Lambda_2(m,t) - \varepsilon\right)^2 dt \quad (P).$$

Thus we get a classical minimization problem. Thereby minimizing the error functional J with respect to $F(t), G(t)$ is equivalent to minimizing J relative to the constant parameter m.

3 Numerical examples

3.1 Application to systems of nonlinear partial differential equations

We are intending to determine the control functions $F(t)$ and $G(t)$ which minimize the functional error

$$J = \int_0^T \left(u(l,t) - \varepsilon\right)^2 dt \quad (P)$$

Where $u(t,x)$ verifies the system of nonlinear partial differential equations following:

$$(S):\begin{cases} u_t = F(t)L(u,v)+F(t)G(t)N(u,v)+\varphi(u,v)\ ;\ 0<t<T \\ v_t = G(t)\tilde{L}(u,v)+F(t)G(t)\tilde{N}(u,v)+\psi(u,v)\ ;\ 0<t<T \\ u(0)=f \\ v(0)=\tilde{f} \end{cases}$$

Where T represents the observation duration of the phenomenon and $\varepsilon > 0$. in the various examples below we take $F(t)=G(t)=e^{mt}$, m is a constant.

Example 1:

Considering the nonlinear partial differential equations as follow:

$$(p):\begin{cases} \dfrac{\partial u}{\partial t} = e^{mt}\dfrac{\partial^2 v}{\partial x^2}+u^2-v^2-\left(a^2-b^2\right)\left(x^4-\dfrac{4}{m^2}\left(e^{mt}-1\right)^2\right) \\ \dfrac{\partial v}{\partial t} = e^{mt}\dfrac{\partial^2 u}{\partial x^2}+u^2-2uv+v^2-\left(a-b\right)^2\left(x^2-\dfrac{2}{m}\left(e^{mt}-1\right)\right)^2 \\ u(x,0)=ax^2 \\ v(x,0)=bx^2 \end{cases}$$

Which solution is?

$$(u,v)=\lim_{k\to\infty}\left(u^k,v^k\right)=\left(ax^2+\frac{2b}{m}\left(e^{mt}-1\right);bx^2+\frac{2a}{m}\left(e^{mt}-1\right)\right)$$

Obtained by the SBA algorithm [1-5].

The error functional J is well

$$J=\int_0^T\left(al^2+\frac{2b}{m}\left(e^{mt}-1\right)-\varepsilon\right)^2 dt \quad (P)$$

Thereby minimizing the error functional J with respect to $F(t),G(t)$ is equivalent to minimizing J relative to constant parameter m.

The minimum of J is obtained by the expression $\dfrac{\partial J}{\partial m}=0$ which solution is written in the following form: $m=f\ (a,b,l,\varepsilon,T)$, where a,b,l,ε,T are real constants.

Example 2:

Considering the nonlinear partial differential equations as follow:

$$(p):\begin{cases} \dfrac{\partial u}{\partial t} = e^{mt}\dfrac{\partial^2 v}{\partial x^2}+u^2-v^2+\cos 2x \\ \dfrac{\partial v}{\partial t} = e^{mt}\dfrac{\partial^2 u}{\partial x^2}+u+v-\sqrt{2}\cos\left(x-\dfrac{\pi}{4}\right)e^{\frac{1-e^{mt}}{m}} \\ u(x,0)=\sin x \\ v(x,0)=\cos x \end{cases}$$

Which solution is?

$$(u,v)=\lim_{k\to\infty}\left(u^k,v^k\right)=\left(\sin x\,ch\left(\frac{e^{mt}-1}{m}\right)-\cos x\,sh\left(\frac{e^{mt}-1}{m}\right);\cos x\,ch\left(\frac{e^{mt}-1}{m}\right)-\sin x\,sh\left(\frac{e^{mt}-1}{m}\right)\right)$$

Obtained by the SBA algorithm, we can determine in the same manner as example1, the controls functions $F(t),G(t)$.

The functional error J is:

$$J = \int_0^T \left(\sin l \, ch\left(\frac{e^{mt} - 1}{m} \right) - \cos l \, sh\left(\frac{e^{mt} - 1}{m} \right) - \varepsilon \right)^2 dt \quad (P)$$

Thereby minimizing the functional error J with respect to $F(t), G(t)$ is equivalent to minimizing J relative to constant parameter m.

The minimum of J is obtained by the expression $\frac{\partial J}{\partial m} = 0$ which solution is written in the following form: $m = f(a, b, l, \varepsilon, T)$, where a, b, l, ε, T are real constants.

4 Discussion

The above examples illustrate the effectiveness of the SBA method in solving optimal control problems. Unlike the methods mentioned above, the SBA algorithm reduces the resolution of an issue of control, problem solving classical minimization. This is of great interest to users.

The SBA method can contribute effectively to solving not only the optimal control, but also the control in general.

5 Conclusion

This work allowed us to show that the SBA method can contribute to solving optimal control problems in one dimension space EDP nonlinear Cauchy-type systems.

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