A Totient Function Inequality

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Abstract: A new unconditional inequality of the totient function is contributed to the literature. This result is associated with various unsolved problems about the distribution of prime numbers.

Mathematics Subject Classifications: 11A25, 11A41, 11Y35.
Keyword: Totient function, Prime numbers, Nicolas inequality, Highly composite numbers, and Riemann hypothesis.

1 Introduction

The totient function $\varphi(N) = \#\{\ n < N : \gcd(n, N) = 1\ \}$, which counts the number of relatively prime integers less than N, is a sine qua non in number theory. It and its various generalizations appear everywhere in the mathematical literature. The product form representation

$$\varphi(N) = N\prod_{p \mid N}(1 - 1/p) \tag{1}$$

unearths its intrinsic link to the distribution of the prime numbers.

The totient function $\varphi(N)$ is an oscillatory function, its value oscillates from its maximum $\varphi(N) = N - 1$ at prime integers N to its minimum $\varphi(N) = N/c_0 \log\log N$, at the primorial integers $N = 2^{v_1} \cdot 3^{v_2} \cdots p_k^{v_k}$, where p_i is the kth prime, $v_i \geq 1$, and $c_0 > 1$ is a constant. The new contributions to the literature are the unconditional estimates stated below.

***Theorem* 1.** Let p_i be the kth prime, and let $N_k = 2 \cdot 3 \cdot 5 \cdots p_k$, $k \geq 1$. Then $N_k / \varphi(N_k) > e^{\gamma} \log\log N_k$ for all sufficiently large primorial integer N_k.

This unconditional result is consistent with the Riemann hypothesis, and seems to prove the Nicolas inequality, Theorem 4 below, for all sufficiently large integers. Just a finite number of cases of primorial integers $N_k \leq N_0$ remain unresolved as possible counterexamples of the inequality.

***Theorem* 2.** The function $\varphi(N)/N > c_0/\log\log\log N$ for almost every integer $N \geq 1$, and $c_0 > 0$ constant.

Currently the best unconditional estimate of this arithmetical function in the literature is the followings:

***Theorem* 3.** ([13]) Let $N \in \mathbb{N}$, then $N/\varphi(N) < e^{\gamma} N \log\log N + 5/(2\log\log N)$ with one exception for $N = 2\cdot3 \cdots 23$.

On the other hand, there are several conditional criteria; one of these is listed below.

***Theorem* 4.** ([12]) Let $N_k = 2\cdot3\cdot5\cdots p_k$ be the product of the first k primes.
(i) If the Riemann Hypothesis is true then $N_k/\varphi(N_k) > e^{\gamma}\log\log N_k$ for all $k \geq 1$.
(ii) If the Riemann Hypothesis is false then both $N_k/\varphi(N_k) < e^{\gamma}\log\log N_k$ and $N_k/\varphi(N_k) > e^{\gamma}\log\log N_k$ occur for infinitely many $k \geq 1$.

Some related and earlier works on this topic include the works of Ramanujan, Erdos, and other on abundant numbers, see [11], [2], and recent related works appeared in [2], [3], [9], [14], and [20].

The next section covers some background materials focusing on some finite sums over the prime numbers and some associated and products. The proofs of Theorems 1 and 2 are given in the last two sections respectively.

2 Background Materials

This section provides a survey of supporting materials. An effort was made to have a self contained paper as much as possible, but lengthy proofs available in the literature are omitted.

2.1 Sums and Products Over the Primes. The most basic finite sum over the prime numbers is the prime harmonic sum $\sum_{n\leq x} p^{-1}$. The refined estimate of this finite sum, stated below, is a synthesis of various results due to various authors. The earliest version $\sum_{n\leq x} p^{-1} = \log\log x + B_1 + O(1/\log x)$ is due to Mertens, see [17].

***Theorem* 5.** Let $x \geq 2$ be a sufficiently large number. Then

$$\sum_{p\leq x}\frac{1}{p} = \begin{cases} \log\log x + B_1 + O(e^{-c(\log\log x)^{1/2}}), & \text{unconditionally}, \\ \log\log x + B_1 + O((\log x)^{-1/2}), & \text{conditional on the Riemann hypothesis}, \\ \log\log x + B_1 + \Omega_{\pm}((\log x)^{-1/2}\log\log\log x/\log x), & \text{unconditional oscillations}, \end{cases} \tag{2}$$

where $B_1 = .2614972128\ldots$.

Proof: Use the integral representation of the finite sum

$$\sum_{p\leq x}\frac{1}{p} = \int_c^x \frac{d\pi(t)}{t}, \tag{3}$$

where $c > 1$ is a small constant. Moreover, the prime counting function $\pi(x) = \#\{ p \le x : p \text{ is prime } \}$ has the form

$$\pi(x) = \begin{cases} li(x) + O(xe^{-c(\log x)^{1/2}}), & \text{unconditionally}, \\ li(x) + O(x^{1/2}\log x), & \text{conditional on the Riemann hypothesis}, \\ li(x) + \Omega_{\pm}(x^{1/2}\,\text{logloglog}\, x/\log x), & \text{unconditional oscillations}. \end{cases} \tag{4}$$

The unconditional part of the prime counting formula arises from the delaVallee Poussin form of the prime number theorem $\pi(x) = li(x) + O(xe^{-c(\log x)^{1/2}})$, see [10, p. 179], the conditional part arises from the Riemann form of the prime number theorem $\pi(x) = li(x) + O(x^{1/2}\log x)$, and the unconditional oscillations part arises from the Littlewood form of the prime number theorem $\pi(x) = li(x) + \Omega_{\pm}(x^{1/2}\log\text{loglog}\, x/\log x)$, consult [7, p. 51], [10, p. 479] et cetera. Now replace the logarithm integral $li(x) = \int_0^x (t\log t)^{-1}\,dt$, and the appropriate prime counting measure $d\pi(t)$, and simplify the integral. ■

The proof of the unconditional part of this result is widely available in the literature, see [6], [10], [16], et cetera. The omega notation $f(x) = g(x) + \Omega_{\pm}(h(x))$ means that both $f(x) > g(x) + c_0 h(x)$ and $f(x) < g(x) - c_0 h(x)$ occur infinitely often as $x \to \infty$, where $c_0 > 0$ is a constant, see [10, p. 5], [18].

As an application of the last result, there is the following interesting product:

***Theorem* 6.** Let $x \in \mathbb{R}$ be a large real number, then

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right)^{-1} = \begin{cases} e^{\gamma}\log x + O(e^{-c(\log x)^{1/2}}), & \text{unconditionally}, \\ e^{\gamma}\log x + O(x^{-1/2}), & \text{conditional on the Riemann hypothesis}, \\ e^{\gamma}\log x + \Omega_{\pm}(x^{-1/2}\,\text{logloglog}\, x/\log x), & \text{unconditional oscillations}, \end{cases} \tag{5}$$

Proof: Consider the logarithm of the product

$$\begin{aligned} \log\prod_{p \le x}\left(1 - \frac{1}{p}\right)^{-1} &= \sum_{p \le x}\frac{1}{p} + \sum_{p \le x}\sum_{n \ge 2}\frac{1}{np^n} \\ &= \sum_{p \le x}\frac{1}{p} + \gamma - B_1 + O(\frac{1}{x}), \end{aligned} \tag{6}$$

where the Euler constant is defined by $\gamma = \lim_{x\to\infty}\sum_{n \le x}\left(n^{-1} - \log n\right) = 0.577215665\ldots$, and the Mertens constant is defined by $B_1 = \gamma + \sum_{p \ge 2}\left(\log(1 - 1/p) + 1/p\right) = .2614972128\ldots$, see [6, p. 466]. The last equality in (6) stems from the power series expansion $B_1 = \gamma - \sum_{p \ge 2}\sum_{n \ge 2}(np^n)^{-1}$, which yields

$$\sum_{p\le x}\sum_{n\ge 2}\frac{1}{np^n}=\gamma-B_1-\sum_{p>x}\sum_{n\ge 2}\frac{1}{np^n}$$
$$=\gamma-B_1+O(\frac{1}{x}), \tag{7}$$

Applying Theorem 5 returns

$$\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=\begin{cases}\gamma+\log\log x+O(e^{-c(\log x)^{1/2}}), & \text{unconditionally},\\ \gamma+\log\log x+O(x^{-1/2}), & \text{conditional on the Riemann hypothesis},\\ \gamma+\log\log x+\Omega_{\pm}(x^{-1/2}\log\log\log x/\log x), & \text{unconditional oscillations},\end{cases} \tag{8}$$

and reversing the logarithm completes the verification. ■

The third part in (5) above simplifies the proof given in [5] of the following result:

The quantity

$$\left(\ x^{1/2}\log x\ \right)\left(\prod\nolimits_{p\le x}(1-1/p)^{-1}-e^{\gamma}\log x\right) \tag{9}$$

attains arbitrary large positive and negative values as $x\to\infty$.

3 An Estimate of the Totient Function

The proof of Theorem 3 on the extreme values of the arithmetic function $N/\varphi(N)$ relies on the oscillation theorem of the finite prime product $\prod_{p\le x}(1-1/p)^{-1}$. This technique leads to a concise and simpler proof. A completely elementary proof, but longer, and not based on the oscillation theorem was presented in the earlier version of this paper.

***Theorem* 1.** Let $N\in\mathbb{N}$ be a primorial integer, then $N/\varphi(N)>6\pi^{-2}e^{\gamma}\log\log N$ holds unconditionally for all sufficiently large $N=2\cdot3\cdot5\cdots p_k$.

Proof : Theorem 6 implies that the product

$$\prod_{p\le x}(1-1/p)^{-1}=e^{\gamma}\log x+\Omega_{\pm}\left(\frac{\log\log\log x}{x^{1/2}\log x}\right). \tag{10}$$

In particular, it follows that

$$\prod_{p\le x}(1-1/p)^{-1}>e^{\gamma}\log x+c_0\frac{\log\log\log x}{x^{1/2}\log x} \tag{11}$$

and

$$\prod_{p \le x} \left(1 - 1/p\right)^{-1} < e^{\gamma} \log x - c_0 \frac{\log\log\log x}{x^{1/2} \log x}$$

occur infinitely often as $x \to \infty$, where $c_0 > 0$, $c_1 > 0$, and $c_2 > 0$ are constants. It shows that $\prod_{p \le x}(1-1/p)^{-1}$ oscillates infinitely often, symmetrically about the line $e^{\gamma} \log x$ as $x \to \infty$.

To rewrite the variable $x \ge 1$ in terms of the integer N, recall that the Chebychev function satisfies $\vartheta(x) = \sum_{p \le x} \log p \le c_1 x$, $c_1 > 1$, see [15]. The properties of this function lead to

$$\log N_k = \sum_{p \le p_k} \log p = \vartheta(p_k), \quad \text{and} \quad \vartheta(p_k) = p_k + o(p_k) \le c_1 \log N_k . \tag{12}$$

So it readily follows that $p_k \le x = c_1 \log N_k$. Moreover, since the maxima of the sum of divisors function

$$\frac{\sigma(N)}{N} = \prod_{p^{\alpha} \| N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \frac{1}{p^{\alpha}}\right) \le \prod_{p \mid N} \left(1 - 1/p\right)^{-1}, \tag{13}$$

where the symbol $p^{\alpha} \| N$ denotes the maximal prime power divisor of N, occur at the colossally abundant integers $N = 2^{v_1} \cdot 3^{v_2} \cdots p_k^{v_k}$, and $v_1 \ge v_2 \ge \cdots \ge v_k \ge 1$, see [2], [3], [9], [11], it follows that the maxima of the inverse totient function $N/\varphi(N)$ occur at the squarefree primorial integers $N_k = 2 \cdot 3 \cdot 5 \cdots p_k$. Therefore, expressions (10) and (11) implies that

$$\begin{aligned} \frac{N_k}{\varphi(N_k)} &= \prod_{p \le c_1 \log N_k} \left(1 - 1/p\right)^{-1} \\ &> e^{\gamma} \log\log N_k + c_2 \frac{\log\log\log\log N_k}{(\log N_k)^{1/2} \log\log N_k} \\ &> e^{\gamma} \log\log N_k \end{aligned} \tag{14}$$

as the primorial integer $N_k = 2 \cdot 3 \cdot 5 \cdots p_k$ tends to infinity. ■

4. Probabilistic Properties

The natural density function

$$B(t) = \lim_{N \to \infty} \#\{\, n \le N : N/\varphi(N) \ge t \,\}/N \tag{15}$$

is known to be a continuous function of $t \ge 0$. Some recent works have established the exact asymptotic expression

$$B(t) = \exp\left(-e^{e^{-\gamma} t}\left(1 + O(1/t^2)\right)\right) \tag{16}$$

as t tends to infinity, see [19], [20].

The evaluation of the natural density (15) at $t = e^{\gamma} \log\log N$ as $N \to \infty$ suggests that the Nicolas inequality should be $e^{\gamma} \log\log N_k > N_k / \varphi(N_k) > e^{\gamma+\alpha(k)} \log\log N_k^{1-\varepsilon}$. The numerical data are compiled in [8].

Now, note that the evaluation at $t = e^{\gamma} \log\log\log N$ as $N \to \infty$ yields the density function

$$\begin{aligned} B(t = e^{\gamma} \log\log\log N) &= \exp\left(- e^{e^{-\gamma} t}\left(1 + O(1/t^2)\right)\right) \\ &= \frac{1}{\log N}\left(1 + O(1/(\log\log\log N)^2)\right). \end{aligned} \tag{17}$$

Consequently, the subset of integers $N \geq 1$ such that $N / \varphi(N) > c_0 \log\log\log N$ has zero density with respect to the set of integers $\mathbb{N}$. A simple proof of this result is included here.

***Theorem* 2.** The function $\varphi(N) > c_0 N / \log\log\log N$ for almost every integer $N \geq 1$, and $c_0 > 0$ constant.

Proof: The prime divisors counting function satisfies $\omega(N) = \#\{ p|N \} \leq c_1 \log\log N$ for almost every integer $N \geq 1$, this is Ramanujan Theorem. In addition,

$$\prod_{p|N} (1 - 1/p) \geq \prod_{p \leq x} (1 - 1/p) \geq \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right), \tag{18}$$

where $x \geq 2$ is a suitable real number, holds for every integer $N \geq 1$, this is Mertens Theorem. Furthermore, by the Prime Number Theorem, the nth prime $p_n \leq x = c_2 n \log n$. In light of these facts, put

$$\begin{aligned} p_n &\leq c_2 (c_1 \log\log N) \log(c_1 \log\log N) \\ &\leq c_3 \log\log N \log\log\log N, \end{aligned} \tag{19}$$

where $c_1, c_2, c_3, c_4, \ldots$ are constants. Substituting (19) into the previous relation (18) implies that

$$\begin{aligned} \prod_{p|N} (1 - 1/p) &\geq \prod_{p \leq p_n} (1 - 1/p) \\ &\geq \frac{c_4}{\log\log\log N} \end{aligned} \tag{20}$$

holds for almost every integer $N \geq 1$. Ergo, the ratio $\varphi(N)/N = \prod_{p|N} (1 - 1/p) \geq c_4 / \log\log\log N$ holds for almost every integer $N \geq 1$ as claimed. ■

***Corollary* 7.** The function $\sigma(N) < c_5 N \log\log\log N$ for almost every integer $N \geq 1$, and $c_5 > 0$ constant.

Proof: The sigma-phi identity, on the first line below, coupled with Theorem 4 lead to

$$\begin{aligned}\frac{\sigma(N)}{N} &= \frac{N}{\varphi(N)} \prod_{p^\alpha \| N} \left((1 - 1/p^{\alpha+1}\right) \\ &< \frac{N}{\varphi(N)} \\ &< c_5 N \log\log\log N ,\end{aligned} \tag{21}$$

where c_5 is a constant, holds for almost every integer $N \geq 1$. ■

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