Existence and stability of triangular points in the relativistic R3BP when the bigger primary is a triaxial rigid body and a source of radiation

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Abstract

We study the effect of triaxiality and radiation of the bigger primary on the positions and stability of the triangular points in the relativistic R3BP. It is found that the locations of the triangular points are affected by the relativistic terms apart the radiation force and the triaxiality of the bigger primary. It is also seen that for these points, the range of stability region increases or decreases according as without which depends upon the relativistic terms, the radiation and triaxiality coefficient is greater than or less than zero. A practical application of this model could be the study of the motion of a dust grain particle near the Sun-Earth system. A practical application of this model could be the study of the motion of a dust grain particle near the Sun-Earth system.

Keywords: Celestial Mechanics; Radiation; Triaxiality; Relativity; R3BP.

1. Introduction

The circular restricted three-body problem (CR3BP) describes the dynamics of a body having infinitesimal mass and moving in the gravitational field of two massive bodies called primaries, which revolve around their common center of mass in circular orbits on account of their mutual attraction. It is originally formulated due to the approximately circular motion of the planets around the Sun, and the small masses of asteroids and the satellites of planets compared to the planets’ masses.

The infinitesimal mass can be at rest in a rotating coordinate frame, at five equilibrium points, where the gravitational and centrifugal forces just balance each other. Three of them are the colinear points L1, L2, L3 lying on the line connecting the primaries, while the other two are the triangular points L4, L5, forming equilateral triangles with the primaries. The latters are linearly stable for the mass ratio μ of the primaries less than μ < 0.03852... (Szabóbelgy [1]). Their stability occurs although the potential energy has a maximum rather than a minimum at L4 and L5. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotated (Wintern [2]; Contopoulos [3]). The bodies in the R3BP are strictly spherical in shape, but in nature, celestial bodies are not perfect spheres. They are either oblate or triaxial.

The lack of sphericity or the oblateness of the planets causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. This motivates many investigators (SubbaRao and Sharma [4]; AbdulRahem and Singh [5]; Sharma [6]; Idrisi et al. [7]). The effect of triaxiality and radiation of the primaries on the existence and stability of libration points in the CR3BP was analyzed by e.g. El-Shaboury [8], Sharma et al. [9], [10], Khanna and Bhatnagar [11], Singh [12] to study CR3BP with oblateness or triaxiality of the bodies.

In general relativity, even writing down the equations of motion in the simplest case N=2 is difficult. Unlike in Newton’s theory, it is impossible to express the acceleration by means of the positions and velocities, in a way which would be valid within the “Exact” theory. Therefore, the approximation method is needed.

Historically, the equations of motion of the problem of N bodies considered as point masses were first obtained by generalizing the geodesic principle. By the use of this method, De sitter [13] first derived the relativistic equation of N-body problem. Some arithmetic errors occurred in these equations are reproduced in the encyclopedic paper of Kottler [14] and treatises by Chazy [15], [16], but were corrected by Eddington and Clark [17]. Brumberg [18], [19] studied the problem in more details and collected most of the important results of relativistic celestial mechanics. He has not only obtained the equations of motion for the general problem of the three bodies, but also deduced the equations of motion for the restricted problem of three bodies.

Bhatnagar and Hallan [20] studied the existence and linear stability of the triangular points L4,5 in the relativistic R3BP, they concluded that L4,5 are always unstable in the whole range 0 ≤ μ ≤ 12

in contrast to the classical R3BP where they are stable for 0 < μ < μ0, where μ is the mass ratio and μ0 = 0.03852... is the Routh’s value. Douskos and Perdios [21] examined the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan [20], they obtained a region of linear stability in the parameter space 0 ≤ μ < μ0 - 17.609

where μ0 = 0.03852... is Routh’s value.
In recent times, many perturbing forces, i.e., oblateness and radiation forces of the primaries, Coriolis and centrifugal forces, have been included in the study of the relativistic R3BP.

The locations of libration points in the relativistic R3BP, when one or more additional effects are included in the potential due to radiation pressure and the oblateness of the primaries, were studied by Abd El-Salam and Abd El-Bar [22] and Katour et al. [23]. The locations of triangular points and their linear stability when the bigger primary is radiating in the relativistic R3BP were examined by Singh and Bello [24]. The locations of triangular points and their linear stability in the presence of small perturbation given to the centrifugal force were also investigated by Singh and Bello [25].

In all the studies previously mentioned in the relativistic R3BP, no work is performed in the direction of linear stability of the triangular point in the presence of both radiation and triaxiality. Hence, the idea of the radiation pressure force together with triaxiality of bigger primary raises a curiosity in our mind to study the “stability of triangular points in the relativistic R3BP”.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect. 4; a discussion of these results is given in Sect. 5, finally Sect. 6 summarizes the conclusions and findings of our paper.

### 2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system ($\xi, \eta$) and dimensionless variables can be written as Brumberg [18] and Bhatnagar and Hallan [20]:

$$
\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)
$$

$$
\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)
$$

(1)

With

$$
W = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{R^2}\left[-\frac{3}{2}\left(\frac{1}{3}\right)\mu(1-\mu)\right](\xi^2 + \eta^2) +
\frac{1}{8}\left[\xi^2 + \eta^2 + 2(\xi\eta - \eta\xi) + (\xi^2 + \eta^2)\right]^2
+ \frac{1}{2}\left[1 + \frac{1}{p_1} + \frac{1}{p_2}\right]\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}(\xi^2 + \eta^2) -
\frac{1}{2}\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
\right]
$$

(2)

$$
\frac{n^2}{R^2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right) + \frac{1}{p_1} + \frac{1}{p_2} + 3\mu - \frac{3}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

(3)

$$
\rho_1^2 = (\xi + \mu)^2 + \eta^2
$$

$$
\rho_2^2 = (\xi - \mu)^2 + \eta^2
$$

(4)

Where $0 < \mu \leq \frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, $\rho_1$ and $\rho_2$ are distances of the infinitesimal mass from the bigger and smaller primary, respectively; $n$ is the mean motion of the primaries; $c$ is the velocity of light.

We now introduce the triaxiality factors of the bigger primary with the help of the parameter $\theta_1 \ll (i=1,2)$, where $\theta_1 = \frac{a^2 - c^2}{2R^2}$.

$$
\sigma_2 = \frac{a^2 - c^2}{2R^2}. 
$$

(Mcuskey [26]). Here $\sigma_1, \sigma_2$ characterize the triaxiality of the bigger primary with $a, b, c$ as lengths of its semi-axes and $R$ is the dimensional distance between the primaries. The radiation factor $q_1$ is given by $q_1(1-q_1)$ such that $0 \leq (1-q_1) < 1$ Radzievskii [27]. For simplicity, putting $q_1 = 1 - (1-q_1) = 1 - \delta$ where $0 \leq \delta < 1$ and neglecting second and higher powers of $\sigma_1(i=1,2)$ and $\delta$, and also their products, we take the equations of motion as:

$$
\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)
$$

$$
\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)
$$

(5)

Where $W$ is the potential-like function of the relativistic R3BP. As Katour et al. [23], we do not include the parameters $\sigma_1(i=1,2)$ and $\delta$ in the relativistic part of $W$ since the magnitude of these terms is so small due to $c^{-2}$.

$$
W = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{R^2}\left[-\frac{3}{2}\left(\frac{1}{3}\right)\mu(1-\mu)\right](\xi^2 + \eta^2) +
\frac{1}{8}\left[\xi^2 + \eta^2 + 2(\xi\eta - \eta\xi) + (\xi^2 + \eta^2)\right]^2
+ \frac{1}{2}\left[1 + \frac{1}{p_1} + \frac{1}{p_2}\right]\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}(\xi^2 + \eta^2) -
\frac{1}{2}\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
\right]
$$

(6)

$$
+ \left[4\xi + \frac{7}{2}\eta\right] + \left[4\eta + \frac{7}{2}\xi\right]
$$

$$
+ \left[\frac{1}{p_1} + \frac{1}{p_2}\right] + \left[\frac{3}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right]
$$

$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
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$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

$$
\left[\left(\frac{1}{3}\right)\mu(1-\mu)\right]^{\alpha}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)
$$

And $n$ is the perturbed mean motion of the primaries and is given by

$$
n = 1 + \frac{3}{4}(2\sigma_1 - \sigma_2) - \frac{3}{2c^2}\left[1 - \frac{1}{3}\mu(1-\mu)\right]
$$

(7)

### 3. Location of triangular points

The libration points are obtained from equation (5) after putting

$$
\dot{\xi} = \eta = \dot{\eta} = 0.
$$

These points are the solutions of the equations.
\[ \frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \]  
With \( \dot{\xi} = \dot{\eta} = 0 \).

That is

\[ \xi = \frac{-(1 - \delta)(1 - \mu)(\xi + \mu)}{\rho_1^2} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right) - \frac{3(1 - \mu)(\xi + \mu)(2\sigma_1 - \sigma_2)}{2\rho_1^2} \]

\[ + \frac{15}{2}(1 - \mu)(\xi + \mu)(\sigma_2 - \sigma_1) + \frac{1}{3} + \frac{1}{2}(\xi^2 + \eta^2) \]

\[ + \frac{1}{2} \left[ \left(1 - \frac{1}{2}\sigma_2 - \sigma_1\right) - \frac{3(1 - \mu)(\xi + \mu)}{\rho_1^2} + \frac{3(1 - \mu)(\xi + \mu)}{\rho_2^2} \right] \]

\[ = \frac{1}{2} \left[ \left(1 - \frac{1}{2}\sigma_2 - \sigma_1\right) - \frac{3(1 - \mu)(\xi + \mu)}{\rho_1^2} + \frac{3(1 - \mu)(\xi + \mu)}{\rho_2^2} \right] \]

\[ + \frac{1}{2} \left[ \left(1 - \frac{1}{2}\sigma_2 - \sigma_1\right) - \frac{3(1 - \mu)(\xi + \mu)}{\rho_1^2} + \frac{3(1 - \mu)(\xi + \mu)}{\rho_2^2} \right] \]

\[ + \frac{1}{2} \left[ \left(1 - \frac{1}{2}\sigma_2 - \sigma_1\right) - \frac{3(1 - \mu)(\xi + \mu)}{\rho_1^2} + \frac{3(1 - \mu)(\xi + \mu)}{\rho_2^2} \right] \]

\( \eta = \pm \frac{\sqrt{\eta}}{2} \left[ 1 + \frac{1}{12\mu} \left( -5 + 6\mu - 9\sigma_2 + \alpha \right) \right] \]

\[ 1 + \frac{23}{8} \left( \frac{1}{8} - \frac{1}{2\mu} \right) \frac{\sigma_2}{\sigma_1} - \frac{2}{9} \]

These points are denoted by \( L_4 \) and \( L_5 \) respectively.

### 4. Stability of triangular points

Since the nature of linear stability about the point \( L_5 \) will be similar to that about \( L_4 \), it will be sufficient to consider here the stability only near \( L_4 \).

Let \( (a, b) \) be the coordinates of the triangular point \( L_4 \).

We set \( \xi = a + \alpha, \eta = b + \beta, (\alpha, \beta) < 1 \) in the equations (5) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

\[ \frac{\partial W}{\partial \xi} \bigg|_{\xi=a+\alpha, \eta=b+\beta} = A\alpha + B\beta + C\alpha + D\beta \]

Where,

\[ A = \frac{3}{4} \left( 1 + \frac{1}{2\mu} \left( 2 - 19\mu + 9\mu^2 \right) \right) \frac{3(15\mu^2 + 19\mu - 8)}{16\mu} \frac{\sigma_1}{\sigma_2} \]

\[ B = \frac{3}{4} \left( 1 + \frac{1}{2\mu} \right) \left( 1 - \frac{2}{3\mu} \right) \frac{\sqrt{37\mu^2 - 9\mu + 8}}{16\mu} \frac{\sigma_2}{\sigma_1} \]

\[ C = \frac{\sqrt{2}}{2\mu} \left( 1 - 2\mu \right) \]

\[ D = \frac{6 - 5\mu + 5\mu^2}{2\mu^2} \]

Similarly, we obtain

\[ \frac{\partial W}{\partial \eta} \bigg|_{\xi=a+\alpha, \eta=b+\beta} = A\eta + B\beta + C\alpha + D\beta \]

Where,

\[ A_1 = \frac{3\sqrt{2}}{4} \left( 1 - 2\mu \right) \left( 1 - \frac{2}{3\mu} \right) \frac{\sqrt{37\mu^2 - 9\mu + 8}}{16\mu} \frac{\sigma_1}{\sigma_2} \]

\[ + \frac{\sqrt{37\mu^2 - 9\mu + 8}}{16\mu} \frac{\sigma_2}{\sigma_1} - \frac{\sqrt{2}}{6} \left( 1 - \mu \right) \delta, \]

Following as Singh and Bello [24], from the system (8) with \( \eta = 0 \), we have obtained the coordinates of the triangular points \( (\xi, \eta) \) as

\[ \xi = \frac{1 - 2\mu}{2} \left( 1 + \frac{5}{4\mu} \right) \left( \frac{1}{8} - \frac{1}{2\mu} \right) + \frac{1}{3\sigma_1 - \frac{3}{2}\sigma_2} \left( \frac{1}{8} + \frac{3}{8} \right) \sigma_2 + \frac{1}{3} \delta \]

(9)
\[ B_1 = \frac{9}{4} \left[ 1 - \frac{7}{6c^2} (-2 + 3\mu - 3\mu^2) \right] - \frac{3(15\mu^2 - 29\mu - 8)}{16\mu} \sigma_1 \]

\[ 3 \left( 15\mu^2 - 7\mu - 8 \right) \sigma_2 + \frac{1}{2} (1 - 3\mu) \delta. \]

\[ C_1 = \frac{1}{2c^2} \left( -4 + \mu - \mu^2 \right). \]

\[ D_1 = -\sqrt{(1 - 2\mu)} \frac{c}{2c^2}. \]

\[ \frac{d}{dt} \left[ \frac{\partial W}{\partial \tilde{\nu}} \right]_{\tilde{\nu} = a + \alpha, \tilde{\eta} = b + \beta} = A_2 \delta + B_2 \beta + C_2 \alpha + D_2 \beta \]

Where,

\[ A_2 = \frac{\sqrt{\lambda}}{2c^2} (1 - 2\mu). \]

\[ B_2 = \frac{1}{2c^2} \left( -4 + \mu - \mu^2 \right). \]

\[ C_2 = \frac{1}{4c^2} \left( 17 - 2\mu + 2\mu^2 \right). \]

\[ D_2 = -\frac{\sqrt{\lambda}}{4c^2} (1 - 2\mu). \]

\[ \frac{d}{dt} \left[ \frac{\partial W}{\partial \xi} \right]_{\xi = a + \alpha, \eta = b + \beta} = A_3 \alpha + B_3 \beta + C_3 \alpha + D_3 \beta \]

Where

\[ A_3 = \frac{1}{2c^2} \left( -6 + 5\mu + 5\mu^2 \right), \]

\[ B_3 = -\frac{\sqrt{\lambda}}{2c^2} (1 - 2\mu), \]

\[ C_3 = -\frac{\sqrt{\lambda}}{4c^2} (1 - 2\mu), \]

\[ D_3 = \frac{3(5 - 2\mu + 2\mu^2)}{4c^2}. \]

The characteristic equation of the variational equations of motion corresponding to (5) can be expressed as

\[ \lambda^4 + \frac{9}{c^2} \left[ 1 - \frac{9}{c^2} + 3\eta_1 + \frac{1}{2} (2\mu - 3) \sigma_2 \right] \lambda^2 + \left[ \frac{27}{4} \mu (1 - \mu) \right] + \frac{(108\mu^4 - 216\mu^3 + 693\mu^2 - 585\mu)}{8c^2} = 0 \]

For \( \frac{1}{c^2} \to 0 \) and when the bigger primary is non-luminous and non-triaxial (i.e. \( \sigma_1 = \sigma_2 = \delta = 0 \)), this reduces to its well-known classical restricted problem form (see e.g. Szehely, [1]):

\[ \lambda^4 + \lambda^2 + \frac{27}{4} \mu (1 - \mu) = 0. \]

The discriminant of (10) is

\[ \Delta = \left( \frac{54}{c^2} \right)^4 \left( \frac{108}{c^2} \right)^3 \left( \frac{27 + \frac{801}{4} \sigma_1 - \frac{333}{4} \sigma_2 + 65 - \frac{693}{2c^2} \mu}{c^2} \right)^2 + \left( \frac{-27 - \frac{891}{4} \sigma_1 + \frac{447}{4} \sigma_2 - 65 + \frac{585}{2c^2} \mu}{c^2} \right)^4 \mu + \frac{1}{c^2} \left( \frac{47}{2} \sigma_1 - \frac{63}{2} \sigma_2 \right)^2. \]

Its roots are

\[ \lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \]

Where

\[ b = \left( 1 - \frac{9}{c^2} \right) + 3\sigma_1 + \frac{3}{2} (2\mu - 3) \sigma_2. \]

From (11), we have

\[ \frac{d\Delta}{d\mu} = -\frac{216}{c^2} \mu^3 + \frac{324}{c^2} \mu^2 + 2 \left[ \frac{27}{4} - \frac{801}{4} \sigma_1 - \frac{333}{4} \sigma_2 + 65 - \frac{693}{2c^2} \mu \right] \mu \]

\[ \left( \frac{-27 - \frac{891}{4} \sigma_1 + \frac{447}{4} \sigma_2 - 65 + \frac{585}{2c^2} \mu}{c^2} \right) < 0 \forall \mu \in \left( 0, \frac{1}{2} \right). \]

From (13), it can be easily seen that \( \Delta \) is monotone decreasing in \( \left( 0, \frac{1}{2} \right) \).

But

\[ (\Delta)_{\mu=0} = 1 + \frac{57}{2} \sigma_1 - \frac{63}{2} \sigma_2 - \frac{18}{c^2} > 0 \]

\[ (\Delta)_{\mu=1/2} = -\frac{23}{4} \sigma_1 + \frac{57}{16} \sigma_2 - \frac{3}{2} \delta + \frac{207}{4c^2} < 0 \]

Since \( (\Delta)_{\mu=0} \) and \( (\Delta)_{\mu=1/2} \) are of opposite signs, and \( \Delta \) is monotone decreasing and continuous, there is one value of \( \mu \), e.g. \( \mu_c \) in the interval \( \left( 0, \frac{1}{2} \right) \) for which \( \Delta \) vanishes.

Solving the equation \( \Delta = 0 \), using (11), we obtain critical value of the mass parameter as
\[ \mu_c = \mu_0 - \frac{17 \sqrt{69}}{486 c^2} + \frac{1}{2} \left( \frac{19}{18} + \frac{59}{9 \sqrt{69}} \right) \sigma_1 - \frac{1}{2} \left( \frac{19}{18} + \frac{85}{9 \sqrt{69}} \right) \sigma_2 - \frac{2}{27 \sqrt{69}} \delta \]

(15)

Where \( \mu_0 = 0.03852 \ldots \) is the Routh's value.

5. Discussion

Equations (5) - (6) describe the motion of a third body under the influence of relativistic terms and triaxiality and radiation of the bigger primary. Equations (9) and (15) give respectively the positions of triangular points and critical mass parameter. Equation (16) describes the region of stability. It can be seen both positions, and critical mass depend upon relativistic terms, triaxiality and radiation factors. It may be noted here that in this problem, the triangular points no longer form equilateral triangles with the primaries as they do in the classical case. Rather, they form scalene triangles with the primaries. It can also be seen from (16) that the relativistic, radiation and triaxiality terms all reduce the size of stability region.

In the absence of radiation and triaxiality (i.e. \( \delta = \sigma_1 = \sigma_2 = 0 \)), the positions of triangular points obtained in this study correspond to those of Bhattachar and Hallan [20], Douskos and Perdios [21].

In the absence of triaxiality (i.e. \( \sigma_1 = \sigma_2 = 0 \)) the results of the present study are in accordance with those of Singh and Bello [24] when the coupling terms \( \frac{\delta}{c^2} \) neglected in their study.

In the absence of radiation and triaxiality (i.e. \( \delta = \sigma_1 = \sigma_2 = 0 \)), the stability results obtained are in agreement with those of Douskos and Perdios [21] and disagree with those of Bhattachar and Hallan [20].

In the absence of relativistic terms, the results of the present study coincide with those of Sharma et al. [10] and with those of Singh [12] when the perturbations are absent and the bigger primary is triaxial and luminous only.

6. Conclusion

By considering the bigger primary as radiating and triaxial rigid body in the relativistic R3BP, we have determined the positions of triangular points and have examined their linear stability. It is found that their positions and stability region are affected by relativistic terms, radiation and triaxiality of the bigger primary. It is also noticed that the expression for A.D.A\(_2\)C\(_2\) in Bhattachar and Hallan [20] differ from the present study when the radiation pressure and triaxiality are absent. Consequently, the characteristic equations are also different. This led them (Bhattachar and Hallan [20]) to conclude that triangular points are unstable, contrary to Douskos and Perdios [21] and our results.

References


