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The short analytic proof of the friendly numbers problem

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Abstract

The notion of a friendly number (or amicable number) (see [1] or [2] or [3] or [7] or [8]) is based on the idea that a human friend is a kind of alter ego. Indeed, Pythagoras wrote (see [7] or [8]): A friend is the other I, such as are 220 and 284. These numbers have a special mathematical property: each is equal to the sum of the other's proper divisors (divisors other than the number itself). The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and they sum to 284; the proper divisors of 284 are 1, 2, 4, 71, and 142, and they sum to 220. So $\{220, 284\}$ is called a pair of friendly numbers [note $\{17296, 18416\}$ is also a pair of friendly numbers (see [7] or [8])]. More precisely, we say that a number a' is a friendly number or amicable number, if there exists a number $a'' \neq a'$ such that $\{a', a''\}$ is a pair of friendly numbers [example. 220, 284, 17296 and 18416 are amicable numbers]. The friendly numbers problem states that there are infinitely many friendly numbers. It is trivial to see that a friendly number is a composite number (we recall that a composite number is a non prime number. Original characterizations of composite numbers via divisibility are given in [5] and [6]). The first proof of the friendly numbers problem is given in [4]; this proof use a logical (non recursive) reasoning. That being so, in this paper, we give the short analytic simple proof of the friendly numbers problem, by using only elementary combinatoric, elementary arithmetic calculus, elementary complex analysis, and an elementary Theorem of Erdos on prime numbers (this new proof is analytic, simple, and is different from the first given in [4]). Moreover, our paper clearly shows that divisibility helps to characterize friendly numbers as we did in [5] and [6], and divisibility does not help to give the simple proof of the friendly numbers problem.

Keywords: *Friendly numbers.*

Prologue: This paper is devided into four simple Sections. In Section.**1**, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions. In Section.**2**, we state an elementary Theorem of Erdos on prime numbers coupled with elementary properties of Section.**1**, and we use them in a simple proposition that we will apply in Section.**4** to give the simple analytic proof of the friendly numbers problem. In Section.**3**, we prove elementary properties linked to prime numbers, elementary complex analysis and arithmetic calculus; we will use them in Section.**4**. In Section.**4**, using a simple proposition proved in Section.**2**, elementary arithmetic calculus and some properties of Section.**3**, we give the analytic simple proof of the friendly numbers problem.

1 Introduction

Definitions 1.0. For every integer $n \geq 2$, we define $\mathcal{A}(n)$, a_n, and $a_{n.1}$ as follows: $\mathcal{A}(n) = \{x; 1 < x < 2n\, and\, x\, is\, a\, friendly\, number\}$, $a_n = \max_{a \in \mathcal{A}(n)} a$, and $a_{n.1} = 4a_n^{a_n}$ [observing (see Abstract and Definitions) that 220 and 284 are friendly numbers, then it becomes immediate to deduce that for every integer $n \geq 284$, $\{220, 284\} \subseteq \mathcal{A}(n)$].
Using the previous definitions and denotations, let us remark.

REMARK 1.1.Let n be an integer ≥ 284; look at $\mathcal{A}(n)$, a_n, and $a_{n.1}$ introduced in Definitions 1.0. Then we

have the following three simple properties.
(1.1.0.) $283 < a_n < a_{n.1}$; $a_{n.1} = 4a_n^{a_n}$; $a_{n.1} > 284^{284}$; and $a_{n.1}$ is even.
(1.1.1.) **If** $a_n < n$, then: $a_n = a_{n-1}$ and $a_{n.1} = a_{n-1.1}$.
(1.1.2.) **If** $a_{n.1} \leq 2n$, then $a_n < n$ and $a_{n.1} = a_{n-1.1}$. $\Big($ Proof. Property (1.1.0) is trivial [**Indeed**, it suffices to use the definition of a_n and $a_{n.1}$, and the fact that $284 \in \mathcal{A}(n)$ (note that 284 is a friendly number (use Abstract and definitions), and observe that n is an integer ≥ 284; so $284 \in \mathcal{A}(n)$)]. Property (1.1.1) is immediate [**Indeed**, **if** $a_n < n$, clearly $n > 284$ (use the definition of a_n and observe that $284 \in \mathcal{A}(n)$, since n is an integer ≥ 284), and so $a_n < n < 2n-2$ (since $n > 284$ (by the previous) and $a_n < n$ (by the hypotheses)); consequently

$$a_n < 2n - 2 \qquad (1.1.1.0).$$

Inequality (1.1.1.0) immediately implies that $\mathcal{A}(n) = \mathcal{A}(n-1)$ and therefore

$$a_n = a_{n-1} \qquad (1.1.1.1).$$

Equality (1.1.1.1) immediately implies that $a_{n.1} = a_{n-1.1}$. Property (1.1.1) follows]. Property (1.1.2) is trivial [**Indeed**, clearly

$$a_n < n \qquad (1.1.2.0);$$

(otherwise

$$a_n \geq n \qquad (1.1.2.1).$$

Now look at $a_{n.1}$ and observe (by using property (1.1.0)) that

$$a_{n.1} = 4a_n^{a_n} \qquad (1.1.2.2).$$

Noticing (by the hypotheses) that $n \geq 284$, then, using (1.1.2.1) and (1.1.2.2), it becomes trivial to deduce that $a_{n.1} > -1 + 4n^n > 2n$; so $a_{n.1} > 2n$ and we have a contradiction, since $a_{n.1} \leq 2n$ (by the hypotheses)). Clearly $a_{n.1} = a_{n-1.1}$ (by using inequality (1.1.2.0) and property (1.1.1)). Property (1.1.2) follows]. Remark 1.1 follows. Using the definition of $a_{n.1}$ (see Definitions 1.0) , then the following remark and corollary become immediate.

REMARK 1.2. **If** $\lim_{n \to +\infty} a_{n.1} = +\infty$, then there are infinitely many friendly numbers.
Proof. Immediate [indeed, it suffices to use definitions of $a_{n.1}$ (see Definitions 1.0)

COROLLARY 1.3. **If** for every integer $n \geq 284$, we have $a_{n.1} > n$, then there are infinitely many friendly numbers.
Proof. Clearly $\lim_{n \to +\infty} a_{n.1} = +\infty$; therefore there are infinitely many friendly numbers [by using the previous equality and by applying Remark 1.2

2 Prime numbers and some consequences

In this section, we state an elementary Theorem of Erdos on prime numbers, and we use it in a simple proposition that we will apply in Section.**4** to give the simple analytic proof of the friendly numbers problem. That being so, the following Theorem is well known

THEOREM 2.0 (The Postulate of Bertrand or Erdos Theorem). Let n be an integer ≥ 1, then there exists a prime between n and $2n$.
Based on Theorem 2.0, let us define.

Definitions 2.1. For every integer $n \geq 2$, we define $\mathcal{P}(n)$ and p_n as follows:
$\mathcal{P}(n) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n\}$, and $p_n = \max_{p \in \mathcal{P}(n)} p$. Using the definition of $\mathcal{P}(n)$ and p_n, then it becomes immediate to deduce that for every integer $n \geq 3$, we clearly have $\mathcal{P}(n-1) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n-2\}$ and $p_{n-1} = \max_{p \in \mathcal{P}(n-1)} p$. By using again the definition of $\mathcal{P}(n)$ and p_n, then it is also immediate to deduce that for every integer $n \geq 4$, $\mathcal{P}(n-2) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n-4\}$ and $p_{n-2} = \max_{p \in \mathcal{P}(n-2)} p$ $\Big($ For every integer $n \geq 2$, it is immediate to see that p_n is odd and $3 \leq p_n \leq 2n-1$; and it is also immediate to see that for every

integer $n \geq 3$, p_{n-1} is odd and $3 \leq p_{n-1} \leq 2n-3$. It is trivial to see that for every integer $n \geq 4$, p_{n-2} is odd and $3 \leq p_{n-2} \leq 2n-5$. Now using the definitions of $\mathcal{P}(n)$, $\mathcal{P}(n-1)$, $\mathcal{P}(n-2)$, p_n, p_{n-1} and p_{n-2} introduced above, then it becomes trivial to deduce that, for every integer $n \geq 4$, we have $\mathcal{P}(n-2) \subseteq \mathcal{P}(n-1) \subseteq \mathcal{P}(n)$, and therefore $p_{n-2} \leq p_{n-1} \leq p_n\big)$.
Now using the definition of p_n (see Definitions 2.1), and using Theorem 2.0, then the following corollary becomes immediate:

COROLLARY 2.2. For every integer $n \geq 2$, $p_n \geq n$.
Proof. Use definition of p_n and Theorem 2.0.
The following proposition is only an immediate consequence of Corollary 2.2 and Corollary 1.3.

PROPOSITION 2.3. **If** for every integer $n \geq 284$ we have $a_{n.1} > p_n$, then, there are infinitely many friendly numbers .
Proof. Immediate, by using Corollary 2.2 and Corollary 1.3.
Proposition 2.3 clearly says that: **if** for every integer $n \geq 284$, we have $a_{n.1} > p_n$, then, there are infinitely many friendly numbers; this is what we will do in Section.**4**, by using only Proposition 2.3, elementary combinatoric, elementary arithmetic calculus, elementary complex analysis, elementary induction and reasoning by reduction to absurd. Proposition 2.3 is stronger than all the investigations that have been done on the friendly numbers problem in the past. Morerover, the reader can easily see that Proposition 2.3 does not use divisibility and is completely different from all the investigations that have been done on the friendly numbers problem in the past. So, in Section.**4**, when we will give the analytic simple proof of the friendly numbers problem, we will not need divisibility and we will not need strong investigations that have been done on the previous problem in the past.

3 Some simple properties linked to primes, elementary complex analysis and arithmetic calculus

In this section, we prove elementary properties linked to prime numbers, complex analysis and elementary arithmetic calculus; we will use them in Section.**4**, where we will give the analytic simple proof of the friendly numbers problem. In this section, the definitions of $\mathcal{A}(n)$, a_n, and $a_{n.1}$ (see Definitions 1.0) and the definitions of $\mathcal{P}(n-1)$, $\mathcal{P}(n)$, p_{n-1}, and p_n (see Definitions 2.1), are crucial.

PROPOSITION 3.1. Let n be an integer ≥ 3, and look at p_n and p_{n-1} (see Definitions 2.1). We have the following two simple properties.
(3.1.1). If $2n-1$ is not prime, then $p_n = p_{n-1}$.
(3.1.2). If $2n-1$ is prime, then $p_n = 2n-1$.$\big($Proof. (3.1.1) Indeed, look at $\mathcal{P}(n)$ and $\mathcal{P}(n-1)$ (see Definitions 2.1); clearly

$$\mathcal{P}(n) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n\} \text{ and } \mathcal{P}(n-1) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n-2\} \qquad (3.1.1.0),$$

by using the definitions of $\mathcal{P}(n)$ and $\mathcal{P}(n-1)$. That being so, **if** $2n-1$ is not prime, clearly

$$p_n \leq 2n-3 \qquad (3.1.1.1),$$

since $p_n \leq 2n-1$ and since $2n-1$ is not prime (note that $n \geq 3$, by the hypotheses). Now using inequality (3.1.1.1) and the definition of $\mathcal{P}(n)$, then it becomes trivial to deduce that

$$\mathcal{P}(n) = \{p;\, p\, is\, prime\, and\, 1 < p < 2n\} = \{p;\, p\, is\, prime\, and\, 1 < p < 2n-2\} \qquad (3.1.1.2).$$

Now using (3.1.1.0) and (3.1.1.2), then it becomes trivial to deduce that $\mathcal{P}(n) = \mathcal{P}(n-1)$; the previous equality clearly says that $p_n = p_{n-1}$. Property (3.1.1) follows.
(3.1.2) Indeed, **if** $2n-1$ is prime, clearly $p_n = 2n-1$ (by recalling that $p_n \leq 2n-1$ (use the definition of p_n and remark that $n \geq 3$) and by observing that $2n-1$ is prime). Property (3.1.2) follows and Proposition 3.1 immediately follows.
The previous elementary proposition made, let us recall.

Recalls 3.2. We recall that θ is a complex number, if $\theta = x + iy$, where x and y are real, and where i is the complex entity satisfying $i^2 = -1$. That being so, let $\theta = x + iy$ be a complex number; we recall that x is called the real part of θ and y is called the imaginary part of θ. The real part of θ is denoted by $R[\theta]$ and the imaginary part of θ is denoted by $I[\theta]$ $\Big($ Example.**1**. Let n be an integer ≥ 284, and let $a_{n.1}$ (see Definitions 1.0); now consider equations $\rho_{n.0}$, $\rho_{n.1}$, and $\rho_{n.2}$, where

$$\rho_{n.0} = (\, ia_{n.1}^2 - 2ina_{n.1} - 2ia_{n.1} + 1 - a_{n.1}^2\,)^2 \qquad (3.2.0);$$

$$\rho_{n.1} = (\, 4na_{n.1} + 4a_{n.1}\,)(\, ia_{n.1}^2 - 2ina_{n.1} - 2ia_{n.1} + 1 - a_{n.1}^2\,) \qquad (3.2.1);$$

and

$$\rho_{n.2} = (\, 2na_{n.1} + 2a_{n.1}\,)^2 \qquad (3.2.2).$$

If $a_{n.1} = 2n+2$, then $\rho_{n.0} + \rho_{n.1} + \rho_{n.2} = R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = 1$. Proof. Indeed, observing (via the hypotheses) that $a_{n.1} = 2n+2$, then it becomes easy to check (by elementary calculation and the fact $i^2 = -1$) that $\rho_{n.0}+\rho_{n.1}+\rho_{n.2} = R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = 1$. Example.**1** follows. Example.**2**. Let n be an integer ≥ 284 and let $a_{n.1}$; now look at equations $\rho_{n.0}$, $\rho_{n.1}$ and $\rho_{n.2}$ introduced in Example.**1**. **If** $a_{n.1} = 2n$, then $R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = 1+4a_{n.1}$. Proof. Indeed, observing (via the hypotheses) that $a_{n.1} = 2n$, then it becomes trivial to check (by elementary calculation and the fact $i^2 = -1$) that $R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = 1 + 4a_{n.1}$. Example.**2** follows. Example.**3**. Let n be an integer ≥ 284 and let $a_{n.1}$ (see Definitions 1.0); now consider equation $\phi_{n.0}$, where

$$\phi_{n.0} = (\, 2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1}\,)(\, 2ina_{n.1} + 2ia_{n.1} - a_{n.1}^2\,).$$

If $a_{n.1} = 2n+2$, then $\phi_{n.0} = I[\phi_{n.0}] = 0$; where $I[\phi_{n.0}]$ is the imaginary part of $\phi_{n.0}$. Proof. Indeed, observing (via the hypotheses) that $a_{n.1} = 2n+2$, clearly $2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1} = 0$, and using the previous equality, it becomes immediate to deduce that $\phi_{n.0} = I[\phi_{n.0}] = 0$. Example.**3** follows. Example.**4**. Let n be an integer ≥ 284 and let $a_{n.1}$; now look at equation $\phi_{n.0}$ introduced in Example.**3**. **If** $a_{n.1} = 2n$, then $I[\phi_{n.0}] = 4a_{n.1}^3$. Proof. Indeed, observing (via the hypotheses) that $a_{n.1} = 2n$, then it becomes trivial to check (by elementary calculation and the fact $i^2 = -1$) that $I[\phi_{n.0}] = 4a_{n.1}^3\Big)$.
The four Examples of Recalls 3.2 will help us later. Now, via Recalls 3.2, let us define:

Definitions 3.3 (Fundamental). Let n be an integer ≥ 284 and let $a_{n.1}$ (see Definitions 1.0), then $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$ and $\phi_{n.1}$ are defined as follow.

$$\rho_{n.0} = (\, ia_{n.1}^2 - 2ina_{n.1} - 2ia_{n.1} + 1 - a_{n.1}^2\,)^2;$$

$$\rho_{n.1} = (\, 4na_{n.1} + 4a_{n.1}\,)(\, ia_{n.1}^2 - 2ina_{n.1} - 2ia_{n.1} + 1 - a_{n.1}^2\,);$$

$$\rho_{n.2} = (\, 2na_{n.1} + 2a_{n.1}\,)^2;$$

$$\phi_{n.0} = (\, 2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1}\,)(\, 2ina_{n.1} + 2ia_{n.1} - a_{n.1}^2\,);$$

and

$$\phi_{n.1} = (\, a_{n.1} - 2n - 2\,)(\, 2a_{n.1}^2 - 2a_{n.1}\,).$$

It is immediate that for every integer $n \geq 284$, equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$ and $\phi_{n.1}$ are well defined and get sense (see Example.**1** of Recalls 3.2 for $\rho_{n.0}$, $\rho_{n.1}$, and $\rho_{n.2}$; and see Example.**3** of Recalls 3.2 for $\phi_{n.0}$). Now using Definitions 3.3, then we have the following three elementary Propositions.

PROPOSITION 3.4. Let n be an integer ≥ 285 and let $a_{n.1}$ (see Definitions 1.0 for the meaning of $a_{n.1}$); now look at equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$ introduced in Definitions 3.3, and via equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$, consider equations $\rho_{n-1.0}$, $\rho_{n-1.1}$, $\rho_{n-1.2}$, $\phi_{n-1.0}$ and $\phi_{n-1.1}$ (these considerations get sense, since $n \geq 285$, and therefore $n-1 \geq 284$). **If** $a_{n.1} \leq 2n$, then we have the following eight simple properties.
(3.4.0) $a_{n.1} = a_{n-1.1}$.
(3.4.1) $\rho_{n-1.0} = \rho_{n.0} + 8na_{n.1}^2 + 4a_{n.1}^2 - 4a_{n.1}^3 + 4ia_{n.1} - 4ia_{n.1}^3$.
(3.4.2) $\rho_{n-1.1} = \rho_{n.1} - 4a_{n.1} + 4a_{n.1}^2 + 16ina_{n.1}^2 + 8ia_{n.1}^2 - 4ia_{n.1}^3$.
(3.4.3) $\rho_{n-1.2} = \rho_{n.2} - 8na_{n.1}^2 - 4a_{n.1}^2$.
(3.4.4) $R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$.
(3.4.5) $\phi_{n-1.0} = \phi_{n.0} - 16na_{n.1}^2 - 8a_{n.1}^2 + 4a_{n.1}^3 - 4ia_{n.1}^3$.
(3.4.6) $I[\phi_{n-1.0}] = I[\phi_{n.0}] - 4a_{n.1}^3$.
(3.4.7) $I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$.

Proof. (3.4.0) Indeed, observe (by the hypotheses) that $a_{n.1} \leq 2n$; now using the previous inequality and property (1.1.2) of Remark 1.1, then it becomes trivial to deduce that $a_{n.1} = a_{n-1.1}$. Property (3.4.0) follows.
(3.4.1) Indeed, look at $\rho_{n.0}$ and observe (by using Definitions 3.3) that

$$\rho_{n.0} = (\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1}\,)^2 \qquad (3.4.1.0).$$

Using equation (3.4.1.0), then it becomes trivial to deduce that

$$\rho_{n-1.0} = (\, ia^2_{n-1.1} - 2ina_{n-1.1} - 2ia_{n-1.1} + 1 - a^2_{n-1.1} \, + 2ia_{n-1.1}\,)^2 \qquad (3.4.1.1).$$

Now look at equation (3.4.1.1); noticing (by property (3.4.0)) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equation (3.4.1.1) is of the form

$$\rho_{n-1.0} = (\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1} \, + 2ia_{n.1}\,)^2 \qquad (3.4.1.2).$$

It is trivial to check (by elementary calculation and the fact that $i^2 = -1$) that equation (3.4.1.2) is of the form

$$\rho_{n-1.0} = (\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1}\,)^2 + \alpha_{n.0} \qquad (3.4.1.3),$$

where

$$\alpha_{n.0} = 8na^2_{n.1} + 4a^2_{n.1} - 4a^3_{n.1} + 4ia_{n.1} - 4ia^3_{n.1} \qquad (3.4.1.4).$$

Clearly $\rho_{n-1.0} = \rho_{n.0} + 8na^2_{n.1} + 4a^2_{n.1} - 4a^3_{n.1} + 4ia_{n.1} - 4ia^3_{n.1}$, by using equations (3.4.1.0) and (3.4.1.3) and (3.4.1.4). Property (3.4.1) follows.
(3.4.2) Indeed, look at $\rho_{n.1}$ and observe (by using Definitions 3.3) that

$$\rho_{n.1} = (\, 4na_{n.1} + 4a_{n.1}\,)(\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1}\,) \qquad (3.4.2.0).$$

Using equation (3.4.2.0), then it becomes trivial to deduce that

$$\rho_{n-1.1} = (\, 4na_{n-1.1} + 4a_{n-1.1} \, - 4a_{n-1.1})(\, ia^2_{n-1.1} - 2ina_{n-1.1} - 2ia_{n-1.1} + 1 - a^2_{n-1.1} \, + 2ia_{n-1.1}\,) \qquad (3.4.2.1).$$

Now look at equation (3.4.2.1); noticing (by property (3.4.0)) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equation (3.4.2.1) is of the form

$$\rho_{n-1.1} = (\, 4na_{n.1} + 4a_{n.1} \, - 4a_{n.1})(\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1} \, + 2ia_{n.1}\,) \qquad (3.4.2.2).$$

It is trivial to check (by elementary calculation) that equation (3.4.2.2) is of the form

$$\rho_{n-1.1} = (\, 4na_{n.1} + 4a_{n.1}\,)(\, ia^2_{n.1} - 2ina_{n.1} - 2ia_{n.1} + 1 - a^2_{n.1}\,) + \alpha_{n.1} \qquad (3.4.2.3),$$

where

$$\alpha_{n.1} = -4a_{n.1} + 4a^2_{n.1} + 16ina^2_{n.1} + 8ia^2_{n.1} - 4ia^3_{n.1} \qquad (3.4.2.4).$$

Clearly $\rho_{n-1.1} = \rho_{n.1} - 4a_{n.1} + 4a^2_{n.1} + 16ina^2_{n.1} + 8ia^2_{n.1} - 4ia^3_{n.1}$, by using equations (3.4.2.0) and (3.4.2.3) and (3.4.2.4). Property (3.4.2) follows.
(3.4.3) Indeed, look at $\rho_{n.2}$ and observe (by using Definitions 3.3) that

$$\rho_{n.2} = (\, 2na_{n.1} + 2a_{n.1}\,)^2 \qquad (3.4.3.0)$$

Using equation (3.4.3.0), then it becomes trivial to deduce that

$$\rho_{n-1.2} = (\, 2na_{n-1.1} + 2a_{n-1.1} \, - 2a_{n-1.1}\,)^2 \qquad (3.4.3.1).$$

Now look at equation (3.4.3.1); noticing (by property (3.4.0)) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equation (3.4.3.1) is of the form

$$\rho_{n-1.1} = (\, 2na_{n.1} + 2a_{n.1} \, - 2a_{n.1}\,)^2 \qquad (3.4.3.2).$$

It is trivial to check (by elementary calculation) that equation (3.4.3.2) is of the form

$$\rho_{n-1.2} = (\,2na_{n.1} + 2a_{n.1}\,)^2 - 8na_{n.1}^2 - 4a_{n.1}^2 \qquad (3.4.3.3).$$

Clearly $\rho_{n-1.2} = \rho_{n.2} - 8na_{n.1}^2 - 4a_{n.1}^2$, by using equations (3.4.3.0) and (3.4.3.3). Property (3.4.3) follows.
(3.4.4) Indeed, let $R[\rho_{n-1.0}]$ be the real part of $\rho_{n-1.0}$; observing (by property (3.4.1)) that $\rho_{n-1.0} = \rho_{n.0} + 8na_{n.1}^2 + 4a_{n.1}^2 - 4a_{n.1}^3 + 4ia_{n.1} - 4ia_{n.1}^3$, and using the previous equality, then it becomes trivial to deduce that

$$R[\rho_{n-1.0}] = R[\rho_{n.0}] + 8na_{n.1}^2 + 4a_{n.1}^2 - 4a_{n.1}^3 \qquad (3.4.4.0).$$

Now let $R[\rho_{n-1.1}]$ be the real part of $\rho_{n-1.1}$; observing (by using property (3.4.2)) that
$\rho_{n-1.1} = \rho_{n.1} - 4a_{n.1} + 4a_{n.1}^2 + 16ina_{n.1}^2 + 8ia_{n.1}^2 - 4ia_{n.1}^3$, then it becomes trivial to deduce that

$$R[\rho_{n-1.1}] = R[\rho_{n.1}] - 4a_{n.1} + 4a_{n.1}^2 \qquad (3.4.4.1).$$

Finally, let $R[\rho_{n-1.2}]$ be the real part of $\rho_{n-1.2}$; observing (by property (3.4.3)) that $\rho_{n-1.2} = \rho_{n.2} - 8na_{n.1}^2 - 4a_{n.1}^2$, then it becomes trivial to deduce that

$$R[\rho_{n-1.2}] = R[\rho_{n.2}] - 8na_{n.1}^2 - 4a_{n.1}^2 \qquad (3.4.4.2).$$

Clearly

$$R[\rho_{n-1.0}] + R[\rho_{n-1.1}] + R[\rho_{n-1.2}] = R[\rho_{n.0}] + R[\rho_{n.1}] + R[\rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3 \qquad (3.4.4.3),$$

by using (3.4.4.0) and (3.4.4.1) and (3.4.4.2), and consequently $R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$, by using (3.4.4.3) and the linearity of R. Property (3.4.4) follows.
(3.4.5) Indeed, look at $\phi_{n.0}$ and observe (by using Definitions 3.3) that

$$\phi_{n.0} = (\,2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1}\,)(\,2ina_{n.1} + 2ia_{n.1} - a_{n.1}^2\,) \qquad (3.4.5.0)$$

Using equation (3.4.5.0), then it becomes trivial to deduce that

$$\phi_{n-1.0} = (\,2ia_{n-1.1}^2 - 4ina_{n-1.1} - 4ia_{n-1.1} + 4ia_{n-1.1})(\,2ina_{n-1.1} + 2ia_{n-1.1} - a_{n-1.1}^2 - 2ia_{n-1.1}) \qquad (3.4.5.1)$$

Now look at equation (3.4.5.1); noticing (by property (3.4.0)) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equation (3.4.5.1) is of the form

$$\phi_{n-1.0} = (\,2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1} + 4ia_{n.1})(\,2ina_{n.1} + 2ia_{n.1} - a_{n.1}^2 - 2ia_{n.1}\,) \qquad (3.4.5.2).$$

It is trivial to check (by elementary calculation and the fact that $i^2 = -1$) that equation (3.4.5.2) is of the form

$$\phi_{n-1.0} = (\,2ia_{n.1}^2 - 4ina_{n.1} - 4ia_{n.1}\,)(\,2ina_{n.1} + 2ia_{n.1} - a_{n.1}^2\,) + \nu_{n.0}, \qquad (3.4.5.3),$$

where

$$\nu_{n.0} = -16na_{n.1}^2 - 8a_{n.1}^2 + 4a_{n.1}^3 - 4ia_{n.1}^3 \qquad (3.4.5.4).$$

Clearly $\phi_{n-1.0} = \phi_{n.0} - 16na_{n.1}^2 - 8a_{n.1}^2 + 4a_{n.1}^3 - 4ia_{n.1}^3$, by using equations (3.4.5.0) and (3.4.5.3) and (3.4.5.4). Property (3.4.5) follows.
(3.4.6) Indeed, let $I[\phi_{n-1.0}]$ be the imaginary part of $\phi_{n-1.0}$; observing (by property (3.4.5)) that $\phi_{n-1.0} = \phi_{n.0} - 16na_{n.1}^2 - 8a_{n.1}^2 + 4a_{n.1}^3 - 4ia_{n.1}^3$, and using the previous equality, then it becomes trivial to deduce that $I[\phi_{n-1.0}] = I[\phi_{n.0}] - 4a_{n.1}^3$. Property (3.4.6) follows.
(3.4.7) Indeed look at $\phi_{n.1}$ and observe (by using Definitions 3.3) that

$$\phi_{n.1} = (\,a_{n.1} - 2n - 2\,)(\,2a_{n.1}^2 - 2a_{n.1}\,) \qquad (3.4.7.0)$$

Using equation (3.4.7.0), then it becomes trivial to deduce that

$$\phi_{n-1.1} = (\,a_{n-1.1} - 2n - 2 + 2)(\,2a_{n-1.1}^2 - 2a_{n-1.1}\,) \qquad (3.4.7.1).$$

Now look at equation (3.4.7.1); noticing (by property (3.4.0)) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equation (3.4.7.1) is of the form

$$\phi_{n-1.1} = (\,a_{n.1} - 2n - 2 + 2)(\,2a_{n.1}^2 - 2a_{n.1}\,) \qquad (3.4.7.2).$$

It is trivial that equation (3.4.7.2) is of the form

$$\phi_{n-1.1} = (\, a_{n.1} - 2n - 2)(\, 2a_{n.1}^2 - 2a_{n.1}\,) + 4a_{n.1}^2 - 4a_{n.1} \qquad (3.4.7.3).$$

Clearly

$$\phi_{n-1.1} = \phi_{n.1} + 4a_{n.1}^2 - 4a_{n.1} \qquad (3.4.7.4),$$

by using equations (3.4.7.0) and (3.4.7.3). That being so, look at $I[\phi_{n-1.0}]$, and observe (by property (3.4.6)) that

$$I[\phi_{n-1.0}] = I[\phi_{n.0}] - 4a_{n.1}^3 \qquad (3.4.7.5).$$

Now look at $I[\phi_{n-1.0}] + \phi_{n-1.1}$; clearly $I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$, by using equations (3.4.7.5) and (3.4.7.4). Property (3.4.7) follows and Proposition 3.4 immediately follows.

PROPOSITION 3.5. Let n be an integer ≥ 284 and let $a_{n.1}$ (see Definitions 1.0); now look at equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$ introduced in Definitions 3.3. **If** $a_{n.1} = 2n$, then

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1},$$

where $R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$ is the real part of $\rho_{n.0} + \rho_{n.1} + \rho_{n.2}$ and $I[\phi_{n.0}]$ is the imaginary part of $\phi_{n.0}$.
Proof. Indeed, look at $R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$; noticing (by the hypotheses) that $a_{n.1} = 2n$, then

$$R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = 1 + 4a_{n.1} \qquad (3.5.1),$$

by using Example.**2** of Recalls 3.2; consequently

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = 4a_{n.1} \qquad (3.5.2),$$

by using (3.5.1). Now let $\phi_{n.1}$ and observe (by Definitions 3.3) that

$$\phi_{n.1} = (\, a_{n.1} - 2n - 2\,)(\, 2a_{n.1}^2 - 2a_{n.1}\,) \qquad (3.5.3).$$

Noticing (by the hypotheses) that $a_{n.1} = 2n$, clearly $a_{n.1} - 2n - 2 = -2$ and it becomes trivial to deduce that equation (3.5.3) is of the form

$$\phi_{n.1} = -4a_{n.1}^2 + 4a_{n.1} \qquad (3.5.4).$$

That being so, look at $I[\phi_{n.0}]$; noticing (by the hypotheses) that $a_{n.1} = 2n$, then

$$I[\phi_{n.0}] = 4a_{n.1}^3 \qquad (3.5.5),$$

by using Example.**4** of Recalls 3.2; consequently

$$I[\phi_{n.0}] + \phi_{n.1} = 4a_{n.1}^3 - 4a_{n.1}^2 + 4a_{n.1} \qquad (3.5.6),$$

by using (3.5.5) and (3.5.4). Now, using equations (3.5.2) and (3.5.6), then it becomes trivial to deduce that $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1}$ $\Big($ otherwise (we reason by reduction to absurd), $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1}$, and using equations (3.5.2) and (3.5.6), we immediately deduce that $4a_{n.1} = 4a_{n.1}^3 - 4a_{n.1}^2 + 4a_{n.1}$; consequetly $0 = 4a_{n.1}^3 - 4a_{n.1}^2$, and this is impossible, since $a_{n.1} > 284^{284}$, by remarking that $n \geq 284$ and by using property (1.1.0) of Remark 1.1 $\Big)$. Proposition 3.5 follows.

PROPOSITION 3.6.Let n be an integer ≥ 284 and let $a_{n.1}$; now look at equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$ introduced in Definitions 3.3. **If** $a_{n.1} \leq 2n$, then

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1}.$$

Proof. Otherwise (we reason by reduction to absurd), let n be a minimun counter-example, clearly

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \qquad (3.6.0),$$

and we observe the following.
Observation.3.6.1. $a_{n.1} \neq 2n$. Otherwise $a_{n.1} = 2n$; now using the previous equality and Proposition 3.5, then it becomes trivial to deduce that

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1},$$

and this contradicts (3.6.0). Observation.3.6.1 follows.
Observation.3.6.2. $a_{n.1} \leq 2n - 2$. Otherwise

$$a_{n.1} > 2n - 2 \qquad (3.6.2.0).$$

Now noticing that $a_{n.1}$ and $2n$ are even ($a_{n.1}$ is trivially even, by using the definition of $a_{n.1}$, and $2n$ is clearly even), then, using the previous, it becomes trivial to see that inequality (3.6.2.0) implies that $a_{n.1} \geq 2n - 2 + 2$; so

$$a_{n.1} \geq 2n \qquad (3.6.2.1).$$

Remarking (by the hypotheses) that $a_{n.1} \leq 2n$ and using inequality (3.6.2.1), then it becomes trivial to deduce that $a_{n.1} = 2n$ and this contradicts Observation.3.6.1. Observation.3.6.2 follows.
Observation.3.6.3. $n > 284$. Otherwise, $n \leq 284$; now remarking (by the hypotheses) that $n \geq 284$, then the previous two inequalities trivially imply that

$$n = 284 \qquad (3.6.3.0).$$

Now look at $a_{n.1}$ and observe (by using property (1.1.0) of Remark 1.1) that

$$a_{n.1} = 4a_n^{a_n} \textit{ and } 283 < a_n < a_{n.1} \qquad (3.6.3.1).$$

Using (3.6.3.0) and (3.6.3.1), then it becomes trivial to deduce that

$$a_{n.1} > -1 + 4a_n^{a_n} > -1 + 4(283)^{283} > 900 > 2n + 2 \qquad (3.6.3.2);$$

so $a_{n.1} > 2n+2$ (use (3.6.3.2)), and we have a contradiction, since $a_{n.1} \leq 2n$ (by the hypotheses). Observation.3.6.3 follows.
Observation.3.6.4. Look at $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$ and $I[\phi_{n.0}] + \phi_{n.1}$, and via $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$ and $I[\phi_{n.0}] + \phi_{n.1}$, consider equations $-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}]$ and $I[\phi_{n-1.0}] + \phi_{n-1.1}$ $\Big($ these considerations get sense, since $n > 284$ (by Observation.3.6.3) and therefore $n - 1 \geq 284$ $\Big)$. Then

$$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] \neq I[\phi_{n-1.0}] + \phi_{n-1.1}.$$

Indeed, since $n - 1 < n$, then, by the minimality of n, $n - 1$ is not a counter-example; so
$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] \neq I[\phi_{n-1.0}] + \phi_{n-1.1}$ $\Big($ the using of $n - 1$ via the minimality of n gets sense, since $n > 284$ (by Observation.3.6.3) and $a_{n.1} \leq 2n - 2$ (by Observation.3.6.2)$\Big)$. Observation.3.6.4 follows.
Observation.3.6.5. Consider equation $R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}]$ $\Big($ this consideration gets sense, since $n > 284$ (by Observation.3.6.3) and therefore $n - 1 \geq 284$ $\Big)$. Then

$$R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3.$$

Clearly $R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$, by remarking (by the hypotheses) that $a_{n.1} \leq 2n$ and by using property (3.4.4) of Proposition 3.4 (note that $n > 284$, by Observation.3.6.3; so $n \geq 285$ and the using of property (3.4.4) of Proposition 3.4 gets sense). Observation.3.6.5 follows.
Observation.3.6.6. Consider equation $I[\phi_{n-1.0}] + \phi_{n-1.1}$ $\Big($ this consideration gets sense, since $n > 284$ (by Observation.3.6.3) and therefore $n - 1 \geq 284$ $\Big)$. Then

$$I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3.$$

Clearly $I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$, by remarking (by the hypotheses) that $a_{n.1} \leq 2n$ and by using property (3.4.7) of Proposition 3.4 (note that $n > 284$, by Observation.3.6.3; so $n \geq 285$ and the using of property (3.4.7) of Proposition 3.4 gets sense). Observation.3.6.6 follows.

Observation.3.6.7. Look at equations $-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}]$ and $I[\phi_{n.0}]+\phi_{n.1}$. Then

$$-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] \neq I[\phi_{n.0}]+\phi_{n.1}.$$

Indeed, look at equations $-1+R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}]$ and $I[\phi_{n-1.0}]+\phi_{n-1.1}$, and observe (by Observation.3.6.4) that

$$-1+R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] \neq I[\phi_{n-1.0}]+\phi_{n-1.1} \qquad (3.6.7.0).$$

Noticing that $R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}]-4a_{n.1}+4a_{n.1}^2-4a_{n.1}^3$ (use Observation.3.6.5), and remarking that $I[\phi_{n-1.0}]+\phi_{n-1.1} = I[\phi_{n.0}]+\phi_{n.1}-4a_{n.1}+4a_{n.1}^2-4a_{n.1}^3$ (use Observation.3.6.6), then, using the previous two equalities, it becomes trivial to deduce that (3.6.7.0) clearly says that $-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] \neq I[\phi_{n.0}]+\phi_{n.1}$. Observation.3.6.7 follows.

These simple observations made, noticing (by (3.6.0)) that

$$-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = I[\phi_{n.0}]+\phi_{n.1} \qquad (3.6.8.0),$$

and observing (by Observation.3.6.7) that

$$-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] \neq I[\phi_{n.0}]+\phi_{n.1} \qquad (3.6.8.1),$$

then (3.6.8.0) and (3.6.8.1) give rise to a serious contradiction. Proposition 3.6 follows.

The previous simple Propositions made, we are now ready to give the analytic simple proof of the friendly numbers problem.

4 The analytic proof of the friendly numbers problem

In this Section, using Proposition 2.3 proved in Section.**2** and simple Propositions of Section.**3**, we give the analytic simple proof of the friendly numbers problem. In this section, the definitions of $\mathcal{A}(n)$, a_n, $a_{n.1}$, (see Definitions 1.0), and the definitions of $\mathcal{P}(n-2)$, $\mathcal{P}(n-1)$, $\mathcal{P}(n)$, p_{n-2}, p_{n-1}, and p_n (see Definitions 2.1), are fundamental and crucial.

Now the following Theorem immediately implies the friendly numbers problem.

THEOREM 4.1.Let n be an integer ≥ 284 and let $a_{n.1}$; consider p_{n-1} and p_n (p_{n-1} and p_n are introduced in Definitions 2.1). Now look at equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$ introduced in Definitions 3.3 $\Big($recall (see Recalls 3.2) that $R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}]$ is the real part of $\rho_{n.0}+\rho_{n.1}+\rho_{n.2}$ and $I[\phi_{n.0}]$ is the imaginary part of $\phi_{n.0}$ $\Big)$.

If $a_{n.1} > p_{n-1}$, then at least one of the following two properties $(i.n)$ and $(ii.n)$ is satisfied by $a_{n.1}$.

$(i.n)$. $a_{n.1} \geq 2n+4$.

$(ii.n)$. $-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = I[\phi_{n.0}]+\phi_{n.1} \quad \Leftrightarrow \quad a_{n.1} > p_n$.

We are going to prove simply Theorem 4.1. But before, let us remark.

REMARK 4.2. Let n be an integer ≥ 284 and let $a_{n.1}$; **fix** once and for all $a_{n.1}$, and let p_{n-1} and p_n. Now consider equations $\rho_{n.0}$, $\rho_{n.1}$, $\rho_{n.2}$, $\phi_{n.0}$, and $\phi_{n.1}$ introduced in Definitions 3.3. We have the following three trivial properties.

(4.2.0.) **If** $a_{n.1} \geq 2n+4$, then Theorem 4.1 is satisfies by $a_{n.1}$.

(4.2.1.) **If** $a_{n.1} = 2n+2$, then Theorem 4.1 is satisfied by $a_{n.1}$.

(4.2.2.) **If** $n \leq 286$, then Theorem 4.1 is satisfied by $a_{n.1}$

Proof. Property (4.2.0) is trivial. Property (4.2.1) is immediate $\Big($ **indeed**, let n be an integer ≥ 284, observing (by the hypotheses) that $a_{n.1} = 2n+2$, then

$$R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = 1 \qquad (4.2.1.0),$$

by using Example.**1** of Recalls 3.2 and the fact that $a_{n.1} = 2n+2$, and

$$I[\phi_{n.0}] = 0 \qquad (4.2.1.1),$$

by using Example.**3** of Recalls 3.2 and the fact that $a_{n.1} = 2n+2$. Now look at equation $\phi_{n.1}$ and observe (by using Definitions 3.3) that

$$\phi_{n.1} = (\,a_{n.1}-2n-2\,)(\,2a_{n.1}^2-2a_{n.1}\,) \qquad (4.2.1.2).$$

Let equation (4.2.1.2); observing (by the hypotheses) that $a_{n.1} = 2n + 2$, clearly

$$\phi_{n.1} = 0 \qquad (4.2.1.3),$$

by using equation (4.2.1.2) and the fact that $a_{n.1} = 2n+2$. Now look at $-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}]$ and $I[\phi_{n.0}]+\phi_{n.1}$, then

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = 0 \qquad (4.2.1.4),$$

by using (4.2.1.0), and

$$I[\phi_{n.0}] + \phi_{n.1} = 0 \qquad (4.2.1.5),$$

by using (4.2.1.1) and (4.2.1.3); consequently

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \qquad (4.2.1.6),$$

by using (4.2.1.4) and (4.2.1.5). That being so, look at p_n; it is trivial (by using the definition of p_n and the fact that $n \geq 284$) that $p_n \leq 2n - 1$, and consequently

$$a_{n.1} > p_n \qquad (4.2.1.7),$$

by observing (by the hypotheses) that $a_{n.1} = 2n + 2$ and by recalling that $p_n \leq 2n - 1$. Clearly

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \ \textit{and} \ \ a_{n.1} > p_n \qquad (4.2.1.8),$$

by using (4.2.1.6) and (4.2.1.7). (4.2.1.8) trivially implies that Theorem 4.1 is satisfies by $a_{n.1}$. Property (4.2.1) follows$\Big)$.

Property (4.2.2.) is immediate $\Big($ **indeed**, observing (by using property (1.1.0) of Remark 1.1) that $a_{n.1} > 284^{284}$, and remarking (by the hyptheses) that $n \leq 286$, then, using the previous two inequalities, it becomes trivial to deduce that

$$a_{n.1} > 284^{284} > 900 > 2n + 4 \qquad (4.2.2.0);$$

so

$$a_{n.1} > 2n + 4 \qquad (4.2.2.1),$$

by using (4.2.2.0). Clearly Theorem 4.1 is satisfies by $a_{n.1}$, by using inequality (4.2.2.1) and property (4.2.0)$\Big)$.
Using Proposition 4.2, let us Remark.

REMARK 4.3. Suppose that Theorem 4.1 is false; then there exists an integer $n \geq 284$ such that $a_{n.1}$ does not satisfied Theorem 4.1. (Proof. Immediate.)
From Remark 4.3, let us define

Definitions 4.4 (Fundamental). **(i).** We say that n is a counter-example to Theorem 4.1, if $n \geq 284$ and if $a_{n.1}$ does not satisfied Theorem 4.1 (If Theorem 4.1 is false, then such a n exists, by using Remark 4.3. Moreover, it is trivial that $a_{n.1}$ does not satisfied Theorem 4.1 **means** none of the two properties $(i.n)$ and $(ii.n)$ of Theorem 4.1 is satisfied by $a_{n.1}$) .
(ii). We say that n is a minimum counter-example to Theorem 4.1 if n is a counter-example to Theorem 4.1 with n minimum (If Theorem 4.1 is false, then such a n exists, by using **(i)**).
The previous simple remarks and definitions made, we now prove simply Theorem 4.1.

Proof of Theorem 4.1. Otherwise (we reason by reduction to absurd), let n be a minimum counter-example to Theorem 4.1 (such a n exists, by using Remark 4.3 and Definitions 4.4). We observe the following.
Observation.4.1.i. Look at n (recall n is a minimum counter-example to Theorem 4.1), and let $a_{n.1}$. Then $n > 286$ and $a_{n.1} \leq 2n + 2$.
Clearly $n > 286$ $\Big($ Otherwise $n \leq 286$ and clearly Theorem 4.1 is satisfies by $a_{n.1}$ (by using the previous inequality and property (4.2.2) of Remark 4.2); a contradiction, since in particular $a_{n.1}$ does not satisfied Theorem 4.1$\Big)$; and clearly $a_{n.1} \leq 2n + 2$ $\Big($ Otherwise $a_{n.1} > 2n + 2$; noticing that $a_{n.1}$ and $2n + 2$ are even ($a_{n.1}$ is even (by using the

definition of $a_{n.1}$) and $2n+2$ is trivially even), then it becomes trivial to deduce that the previous inequality implies that $a_{n.1} \geq 2n+2+2$; so $a_{n.1} \geq 2n+4$ and this last inequality clearly implies that property $(i.n)$ of Theorem 4.1 is satisfied by $a_{n.1}$; in particular $a_{n.1}$ satisfies Theorem 4.1, and we have a contradiction since $a_{n.1}$ does not clearly satisfied Theorem 4.1. Observation.4.1.*i* follows.

Observation.4.1.*ii*. Look at n (recall n is a minimum counter-example to Theorem 4.1), and let $a_{n.1}$. Then

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \quad \Rightarrow \quad a_{n.1} > p_n.$$

Otherwise, we have

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \;\; and \;\; a_{n.1} \leq p_n \qquad (4.1.ii.0),$$

and we pretend the following.

Pretension.4.1.*ii*.1. $a_{n.1} < 2n$. Indeed, remarking (by using (4.1.ii.0)) that $a_{n.1} \leq p_n$ and since it is trivial that $p_n < 2n$ (use the definition of p_n and the fact that $n > 286$, by Observation.4.1.*i*), then, using the previous, it becomes trivial to deduce that $a_{n.1} < 2n$. Pretension.4.1.*ii*.1 follows.

That being so, noticing (by Pretension.4.1.*ii*.1) that $a_{n.1} < 2n$ and using Proposition 3.6, then it becomes trivial to deduce that $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1}$; the previous clearly contradicts (4.1.ii.0). Observation.4.1.*ii* follows.

Observation.4.1.*iii*. Look at n (recall n is a minimum counter-example to Theorem 4.1), and let $a_{n.1}$. Then $a_{n.1} > p_n$ and $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1}$.

Indeed, observing (by Observation.4.1.*ii*) that

$$-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] = I[\phi_{n.0}] + \phi_{n.1} \;\Rightarrow\; a_{n.1} > p_n \qquad (4.1.iii.0),$$

and since in particular $a_{n.1}$ does not satisfied Theorem 4.1, then using (4.1.iii.0), it becomes trivial to deduce that $a_{n.1} > p_n \;\; \not\Rightarrow \;\; -1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] = I[\phi_{n.0}]+\phi_{n.1}$; so $a_{n.1} > p_n$ and $-1+R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] \neq I[\phi_{n.0}]+\phi_{n.1}$. Observation.4.1.*iii* follows.

Observation.4.1.*iv*. Look at n (recall n is a minimum counter-example to Theorem 4.1), and let $a_{n.1}$ (clearly $a_{n.1}$ does not satisfied Theorem 4.1). Then

$$a_{n.1} \leq 2n \;\; and \;\; a_{n.1} = a_{n-1.1}.$$

Firstly, we are going to show that $a_{n.1} \leq 2n$. Fact: $a_{n.1} \leq 2n$. Otherwise,

$$a_{n.1} > 2n; \qquad (4.1.iv.0);$$

remarking that $a_{n.1}$ and $2n$ are even $\Big($ $a_{n.1}$ is even (by using the definition of $a_{n.1}$) and $2n$ is trivially even$\Big)$, then inequality (4.1.iv.0) immediately implies that $a_{n.1} \geq 2n+2$. Note (by using Observation.4.1.*i*) that $a_{n.1} \leq 2n+2$. Now using the previous two inequalities, then it becomes trivial to see that $a_{n.1} = 2n+2$; so Theorem 4.1 is satisfied by $a_{n.1}$ $\Big($ by using the previous equality and property (4.2.1) of Remark 4.2$\Big)$, and we have a contradiction, since $a_{n.1}$ does not clearly satisfied Theorem 4.1. So

$$a_{n.1} \leq 2n \qquad (4.1.iv.1),$$

and the Fact follows. Now using inequality (4.1.iv.1) and property (1.1.2) of Remark 1.1 ,then it becomes trivial to deduce that $a_{n.1} = a_{n-1.1}$ $\Big($note that $n > 286$ (by using Observation.4.1.*i*) and $a_{n.1} \leq 2n$ (by inequality (4.1.iv.1)); so the using of property (1.1.2) of Remark 1.1 gets sense$\Big)$. Observation.4.1.*iv* follows.

Observation.4.1.*v*. Look at $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$, and via $-1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}]$, consider $-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}]$ $\Big($ this consideration gets sense, since $n > 286$ (by using Observation.4.1.*i*), and therefore $n-1 > 285 > 284$ $\Big)$. Then

$$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = -1 + R[\rho_{n.0} + \rho_{n.1} + \rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3.$$

Indeed, observing (by Observation.4.1.*iv*) that $a_{n.1} \leq 2n$ and using property (3.4.4) of Proposition 3.4, then it becomes trivial to deduce that

$$R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3 \qquad (4.1.v.0)$$

$\Big($ note that $n > 286$ (by using Observation.4.1.*i*) and $a_{n.1} \leq 2n$ (by Observation.4.1.*iv*); so the using of property (3.4.4) of Proposition 3.4 gets sense$\Big)$. Clearly

$$-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = -1 + R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3,$$

by using (4.1.v.0). Observation.4.1.*v* follows.

Observation.4.1.*vi.* Look at $I[\phi_{n.0}] + \phi_{n.1}$ and via $I[\phi_{n.0}] + \phi_{n.1}$, consider $I[\phi_{n-1.0}] + \phi_{n-1.1}$ $\Big($ this consideration gets sense, since $n > 286$ (by Observation.4.1.*i*), and therefore $n - 1 > 285 > 284\Big)$. Then $I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$.

Indeed, observing (by Observation.4.1.*iv*) that $a_{n.1} \leq 2n$ and using property (3.4.7) of Proposition 3.4, then it becomes trivial to deduce that

$$I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$$

$\Big($ note that $n > 286$ (by Observation.4.1.*i*) and $a_{n.1} \leq 2n$ (by Observation.4.1.*iv*); so the using of property (3.4.7) of Proposition 3.4 gets sense$\Big)$. Observation.4.1.*vi* follows.

Observation.4.1.*vii.* Look at $-1 + R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}]$ and $I[\phi_{n.0}] + \phi_{n.1}$; and via $I[\phi_{n.0}] + \phi_{n.1}$ and $-1 + R[\rho_{n.0} + \rho_{n.1}+\rho_{n.2}]$, consider $-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}]$ and $I[\phi_{n-1.0}] + \phi_{n-1.1}$ $\Big($ these considerations get sense, since $n > 286$(by Observation.4.1.*i*), and therefore $n - 1 > 285 > 284\Big)$. Then $-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] \neq I[\phi_{n-1.0}] + \phi_{n-1.1}$.

Indeed, observe (by using Observation.4.1.*iii*) that

$$-1 + R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] \neq I[\phi_{n.0}] + \phi_{n.1} \qquad (4.1.vii.0).$$

Now noticing that $-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = -1 + R[\rho_{n.0}+\rho_{n.1}+\rho_{n.2}] - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$ (use Observation.4.1.*v*), and since $I[\phi_{n-1.0}] + \phi_{n-1.1} = I[\phi_{n.0}] + \phi_{n.1} - 4a_{n.1} + 4a_{n.1}^2 - 4a_{n.1}^3$ (by Observation.4.1.*vi*), then, using the previous two equalities, it becomes trivial to deduce that (4.1.vii.0) clearly says that $-1+R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] \neq I[\phi_{n-1.0}] + \phi_{n-1.1}$. Observation.4.1.*vii* follows.

These simple observations made, look at n (recall n is a minimum counter-example to Theorem 4.1) and let $a_{n.1}$ (clearly $a_{n.1}$ does not satisfied Theorem 4.1). Via n, consider $n-1$ $\Big($ this consideration gets sense, since $n > 286$ (by Observation.4.1.*i*), and therefore $n - 1 > 285 > 284\Big)$. Then, by the minimality of n, $n - 1$ is not a counter-example to Theorem 4.1; so at least one of the following two properties $(i.n - 1)$ and $(ii.n - 1)$ is satisfied by $a_{n-1.1}$.

$$a_{n-1.1} \geq 2(n-1) + 4 \qquad (i.n-1).$$

$$-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \quad \Leftrightarrow \quad a_{n-1.1} > p_{n-1} \qquad (ii.n-1).$$

That being so, we claim the following.

Claim 4.1.*viii*. Let $a_{n-1.1}$; then property $(ii.n - 1)$ is satisfied by $a_{n-1.1}$. Otherwise, remarking just above that at least one of the two properties $(i.n - 1)$ and $(ii.n - 1)$ is satisfied by $a_{n-1.1}$, then it becomes trivial to deduce that property $(i.n - 1)$ is satisfied by $a_{n-1.1}$; so $a_{n-1.1} \geq 2(n - 1) + 4$ and clearly

$$a_{n-1.1} \geq 2n + 2 \qquad (4.1.viii.0).$$

Now observing (by using Observation.4.1.*iv*) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that inequality (4.1.viii.0) clearly says that $a_{n.1} \geq 2n + 2$; this last inequality clearly contradicts Observation.4.1.*iv* (recall that $a_{n.1} \leq 2n$, by using Observation.4.1.*iv*). Claim.4.1.*viii* follows.

Claim.4.1.*ix*. Let $a_{n-1.1}$; then

$$-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \quad \Leftrightarrow \quad a_{n-1.1} > p_{n-1}.$$

Indeed, observing (by Claim.4.1.*viii*) that property $(ii.n - 1)$ is satisfied by $a_{n-1.1}$, then it becomes trivial to deduce that $-1 + R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \Leftrightarrow a_{n-1.1} > p_{n-1}$. Claim.4.1.*ix* follows.

Claim 4.1.x. Look at $a_{n.1}$ (recall that $a_{n.1}$ does not satisfied Theorem 4.1). Then

$$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \quad \Leftrightarrow \quad a_{n.1} > p_{n-1}.$$

Indeed, observing (by using Observation.4.1.iv) that $a_{n.1} = a_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that Claim.4.1.ix clearly says that

$$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \;\Leftrightarrow\; a_{n.1} > p_{n-1}.$$

Claim.4.1.x follows.
These trivial claims made, look at Claim.4.1.x; noticing (by the hypotheses made on the statement of Theorem 4.1) that $a_{n.1} > p_{n-1}$, then, using the previous inequality and using Claim.4.1.x, it becomes trivial to deduce that $1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1}$; this last equality clearly contradicts Observation.4.1.vii. Theorem 4.1 follows.
Recalling that p_{n-2} and p_{n-1} were introduced in Definitions 2.1, then the following Corollary is an immediate consequence of Theorem 4.1.

COROLLARY 4.5 (Immediate consequence of Theorem 4.1). Let m be an integer ≥ 286; consider $a_{m-1.1}$ and p_{m-2}, p_{m-1}, $\rho_{m-1.0}$, $\rho_{m-1.1}$, $\rho_{m-1.2}$, $\phi_{m-1.0}$, and $\phi_{m-1.1}$ $\Big($ these considerations get sense, since $m \geq 286$ and therefore $m-2 \geq 284$ and $m-1 \geq 285$. Recall that $\rho_{m-1.0}$, $\rho_{m-1.1}$, $\rho_{m-1.2}$, $\phi_{m-1.0}$, and $\phi_{m-1.1}$ are introduced in Definitions 3.3; p_{m-2} and p_{m-1} are introduced in Definitions 2.1; and $a_{m-1.1}$ is introduced in Definitions 1.0$\Big)$. **If** $a_{m-1.1} > p_{m-2}$, then at least one of the following two properties $(i.m-1)$ and $(ii.m-1)$ is satisfied by $a_{m-1.1}$.
$(i.m-1)$. $\quad a_{m-1.1} \geq 2(m-1) + 4$.
$(ii.m-1)$. $-1 + R[\rho_{m-1.0} + \rho_{m-1.1} + \rho_{m-1.2}] = I[\phi_{m-1.0}] + \phi_{m-1.1} \quad \Leftrightarrow \quad a_{m-1.1} > p_{m-1}$.
Proof. Immediate $\Big($ indeed, it suffices to replace n of Theorem 4.1 by $m-1$, and Corollary follows.$\Big)$.
Now the friendly numbers problem directly results from the following Theorem.

THEOREM 4.6. Let n be an integer ≥ 284 and let $a_{n.1}$ (see Definitions 1.0 for the meaning of $a_{n.1}$); consider p_{n-1} and p_n (p_{n-1} and p_n are introduced in Definitions 2.1). Then $a_{n.1} > p_n$.
Proof. Immediate and is an obvious consequence of Theorem 4.1 from which Corollary 4.5 is only a special case, when reasoning by reduction to absurd $\Big($ indeed, when reasoning by reduction to absurd, let n be a minimum counter-example, clearly $n > 286$ and

$$a_{n-1.1} > p_{n-1} \qquad (4.6.0),$$

by the minimality of n. Now using Corollary 4.5 and inequality (4.6.0), then it becomes immediate to deduce that

$$-1 + R[\rho_{n-1.0} + \rho_{n-1.1} + \rho_{n-1.2}] = I[\phi_{n-1.0}] + \phi_{n-1.1} \qquad (4.6.1).$$

That being so, observing that $a_{n.1} \leq p_n$ (since $a_{n.1}$ does not satisfy Theorem 4.6) and since it is trivial that $a_{n.1}$ is even and p_n is odd $\leq 2n-1$, then the previous inequality immediately implies that $a_{n.1} \leq 2n-1-1$ and clearly

$$a_{n.1} \leq 2n-2 \qquad (4.6.2).$$

Now using inequality (4.6.2) and Proposition 3.6 where we replace n by $n-1$, then it becomes trivial to deduce that $-1+R[\rho_{n-1.0}+\rho_{n-1.1}+\rho_{n-1.2}] \neq I[\phi_{n-1.0}]+\phi_{n-1.1}$ and this contradicts equality (4.6.1). Theorem 4.6 follows $\Big)$.
Theorem 4.6 immediately implies the friendly numbers problem.

THEOREM 4.7 (The Proof of the friendly numbers problem). There are infinitely many friendly numbers. [
Proof. Observe [by using Theorem 4.6] that

$$For\ every\ integer\ n \geq 284\, we\ have\ a_{n.1} > p_n \qquad (4.7.0);$$

consequently, there are infinitely many friendly numbers, by using (4.7.0) and Proposition 2.3.]

Epilogue. Our simple article clearly shows that divisibility does not help to give a simple proof of the friendly numbers problem; since in our article, we do not use divisibility. Indeed divisibility helps to characterize friendly numbers as we did in [5] and [6], and divisibility does not help to give a simple analytic proof of problem posed by the friendly numbers.

References

[1] Ikorong Anouk Gilbert Nemron. An original abstract over the twin primes, the Goldbach Conjecture, the Friendly numbers, the perfect numbers, the Mersenne composite numbers, and the Sophie Germain primes. Journal of Discrete Mathematical Sciences And Cryptography; Taru Plublications; Vol.11; Number.6, (2008). $715 - 726$.

[2] Ikorong Anouk Gilbert Nemron. An original abstract over the twin primes, the Goldbach Conjecture, the Friendly numbers, the perfect numbers, the Mersenne composite numbers, and the Sophie Germain primes. Journal of Discrete Mathematical Sciences And Cryptography; Taru Plublications; Vol.11; Number.6, (2008). $715 - 726$.

[3] Ikorong Anouk Gilbert Nemron. A Glance At A Different Kinds Of Numbers. International Journal Of Mathematics And Computer Sciences. Vol.4, No.1, 2009. $43 - 53$.

[4] Ikorong Anouk Gilbert Nemron. A Proof Of Eight Famous Number Theory Problems And Their Connection To The Goldbach Conjecture. South Asian Journal Of Mathematics; Vol1 (3); 2011, $87 - 105$.

[5] Ikorong Anouk Gilbert Nemron. Nice Rendez Vous With Primes And Composite Numbers. South Asian Journal Of Mathematics; Vol1 (2); 2012, $68 - 80$.

[6] Ikorong Anouk Gilbert Nemron. Meeting With Primes And Composite Numbers. Will Appear In Asian Journal of Mathematics and Applications; 2013.

[7] Paul Hoffman. Erdös, l'homme qui n'aimait que les nombres. Editions Belin, (2000). $30 - 49$.

[8] Paul Hoffman. The man who loved only numbers. The story of Paul Erdös and the search for mathematical truth. 1998. $30 - 49$.