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Research paper

Location of triangular equilibrium points in the perturbed CR3BP with laser radiation pressure and oblateness

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Abstract

This paper represents a semi-analytical study of the effect of ground-based laser radiation pressure on the location of triangular points in the framework of the planar circular restricted three-body problem (CR3BP). The formulation includes both the effects of oblateness of J2 in addition to laser radiation pressure, where laser's disturbing function expanded in Legendre polynomials up to the first order. Earth-Moon system is considered in which a laser station is located on Earth and sends laser beams toward the infinitesimal body. The model takes into account the effect of Earth's atmosphere on laser beam propagation. The numerical application emphasis that the location of the triangular points affected by the considered perturbations.

Keywords: *Perturbed CR3BP- Laser Radiation Pressure-Triangular Points.*

1. Introduction

The Circular Restricted Three-Body Problem (CR3BP) is considered as one of the most famous problems in celestial mechanics. Many pieces of literature were interested in studying CR3BP [1], [2], [3]. The planar case possesses five equilibrium points; three of them are collinear L_1, L_2 and L_3, however, the other two are triangular points (L_4 and L_5). The classical problem of CR3BP was modified by including additional effects (e.g. oblateness, relativistic effects and radiation). The effect of radiation pressure in CR3BP was first studied by Radzievskii in 1950 after wards Different treatments of radiation effect were presented [4], [5], [6]. Also, the motion were studied under the influences of oblateness, radiation pressure in addition to the combined effect of small perturbations in the Coriolis and centrifugal forces [7-12].
The main issue of the current study is to investigate the effect of an artificial radiation pressure (e.g. laser radiation pressure) on the location of the triangular points of the CR3BP.

2. Formulation of the problem

2.1. Equation of motion

Consider three bodies of masses m_1, m_2 and m such that $m_1 > m_2$ and m is the infinitesimal mass where the primaries revolve in a circular orbits and they are fixed on x-axis at $(\mu, 0)$ and $(\mu - 1, 0)$ where the origin lies on the center of mass "CM" of the two primaries (see figure 1). Assuming that the three bodies are oblate spheroids whatever Coriolis and centrifugal forces are switched off and the equations of motion of the infinitesimal body in synodic coordinates (x, y) are given by:

$$\ddot{x} - 2n\dot{y} = \frac{\partial\Omega}{\partial x} \tag{1}$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial\Omega}{\partial y} \tag{2}$$

Where the potential, "Ω", is given by [7]:

$$\Omega = \frac{1}{2}n^2((1-\mu)r_1^2 + \mu r_2^2) + (1-\mu)[\frac{1}{r_1} + \frac{A_1 + A}{2r_1^3}] + \mu[\frac{1}{r_2} + \frac{A_2 + A}{2r_2^3}], \tag{3}$$

Where A_1, A_2 and A are the oblateness of the three bodies. The mean motion of the perturbed rotating system is given by:

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2),$$

The distances to the infinitesimal mass, are given by (see figure 1):

$$r_1^2 = (x - \mu)^2 + y^2 \tag{6}$$

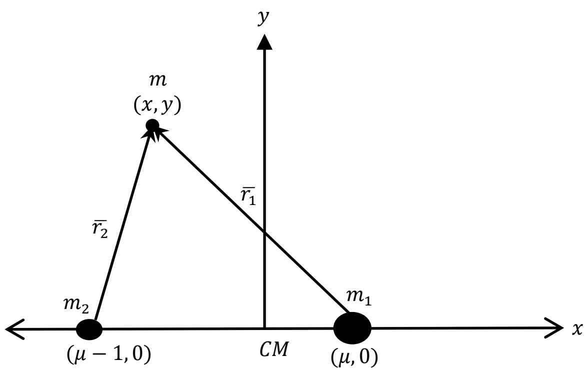

Fig. 1: Planar Restricted Three-Body Problem.

$$r_2^2 = (x + 1 - \mu)^2 + y^2 \tag{7}$$

2.2. Laser disturbing potential

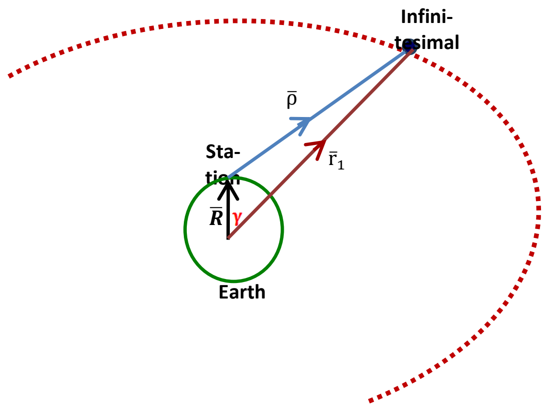

Fig. 2: Laser Intensity Delivered to An Infinitesimal Body.

Assuming that the current three bodies are Earth, Moon and an infinitesimal body. Where a laser station was built on the Earth's surface and sends laser beams towards the infinitesimal body (see figure 2).
For flat non-perfectly reflecting surfaces, the laser force is given by [13]:

$$\bar{F}_l = \frac{S(r_1)\, a}{c} \Psi \cos\Theta\, \hat{r}_1 \tag{8}$$

Where, Θ is the cone angle, $\hat{r}_1$ is the radiation incident direction, a is the projected area, C is the speed of light, $S(r_1)$ is the laser beam intensity and Ψ is a function of infinitesimal thermo-optical properties which is given by:

$$\Psi = \left\{ 4\rho'\beta\cos^4\eta + 2(1+\rho'\beta)\left[B_f\rho'(1-\beta) + \alpha'\left(\frac{\varepsilon_f B_f - \varepsilon_b B_b}{\varepsilon_f + \varepsilon_b}\right)\right]\cos^3\eta + \left[(1+\rho'\beta)^2 + \left(B_f\rho'(1-\beta) + \alpha'\left(\frac{\varepsilon_f B_f - \varepsilon_b B_b}{\varepsilon_f + \varepsilon_b}\right)\right)^2\right]\cos^2\eta \right\}^{\frac{1}{2}}, \tag{9}$$

Where η is the radiation incident angle w.r.t. the surface normal, β is the surface specularity, ρ' is the surface reflectivity, B_f and B_b are the non-Lambartian coefficient of front and back surfaces respectively, α' is the absorption coefficient, and ε_f and ε_b are front and back surface emissivity respectively. The value of Ψ is ranged from 1 to 2; where $\Psi = 0$ for transparent surfaces, $\Psi = 2$ for perfectly reflecting surfaces and $\Psi = 1$ for opaque surfaces (i.e. the object that absorbs all of the incident radiation).
For standard atmospheric conditions, only linear mechanism of laser atmospheric interactions is considered. Atmospheric turbulences are countered by using the adaptive optics and technical capabilities of the laser system. Based on the previous postulates, the laser intensity delivered to the infinitesimal surface is determined using an analytical model in which laser intensity is proportional directly to the laser power and inversely to infinitesimal altitude and laser divergence. This laser intensity is given by [13], [14]:

$$S(r_1) = \frac{P_o}{\pi} \frac{K}{r_1^2}, \tag{10}$$

Where K represents the effect of the Earth's atmosphere on the laser beam propagation and it can be given by:

$$K = \frac{1}{\theta^2} \exp\left[-\left(\frac{\sigma_{scat}^{mol}(0)h}{\sin\phi}\left(1 - \exp -\left(\frac{w\sin\phi}{h}\right) + \sigma_{scat}^{aer}\right)\right]\right. \tag{11}$$

Where P_o is the laser initial power, θ is the beam divergence, ϕ is the elevation angle, $\sigma_{scat}^{mol}(0)$ is the molecule scattering coefficient at sea level, σ_{scat}^{aer} is the aerosols scattering coefficient and h is the sea level altitude which is considered to be 7 km. Since the most effective region on the beam propagation is considered to be the first 50 km of the Earth's atmosphere, w is chosen to be 50 km [15]. For laser beam of wavelength ($\lambda = 532$ nm), the molecular scattering coefficient at the sea level, $\sigma_{scat}^{mol}(0) = 1.7 \times 10^{-5}\mathrm{m}^{-1}$.while for clear weather condition, the aerosols scattering coefficient $\sigma_{scat}^{aer} = 0.0001\ \mathrm{m}^{-1}$ [16].
The laser force which was given by eqn. (8) can be rewritten as:

$$\bar{F}_l = \Psi \frac{\hat{r}_1}{r_1^3} \tag{12}$$

Where $\Psi_1 = \frac{P_o}{\pi c} K A \Psi \cos\Theta$. The problem admits the Jacobi integral. Assuming that $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are three unit vectors of a geocentric coordinate system xyz. In which $\hat{e}_x$ is directed parallel to the Earth equatorial plane, $\hat{e}_y$ is directed in the plane that contains the meridian of the sub-satellite point and $\hat{e}_z$ directed normal to the equatorial plane, as described by [17]. For a single laser shot, the Curl of laser force is given by:

$$\mathrm{Curl}\ \bar{F}_l = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_{lx} & F_{ly} & F_{lz} \end{vmatrix}$$

$$= \left(\frac{\partial F_{lz}}{\partial y} - \frac{\partial F_{ly}}{\partial z}\right)\hat{e}_x + \left(\frac{\partial F_{lz}}{\partial x} - \frac{\partial F_{lx}}{\partial z}\right)\hat{e}_y + \left(\frac{\partial F_{ly}}{\partial x} - \frac{\partial F_{lx}}{\partial y}\right)\hat{e}_z$$

$$= -\frac{3\Psi}{\rho^5}\left[(\rho_y\rho_z - \rho_z\rho_y)\hat{e}_x + (\rho_x\rho_z - \rho_z\rho_x)\hat{e}_y + (\rho_y\rho_x - \rho_x\rho_y)\hat{e}_z = \bar{0}\right. \tag{13}$$

Since, the laser force $\bar{F}$ meets the condition,Curl $\bar{F}_l = \bar{0}$, so it can be considered as a conservative force. Since laser force is a conservative force. So, it is possible to assign a numerical value for the potential at any point. Assuming that the potential function is $u(x, y, z)$ so that

$$\bar{F}_l = -\nabla u(r),$$

Consequently:

$$u(r) = -\int \bar{F}_l . d\bar{r} = \frac{3\Psi}{\rho} \tag{14}$$

As is clear in figure 2, the range vector " $\bar{\rho}$ " is given by:

$$\bar{\rho} = \bar{r}_1 - \bar{R}$$

Where $\bar{r}_1$ is the infinitesimal position vector and $\bar{R}$ is the geocentric station coordinate vector.
Since

$$\rho = \sqrt{\bar{\rho}.\bar{\rho}} = \sqrt{r_1^2 + R^2 - 2\, r_1\, R\cos\gamma} = r_1\sqrt{1 + \left(\frac{R}{r_1}\right)^2 - 2\frac{R}{r_1}\cos\gamma}$$

Let $\chi = \frac{R}{r_1}$, then:

$$u(r_1) = \frac{3\Psi}{r_1}\left(1 - 2\,\chi\cos\gamma + \chi^2\right)^{-\frac{1}{2}}$$

Since $R < r_1$, then $\chi < 1$ and the laser potential can be expanded in a convergent power series as it discussed for gravitational potential [17]:

$$u(r_1) = \frac{3\,\Psi}{r_1}\sum_{n=0}^{\infty}\chi^n\, P_n(\cos\gamma) \tag{15}$$

Where $P_n(x)$ are the conventional Legendre polynomials defined as:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n\left(x^2-1\right)}{dx^n} \qquad \forall\, x\in[-1,1]$$

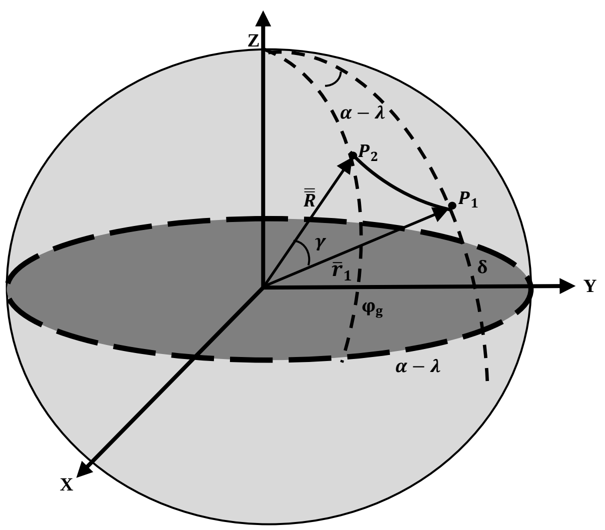

Fig. 3: Spherical trigonometry of satellite position where P_1 is the position of the infinitesimal while P_2 is the position of the ground station.

As illustrated in figure 3, using the spherical triangle ZP_1P_2,

$$\cos\gamma = \sin\varphi_g \sin\delta + \cos\varphi_g \cos\delta\cos(\alpha-\lambda)$$

Where α and δ are the infinitesimal's right ascension and declination of the satellite's position, while φ_g and λ is the ground station latitude and longitude. After simple mathematical treatments, we get:

$$u(r_1) = \frac{3\,\Psi}{r_1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{R}{r_1}\right)^n P_{nk}(\sin\delta)\,(C_{nk}\cos k\alpha + S_{nk}\sin k\alpha) \tag{16}$$

Where $P_{nk}(\sin\delta)$ is the associated Legendre polynomial. The coefficients, C_{nk} and S_{nk} are dimensionless functions of latitude and longitude of the laser firing station and are defined as:

$$C_{nk} = \frac{6}{\delta_k}\frac{(n-k)!}{(n+k)!} P_{nk}\left(Sin\,\varphi_g\right) Cos\, k\,\lambda \tag{17}$$

$$S_{nk} = \frac{6}{\delta_k}\frac{(n-k)!}{(n+k)!} P_{nk}\left(Sin\,\varphi_g\right) Sin\, k\lambda \tag{18}$$

The parameter δ_k has the values:

$$\delta_k = \begin{cases} 2 & for\ k=0 \\ 1 & for\ k>0 \end{cases} \tag{19}$$

For n =0 and k =0, eqn. (16) can be rewritten as:

$$u(r_1) = \frac{\varepsilon\,\Psi}{r_1}\Big[3 + \sum_{n=1}^{\infty}\left(\frac{R}{r_1}\right)^n C_{n0}P_n(\sin\delta) + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{R}{r_1}\right)^n P_{nk}(\sin\delta)\,(C_{nk}\cos k\alpha + S_{nk}\sin k\alpha) \tag{20}$$

As it is clear from equations (20) that the maximum contribution of laser potential is obtained for $n = 1$, then the laser potential can be reduced to the form:

$$u(r_1) = \frac{3\Psi}{r_1}\left[1 + \left(\frac{R}{r_1}\right) C_{10}P_1(\sin\delta)\right]$$

$$= \frac{3\Psi}{r_1}\left[1 + \left(\frac{R}{r_1}\right)\sin\varphi_g \sin I\sin(f+\omega)\right] \tag{21}$$

With I, f, and ω are the inclination, true anomaly and the argument of perigee of the spacecraft's orbit respectively.

2.3. Location of triangular points

The location of equilibrium points can be obtained by taking the laser potential into consideration, the new potential will be:

$$\Omega = \left[\frac{1}{2}n^2r_1^2 + \left(\frac{1}{r_1} + \frac{A_1+A}{2r_1^3}\right)\right](1-\mu) + \frac{3\Psi}{r_1}\left[1 + \left(\frac{R}{r_1}\right)\sin\varphi_g\sin I\sin(f+\omega)\right] +$$

$$+\left[\frac{1}{2}n^2r_2^2 + \left(\frac{1}{r_2} + \frac{A_2+A}{2r_2^3}\right)\right]\mu \tag{22}$$

Where the current problem admit the following Jacobian integral of motion

$$\dot{x}^2 + \dot{y}^2 = 2\,\Omega - J \tag{23}$$

Where J is the integration constant.
The equilibrium points are the solutions of $\Omega_x = \Omega_y = \Omega_{\dot{x}} = \Omega_{\dot{y}} = 0$. Based on eqs. (22) and make use of eqs. (6) and (7), the partial derivatives are given by:

$$\frac{\partial\Omega}{\partial x} = (x-\mu)W_1(r_1) + (x+1-\mu)W_2(r_2) \tag{24}$$

$$\frac{\partial\Omega}{\partial y} = y\,(W_1(r_1) + W_2(r_2)) \tag{25}$$

Where

$$W_1(r_1) = \frac{(1-\mu)}{C_1r_1^5}(r_1^5 + C_2r_1^2 - C_3r_1 - C_4) \tag{26}$$

$$W_2(r_2) = \frac{\mu}{C_1r_2^5}(r_2^5 - C_1r_2^2 - C_5) \tag{27}$$

Where

$$C_1 = \frac{1}{n^2}$$

$$C_2 = C_1\left(1 + \frac{3\Psi}{(1-\mu)}\right)$$

$$C_3 = \frac{6\,C_1\,R\,\Psi\sin\varphi_g\sin I\ \sin(f+\omega)}{(1-\mu)}$$

$$C_4 = \frac{3}{2}C_1(A_1 + A)$$

$$C_5 = \frac{3}{2}C_1(A_2 + A)$$

The triangular points are the solutions of $\Omega_x = \Omega_y = \Omega_{\dot{x}} = \Omega_{\dot{y}} = 0$ with $y \neq 0$, then eqn (15) will be

$$W_1(r_1) = W_2(r_2) = 0 \tag{28}$$

For classical problem where the primaries are neither radiating nor oblate spheroids, the triangular points are obtained by setting $r_1 = r_2 = 1$. Upon these assumptions, the coordinates of the triangular points $L_{4,5}$ are $x_{4,5} = \mu - \frac{1}{2}$ and $y_{4,5} = \pm\frac{\sqrt{3}}{2}$. In the current study we assume that the large primary sends laser radiation toward the infinitesimal body. Moreover, the three bodies are oblate spheroids. Consequently, small perturbations" ϵ_i" will cause changes of r_1 and r_2 as follows:

$$r_i = 1 + \epsilon_i \tag{29}$$

Where $|\epsilon_i| \ll 1$ and $i = 1, 2$. ϵ_1 represents the perturbation due to the oblateness of the large primary in addition to the laser radiation pressure. However, ϵ_2 represents the perturbation due to the oblateness of the smaller primary. Substituting into eqs.(6) and (7), we get

$$(1 + \epsilon_1)^2 = (x - \mu)^2 + y^2 \tag{30}$$

$$(1 + \epsilon_2)^2 = (x + 1 - \mu)^2 + y^2 \tag{31}$$

Using power series expansion to determine an approximate solution of the system (30) and (31) ignoring all the higher order terms of ϵ_i as well as the mixed terms, the coordinates of the perturbed triangular equilibrium points are:

$$x_{4,5} = \mu - \frac{1}{2} - \epsilon_1 + \epsilon_2, \tag{32}$$

$$y_{4,5} = \pm \frac{\sqrt{3}}{2}\left(1 - \frac{2}{3}(\epsilon_1 + \epsilon_2)\right), \tag{33}$$

Substituting the values of r_1, r_2, $x_{4,5}$ and $y_{4,5}$ into eqs. (28), and Using power series expansion ignoring all the higher order terms of ϵ_i as well as the mixed terms, the values of the perturbing parameters ϵ_i are:

$$\epsilon_1 = \frac{-1+C_2+C_3+C_4}{3C_2+4C_3+5C_4}, \ \epsilon_2 = \frac{-1+C_1+C_5}{3C_1+5C_5} \tag{34}$$

Substituting into eqs. (34), the perturbed location of the triangular points are

$$x_{4,5} = \mu - \frac{1}{2} - \left(\frac{-1+C_2+C_3+C_4}{3C_2+4C_3+5C_4}\right) + \left(\frac{-1+C_1+C_5}{3C_1+5C_5}\right), \tag{35}$$

$$y_{4,5} = \pm \frac{\sqrt{3}}{2}\left(1 - \frac{2}{3}\left(\frac{-1+C_2+C_3+C_4}{3C_2+4C_3+5C_4} + \frac{-1+C_1+C_5}{3C_1+5C_5}\right)\right), \tag{36}$$

3. Numerical application

Assuming, a ground station sends laser beam of 5 kW power and 0.1mrad divergence angle towards an infinitesimal body of reflecting surface. Figure 4 represents the displacement of triangular points under the effect laser radiation and oblateness. The dashed line represents the positions of L_4 and L_5 in the classical problem. However, the solid line represents the new positions of these points under the effect of both laser radiation and oblateness.

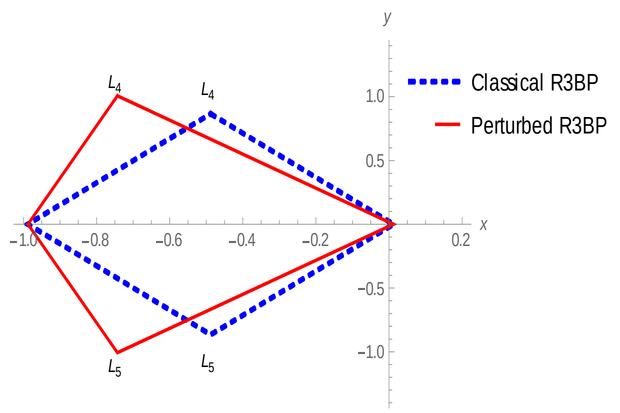

Fig. 4: Effect of Laser Radiation Pressure of 5000 KW Power on the Location of the Triangular Points.

4. Conclusion

For the planar circular restricted three-body problem (CR3BP), we treated the perturbation of the triangular points' location in presence of the effect of an artificial radiation (laser) in addition to the effect of oblatness. The numerical application shows that

1) The results are consistent with the classical problem in absence of any perturbation [2].
2) There are significant changes in the positions of L_4 and L_5 due to the effect of laser radiation pressure and oblateness.

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