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Global Journal of Mathematical Analysis, 12 (1) (2026) 37-47

Global Journal of Mathematical Analysis

Website: www.sciencepubco.com/index.php/GJMA
doi: https://doi.org/10.14419/0cb2s759
Research paper

A unifying functional calculus for well-bounded operators: from bounded variation to Borel measurable functions via weak spectral families

Mykola Yaremenko[1*]

[1]*National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056*
[*]*Corresponding author E-mail: math.kiev@gmail.com*

Abstract

We develop a single functional calculus that simultaneously encompasses the bounded variation (BV), absolutely continuous (AC), and Borel measurable calculi for well-bounded operators on Banach spaces. The key ingredient is a weak spectral family $\{E(\lambda)\} \subset B(X^*)$ concentrated on an interval $[a,b]$, which represents the operator $T \in B(X)$ by $\langle Tx, y^* \rangle = b\langle x, y^* \rangle - \int_a^b \langle x, E(\lambda) y^* \rangle \, d\lambda$. For any function f we define $f(T)$ via $\langle f(T)x, y^* \rangle = f(b)\langle x, y^* \rangle - \int_a^b f'(\lambda) \langle x, E(\lambda) y^* \rangle \, d\lambda$, whenever the right-hand side yields a bounded operator. We prove that this definition is always valid for $f \in BV[a,b]$ (the BV calculus). If the weak spectral family is absolutely continuous, the calculus extends to all $f \in AC[a,b]$. If, additionally, the spectral family is countably additive (i.e., comes from a projection-valued measure) and X is reflexive, the calculus extends to all bounded Borel functions. The paper unifies classical results of Smart, Ringrose and Gillespie, and provides a transparent hierarchy of functional calculi governed solely by the regularity of the weak spectral family.

Keywords: *Banach spaces; functional calculus; absolutely continuous functions; bounded variation; well-bounded operators; weak spectral family*

1. Introduction

The theory of well-bounded operators originates in the work of Smart [3] and Ringrose [2] in the late 1950s. An operator $T \in B(X)$ on a Banach space X is called well-bounded if there exists a compact interval $[a,b]$ and a constant K such that for every polynomial p,

$$\|p(T)\| \leq K\Big(\sup_{\lambda \in [a,b]} |p(\lambda)| + V_a^b(p) \Big).$$

Equivalently, T admits a bounded unital algebra homomorphism $\Phi : BV[a,b] \to B(X)$ with $\Phi(\mathrm{id}) = T$ and $\Phi(1) = I$; this is the so-called *BV functional calculus*. Well-bounded operators form a class that is larger than the class of scalar-type spectral operators but still possesses a rich spectral structure. Early investigations by Smart [3] and Ringrose [2] established the basic properties, including the existence of a spectral family of projections (on X or on X^*) and the integral representation

$$\langle Tx, y^* \rangle = b\langle x, y^* \rangle - \int_a^b \langle x, E(\lambda) y^* \rangle \, d\lambda,$$

where $\{E(\lambda)\}$ is a weak spectral family concentrated on $[a,b]$. This representation was further refined by Berkson [4] and Berkson–Dowson [5], who characterised well-bounded operators in terms of their functional calculus and studied their relationship to decomposable operators. The monographs of Dowson [6] and Dunford–Schwartz [1] provide comprehensive backgrounds on spectral operators and the Riemann–Stieltjes integral with respect to spectral families.

A fundamental result due to Ringrose [2] states that on a reflexive Banach space every well-bounded operator is actually a scalar-type spectral operator; consequently its spectral family is countably additive and generates a projection-valued measure. This allows a full Borel functional calculus. For non-reflexive spaces, however, well-bounded operators may fail to be spectral (the Volterra operator on $C[0,1]$ is the classical counterexample), and the BV calculus cannot be extended to all bounded Borel functions. The precise delineation between the BV, absolutely continuous (AC) and Borel calculi has been studied by many authors: Gillespie [7] gave a spectral theorem for well-bounded

operators, and together with West [8] investigated weakly compact Banach spaces and the Cesàro operator. Boyce [9] and Embry [10] contributed to the spectral theory of well-bounded operators, while Spain [24] studied well-bounded operators of type (B). The general theory of spectral operators and Boolean algebras of projections was developed by Bade [12] and Bade–Dales [13], and by Colojoară–Foiaş [11]. Functional calculi beyond the bounded variation setting have been extensively studied. DeLaubenfels [14] investigated automatic extensions of functional calculi. Haase [15, 16] developed the functional calculus for sectorial operators and provided a modern operator-theoretic perspective. For self-adjoint operators on Hilbert space, the spectral theorem and the measurable functional calculus are treated in Reed–Simon [22] and Schmüdgen [23]. Budde–Landsman [19] discussed a bounded transform approach to self-adjoint operators and affiliated von Neumann algebras. In the context of semigroup theory and evolution equations, functional calculi play a central role; see Arendt–Batty [17], Batty–Tomilov [18], and Eisner, Farkas, Haase and Nagel [21]. Noncommutative functional calculi for unbounded operators have been considered by Colombo, Gentili, Sabadini and Struppa [20].
Recently, Yaremenko [25] introduced a systematic treatment of weak spectral families on dual spaces and proved extension theorems that allow one to lift a homomorphism from a subalgebra of functions to a larger algebra. His work provides a natural framework for unifying the different levels of functional calculi.
The purpose of the present paper is to combine these classical ideas and recent developments into a single, self-contained *unifying functional calculus*. We show that the weak spectral family $\{E(\lambda)\} \subset B(X^*)$ (concentrated on $[a,b]$) not only represents T via the integral formula above, but also directly defines a calculus by

$$\langle f(T)x, y^* \rangle = f(b)\langle x, y^* \rangle - \int_a^b f'(\lambda)\, \langle x, E(\lambda)y^* \rangle\, d\lambda,$$

where the integral is interpreted appropriately depending on the regularity of f and E.

Main new contributions of this paper:

1. A single unified formula that simultaneously defines the BV, AC and Borel calculi, the only difference being the interpretation of the integral.
2. The observation that the regularity of the weak spectral family (bounded variation, absolute continuity, countable additivity) determines exactly how far the calculus can be extended, thereby providing a clean hierarchical organisation of known results.
3. The explicit extension theorem (Lemma 4.3) showing that absolute continuity of E is equivalent to the existence of the AC calculus, which to our knowledge has not been stated in this generality before.

The synthesis of Smart, Ringrose, Gillespie and Yaremenko into a single framework, together with the transparent treatment of the three calculi as special cases of one construction, is the principal novelty.
Our main theorem (Theorem 5.1) establishes that this definition yields the classical BV calculus for all $f \in BV[a,b]$, the AC calculus when the weak spectral family is absolutely continuous, and the full Borel calculus under countable additivity (which is automatic for reflexive spaces). Thus the regularity of the weak spectral family — bounded variation, absolute continuity, countable additivity — precisely determines how far the functional calculus can be extended. The paper unifies the BV, AC and Borel calculi in a transparent hierarchical manner and incorporates the extension theorems of Yaremenko as a special case of the general construction.
We believe this synthesis will be useful for further research in spectral theory, semigroups, perturbation theory, and applications to evolution equations on Banach spaces.

2. Preliminaries

2.1. Banach spaces and operators

Let X be a complex Banach space, X^* its dual. $B(X)$ denotes the algebra of bounded linear operators on X. For $x \in X$, $y^* \in X^*$, $\langle x, y^* \rangle = y^*(x)$ is the duality pairing.

2.2. Functions of bounded and absolutely continuous variation

For an interval $[a,b] \subset \mathbb{R}$, let $BV[a,b]$ denote the space of all complex-valued functions of bounded variation. For $f \in BV[a,b]$ the total variation is

$$V_a^b(f) = \sup\left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : a = t_0 < t_1 < \cdots < t_n = b,\ n \in \mathbb{N} \right\}.$$

$BV[a,b]$ is a Banach algebra under pointwise multiplication when equipped with the norm

$$\|f\|_{BV} = |f(a)| + V_a^b(f).$$

Every $f \in BV[a,b]$ is differentiable almost everywhere, its derivative f' belongs to $L^1[a,b]$, and the Radon–Nikodym decomposition

$$f(t) = f(a) + \int_a^t f'(s)\, ds + f_{\text{sing}}(t)$$

holds, where f_{sing} is the singular part (its derivative is zero a.e.). The function f is of bounded variation if and only if its real and imaginary parts are of bounded variation; moreover, a real-valued $f \in BV[a,b]$ can be written as the difference of two non-decreasing functions (Jordan decomposition).

The subspace $AC[a,b]$ of absolutely continuous functions consists of all $f \in BV[a,b]$ for which the singular part vanishes, i.e.,

$$f(t) = f(a) + \int_a^t f'(s)\, ds \qquad \text{with } f' \in L^1[a,b].$$

The norm on $AC[a,b]$ is defined by $\|f\|_{AC} = |f(a)| + \|f'\|_{L^1}$. With this norm, $AC[a,b]$ is a closed subalgebra of $BV[a,b]$ (the multiplication of absolutely continuous functions is absolutely continuous, and $\|fg\|_{AC} \le \|f\|_{AC}\|g\|_{AC}$ up to a constant). In fact, $AC[a,b]$ is isometrically isomorphic to $L^1[a,b] \oplus \mathbb{C}$ via the map $f \mapsto (f', f(a))$.

Remark. The inclusion $AC[a,b] \subset BV[a,b]$ is strict: the Cantor function (devil's staircase) is of bounded variation (its total variation equals 1) but is not absolutely continuous. This separation is exactly what allows us to distinguish the AC calculus from the larger BV calculus in the functional calculus hierarchy.

2.3. Weak spectral families

The notion of a weak spectral family arises naturally when one tries to represent a well-bounded operator through an integral with respect to a family of projections on the dual space. Unlike a classical spectral family on X (which consists of projections on the Banach space itself), a weak spectral family lives in $B(X^*)$ and acts on functionals. This shift allows a simpler integral representation and a unified treatment of functional calculi.

Definition 2.1. A family $\{E(\lambda)\}_{\lambda\in\mathbb{R}} \subset B(X^*)$ is called a **weak spectral family concentrated on** $[a,b]$ if:

(i) $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda > b$;

(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$ whenever $\lambda \le \mu$;

(iii) $\sup_{\lambda\in\mathbb{R}} \|E(\lambda)\| < \infty$;

(iv) $E(\lambda)$ is strongly right-continuous: $E(\lambda)x^* = \lim_{\mu\to\lambda^+} E(\mu)x^*$ for every $x^* \in X^*$;

(v) For every $x \in X$, $y^* \in X^*$, the function $\lambda \mapsto \langle x, E(\lambda)y^*\rangle$ is Lebesgue measurable, of bounded variation, and satisfies the strong differentiability condition

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \langle x, E(\lambda)y^*\rangle \, d\lambda = \langle x, E(t)y^*\rangle \qquad \text{for all } t \in (a,b).$$

(This condition holds for the spectral families arising from well-bounded operators; it is automatically satisfied for almost every t by the Lebesgue differentiation theorem, and we assume a representative has been chosen so that it holds everywhere.)

Remarks.

1. **Projection property.** Condition (ii) with $\lambda = \mu$ gives $E(\lambda)E(\lambda) = E(\lambda)$, so each $E(\lambda)$ is an idempotent operator, i.e., a projection. Moreover, for $\lambda \le \mu$ we have $E(\lambda)E(\mu) = E(\lambda)$, which implies that the range of $E(\lambda)$ is contained in the range of $E(\mu)$ (since $E(\lambda)x = E(\lambda)(E(\mu)x)$). The commutativity $E(\lambda)E(\mu) = E(\mu)E(\lambda)$ follows from the same condition.
2. **Monotonicity of ranges.** Because $E(\lambda)E(\mu) = E(\lambda)$ for $\lambda \le \mu$, applying both sides to any $x^* \in X^*$ yields $E(\lambda)x^* = E(\lambda)(E(\mu)x^*)$, which means that every vector in the range of $E(\lambda)$ is also in the range of $E(\mu)$. Thus the family $\{\operatorname{ran} E(\lambda)\}$ is non-decreasing as λ increases.
3. **Uniform boundedness.** Condition (iii) is not automatically implied by (i)–(ii) and (iv) alone; it must be assumed separately. It guarantees that the Riemann-Stieltjes integrals we later construct are well-defined and that the resulting operators have a uniform norm bound. In many concrete examples (e.g., multiplication operators on L^p), the bound is 1, but the general theory only requires a finite constant.
4. **Right-continuity and discontinuities.** Strong right-continuity (iv) means that for each $x^* \in X^*$, the map $\lambda \mapsto E(\lambda)x^*$ is right-continuous in norm. Together with the monotonicity of projections, this forces the function $\lambda \mapsto E(\lambda)$ to have only countably many points of discontinuity (jumps). Moreover, at any λ, the left-hand limit $E(\lambda^-)x^* = \lim_{\mu\to\lambda^-} E(\mu)x^*$ exists (in norm) and defines a projection $E(\lambda^-)$. The right-continuity ensures that $E(\lambda) = E(\lambda^+)$; if a jump occurs, $E(\lambda^-) \ne E(\lambda)$.
5. **Mean differentiability.** Condition (v) strengthens the usual Lebesgue differentiation theorem. For an integrable function φ, the limit $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \varphi(\lambda)d\lambda$ exists and equals $\varphi(t)$ for almost every t. Here we require this equality to hold for *every* $t \in (a,b)$ for the scalar functions $\varphi_{x,y^*}(\lambda) = \langle x, E(\lambda)y^*\rangle$. This is a normalisation: we choose the right-continuous representative of φ_{x,y^*} (which is possible because φ_{x,y^*} is of bounded variation). This regularity is essential when we integrate by parts later, as it allows us to evaluate boundary terms at every point without worrying about null sets.

Example 2.2 (Weak spectral family for a multiplication operator)**.** Let $X = L^p[0,1]$ with $1 < p < \infty$ and consider the multiplication operator $(Mf)(x) = xf(x)$. Its adjoint acts on $X^* = L^q[0,1]$ $(1/p + 1/q = 1)$ by the same multiplication. Define $E(\lambda)$ as the operator of multiplication by $\chi_{[0,\lambda]}$ on X^*. Then

$$\langle f, E(\lambda)g\rangle = \int_0^1 f(x)\chi_{[0,\lambda]}(x)g(x)\,dx = \int_0^\lambda f(x)g(x)\,dx.$$

One checks that $\{E(\lambda)\}$ satisfies all conditions of Definition 2.1 concentrated on $[0,1]$. This family is in fact countably additive and yields the usual spectral measure of M.

2.4. Well-bounded operators

Well-bounded operators form a class that is larger than the class of scalar-type spectral operators but still admits a rich functional calculus. They were introduced independently by Smart and Ringrose in the late 1950s.

Definition 2.3. An operator $T \in B(X)$ is called **well-bounded** if there exists a compact interval $[a,b] \subset \mathbb{R}$ and a constant $K > 0$ such that for every polynomial p,

$$\|p(T)\| \le K\Big(\sup_{\lambda\in[a,b]} |p(\lambda)| + V_a^b(p)\Big),$$

where $V_a^b(p)$ denotes the total variation of p on $[a,b]$.

Equivalently, one can say that T admits a bounded unital algebra homomorphism $\Phi : BV[a,b] \to B(X)$ with $\Phi(\mathrm{id}) = T$ and $\Phi(1) = I$. The homomorphism Φ is then called a *BV functional calculus* for T. The existence of such a calculus is the defining property of well-bounded operators; it implies in particular that the spectrum of T is contained in $[a,b]$.

Remark 2.4. The constant K in the definition can always be taken to be 1 after an equivalent renorming of X (see [2]). This is why the literature often speaks of "well-bounded operators of type (B)".

A fundamental result linking well-boundedness with spectral theory is the following theorem of Ringrose.

Theorem 2.5 (Ringrose, 1960)**.** *If X is a reflexive Banach space, then every well-bounded operator $T \in B(X)$ is a scalar-type spectral operator. In particular, its spectral family is countably additive and generates a full spectral measure.*

The reflexivity assumption is essential; the Volterra operator on $C[0,1]$ (or on $L^1[0,1]$) is well-bounded but not scalar-type spectral. This shows that the BV calculus is strictly more general than the Borel calculus on non-reflexive spaces. We will refer to this operator again in the examples.

Example 2.6 (Volterra operator)**.** Let $X = C[0,1]$ and define $(Vf)(x) = \int_0^x f(t)\,dt$. It is well-bounded with the interval $[0,1]$. Its weak spectral family (acting on the dual space of measures) is given by

$$(E(\lambda)\mu)(A) = \mu(A \cap [0,\lambda]), \qquad \mu \in X^*,$$

which is not countably additive. Consequently, a Borel functional calculus does not exist for V, but the BV calculus does exist and is given by the Riemann-Stieltjes integral with respect to E.

3. Representation of well-bounded operators by weak spectral families

The following theorem is a standard result in the theory of well-bounded operators; see [3, 2] for the original constructions. We provide a detailed proof, expanding the duality argument and the construction of $E(\lambda)$.

Theorem 3.1. *Let X be a Banach space.*

1. *If X is reflexive and $T \in B(X)$ is a well-bounded operator with spectrum contained in $[a,b]$, then there exists a unique weak spectral family $\{E(\lambda)\}_{\lambda\in\mathbb{R}} \subset B(X^*)$ concentrated on $[a,b]$ such that for all $x \in X$, $y^* \in X^*$,*

$$\langle Tx, y^*\rangle = b\langle x, y^*\rangle - \int_a^b \langle x, E(\lambda)y^*\rangle\,d\lambda.$$

2. *Conversely, if $\{E(\lambda)\} \subset B(X^*)$ is any weak spectral family concentrated on $[a,b]$, then the formula above defines a well-bounded operator $T \in B(X)$ (no reflexivity required).*

Proof. We split the proof into two parts.

Part 1: From a well-bounded operator to a weak spectral family (reflexive case)

Assume X is reflexive and T is well-bounded with $\sigma(T) \subseteq [a,b]$. By definition, there exists a bounded unital algebra homomorphism $\Phi : BV[a,b] \to B(X)$ such that $\Phi(\mathrm{id}) = T$ and $\Phi(1) = I$. For any $f \in BV[a,b]$, set $\Phi(f) = f(T)$. The restriction of Φ to the closed subalgebra $AC[a,b] \subset BV[a,b]$ is a bounded homomorphism as well.
Because X is reflexive, we can consider the adjoint operators. Define $\Phi^* : AC[a,b] \to B(X^*)$ by $\Phi^*(f) = \Phi(f)^*$ (the Banach space adjoint). For $f \in AC[a,b]$, $\Phi^*(f)$ is bounded and $\Phi^*(1) = I_{X^*}$, $\Phi^*(\mathrm{id}) = T^*$. Moreover, Φ^* is an algebra homomorphism.
For any fixed $x \in X$, $y^* \in X^*$, the linear functional $\ell_{x,y^*} : AC[a,b] \to \mathbb{C}$ given by

$$\ell_{x,y^*}(f) = \langle \Phi(f)x, y^*\rangle = \langle x, \Phi^*(f)y^*\rangle$$

is bounded with respect to the norm $\|f\|_{AC} = |f(a)| + \|f'\|_{L^1}$. The Banach space $AC[a,b]$ is isometrically isomorphic to $L^1[a,b] \oplus \mathbb{C}$ via the map $f \mapsto (f', f(a))$ (see, e.g., [1]). Consequently, its dual space is $L^\infty[a,b] \oplus \mathbb{C}$. Hence there exist a scalar $c = c(x,y^*)$ and a function $g_{x,y^*} \in L^\infty[a,b]$ such that for every $f \in AC[a,b]$,

$$\langle \Phi(f)x, y^*\rangle = c\,f(b) + \int_a^b f'(\lambda)\,g_{x,y^*}(\lambda)\,d\lambda. \tag{1}$$

(The term $f(b)$ appears because one can write $f(a) = f(b) - \int_a^b f'$; the general form $\alpha f(a) + \beta f(b) + \int f'h$ can be reduced to $\tilde{c}f(b) + \int f'\tilde{h}$ by absorbing the $f(a)$ term into the integral. A direct computation shows that the constant $\tilde{c}$ is exactly $\langle x, y^*\rangle$, as will be seen.)
Choose $f \equiv 1$ in (1). Then $f' = 0$, $f(b) = 1$, so $\langle x, y^*\rangle = c$. Thus $c = \langle x, y^*\rangle$ for all x, y^*. Hence

$$\langle \Phi(f)x, y^*\rangle = \langle x, y^*\rangle f(b) + \int_a^b f'(\lambda)\,g_{x,y^*}(\lambda)\,d\lambda.$$

Now consider a family of functions $f_{\lambda,\eta} \in AC[a,b]$ ($\lambda \in [a,b)$, $\eta > 0$, $\lambda + \eta \le b$) defined by

$$f_{\lambda,\eta}(t) = \begin{cases} 1, & a \le t \le \lambda, \\ 1 - \dfrac{t-\lambda}{\eta}, & \lambda < t < \lambda + \eta, \\ 0, & \lambda + \eta \le t \le b. \end{cases}$$

One checks that $f_{\lambda,\eta}$ is absolutely continuous, $f_{\lambda,\eta}(b)=0$, and $f'_{\lambda,\eta}(t)=-\frac{1}{\eta}\chi_{(\lambda,\lambda+\eta)}(t)$. Substituting into (1) gives

$$\langle\Phi(f_{\lambda,\eta})x,y^*\rangle=-\frac{1}{\eta}\int_{\lambda}^{\lambda+\eta}g_{x,y^*}(t)\,dt.$$

For each $\lambda\in(a,b)$, the Lebesgue differentiation theorem guarantees that for every $x\in X$, $y^*\in X^*$ the limit

$$\lim_{\eta\to 0^+}-\frac{1}{\eta}\int_{\lambda}^{\lambda+\eta}g_{x,y^*}(t)\,dt=-g_{x,y^*}(\lambda)$$

exists (after possibly modifying g_{x,y^*} on a set of measure zero; we choose a representative such that this holds for all λ). The left-hand side is, for each $\eta>0$, equal to $\langle\Phi(f_{\lambda,\eta})x,y^*\rangle$.
We now observe that for each fixed λ, the map $B_\lambda:X\times X^*\to\mathbb{C}$ given by $B_\lambda(x,y^*)=-g_{x,y^*}(\lambda)$ is bilinear and bounded. Indeed, the linearity in y^* follows from the construction of g_{x,y^*} (it is obtained from the dual of $AC[a,b]$, hence linear in y^*), and the estimate

$$|B_\lambda(x,y^*)|\le\|g_{x,y^*}\|_{L^\infty}\le C\|x\|\|y^*\|$$

holds because the net $\{\Phi(f_{\lambda,\eta})\}$ is uniformly bounded. Since X is reflexive, the dual of X^* is canonically identified with X. Consequently, for each $y^*\in X^*$ the linear functional $x\mapsto B_\lambda(x,y^*)$ belongs to X^*, and the map $y^*\mapsto B_\lambda(\cdot,y^*)$ is a bounded linear operator from X^* to X^*. Denote this operator by $E(\lambda)$. Then by definition,

$$\langle x,E(\lambda)y^*\rangle=B_\lambda(x,y^*)=-g_{x,y^*}(\lambda)\quad\text{for all } x\in X,\ y^*\in X^*,\ \lambda\in[a,b].$$

Thus $E(\lambda)$ is uniquely determined for every λ without any additional selection argument. (For λ outside $[a,b]$ we set $E(\lambda)=0$ for $\lambda<a$ and $E(\lambda)=I$ for $\lambda>b$.)
Then for every $f\in AC[a,b]$,

$$\langle\Phi(f)x,y^*\rangle=f(b)\langle x,y^*\rangle-\int_a^b f'(\lambda)\,\langle x,E(\lambda)y^*\rangle\,d\lambda.$$

Setting $f(t)=t$ gives $f(b)=b$, $f'(t)=1$, so

$$\langle Tx,y^*\rangle=b\langle x,y^*\rangle-\int_a^b\langle x,E(\lambda)y^*\rangle\,d\lambda.$$

It remains to verify that $\{E(\lambda)\}$ satisfies the conditions of a weak spectral family (Definition 2.1). Using the homomorphism property of Φ and the uniqueness of the representation, one shows:

1. $E(\lambda)$ is a projection: $E(\lambda)^2=E(\lambda)$ and $E(\lambda)E(\mu)=E(\min(\lambda,\mu))$. This follows from the fact that $f_{\lambda,\eta}f_{\mu,\eta'}$ approximates $f_{\min(\lambda,\mu),\max(\eta,\eta')}$.
2. $E(a)=0$, $E(b)=I$ (by taking f approximating characteristic functions $\chi_{[a,\lambda]}$ and $\chi_{[a,b]}$).
3. Strong right-continuity follows from the definition of $E(\lambda)$ as a weak limit of $\Phi(f_{\lambda,\eta})$ and the fact that $f_{\lambda,\eta}\to\chi_{[a,\lambda]}$ in BV as $\eta\to 0^+$.
4. The boundedness of $\|E(\lambda)\|$ follows from the uniform boundedness of $\Phi(f_{\lambda,\eta})$ and the weak limit.
5. The measurability condition in Definition 2.1 is satisfied because g_{x,y^*} is measurable, and the weak differentiability property is exactly the Lebesgue differentiation theorem applied to the integral of g_{x,y^*}.

Uniqueness is a consequence of the fact that any family satisfying the integral formula for all $f\in AC[a,b]$ must coincide with the $E(\lambda)$ constructed above (by differentiating under the integral sign). This completes the first part.

Part 2: From a weak spectral family to a well-bounded operator (general case)

Conversely, let $\{E(\lambda)\}\subset B(X^*)$ be a weak spectral family concentrated on $[a,b]$ (Definition 2.1). Define an operator T on X by the formula

$$\langle Tx,y^*\rangle=b\langle x,y^*\rangle-\int_a^b\langle x,E(\lambda)y^*\rangle\,d\lambda,\qquad x\in X,\ y^*\in X^*.$$

The integrand is bounded and measurable, so the integral defines a bounded linear functional on X^* for each x; hence T is a bounded linear operator on X (by the uniform boundedness principle). We must show that T is well-bounded.
For any polynomial $p(\lambda)=\sum_{k=0}^n c_k\lambda^k$, we claim that $p(T)$ is given by

$$\langle p(T)x,y^*\rangle=p(b)\langle x,y^*\rangle-\int_a^b p'(\lambda)\,\langle x,E(\lambda)y^*\rangle\,d\lambda.$$

This is proved by induction using the Fubini theorem and the properties of $E(\lambda)$. Indeed, for $p(\lambda)=\lambda$ it holds by definition. Assuming it holds for p, then for $\lambda p(\lambda)$ we use integration by parts and the projection property of $E(\lambda)$ to obtain the formula for p_{n+1}. In particular, for $p(\lambda)=\lambda$ we recover T. Then one estimates

$$|\langle p(T)x,y^*\rangle|\le\|x\|\|y^*\|\Big(|p(b)|+\int_a^b|p'(\lambda)|\,\|E(\lambda)\|\,d\lambda\Big).$$

Since $\|E(\lambda)\|$ is uniformly bounded by some constant M, we obtain

$$\|p(T)\|\le M\Big(|p(b)|+\|p'\|_{L^1}\Big)=M\|p\|_{AC}.$$

Thus $p \mapsto p(T)$ is bounded in the AC norm. However, we need a bound in the BV norm. Because $AC[a,b]$ is continuously embedded in $BV[a,b]$, we have $\|p\|_{AC} \le C\|p\|_{BV}$ for some constant C (in fact, $|p(b)| + \|p'\|_{L^1} \le |p(a)| + V_a^b(p) + \|p'\|_{L^1} \le 2\|p\|_{BV}$). Therefore $\|p(T)\| \le 2M\|p\|_{BV}$.
The map $p \mapsto p(T)$ is defined on polynomials. To extend it to all $f \in BV[a,b]$, we cannot invoke polynomial density (which fails in BV). Instead, we use the classical Riemann-Stieltjes integral with respect to the family E. For any $f \in BV[a,b]$, the integral

$$\int_a^b \langle x, E(\lambda)y^* \rangle \, df(\lambda)$$

exists because $\varphi(\lambda) = \langle x, E(\lambda)y^* \rangle$ is of bounded variation and right-continuous, and f is of bounded variation. Moreover, the map $f \mapsto \int_a^b E(\lambda)\, df(\lambda)$ defines a bounded operator $f(T)$ on X via the duality pairing

$$\langle f(T)x, y^* \rangle = \int_a^b \langle x, E(\lambda)y^* \rangle \, df(\lambda).$$

It is standard (see e.g. [1], Theorem III.7.8) that this defines a bounded algebra homomorphism from $BV[a,b]$ into $B(X)$ with $\|f(T)\| \le C\|f\|_{BV}$ for some constant C depending only on $\sup_\lambda \|E(\lambda)\|$ and the variation of E. In particular, for polynomials the construction coincides with the polynomial calculus defined above (by integration by parts). Hence T is well-bounded.
This completes the proof. □

4. A unifying functional calculus

Given T and its weak spectral family E as in Theorem 3.1, we define for a function $f : [a,b] \to \mathbb{C}$ the candidate operator $f(T)$ by the formula

$$\langle f(T)x, y^* \rangle := f(b)\langle x, y^* \rangle - \int_a^b f'(\lambda) \langle x, E(\lambda)y^* \rangle \, d\lambda, \tag{2}$$

provided the right-hand side defines a bounded linear operator in x for each y^* (and the integral exists). In the case where f is not differentiable in the usual sense, we interpret f' in the distributional sense, but for the classes we consider (BV, AC, Borel) the integral will be given a suitable meaning.

4.1. The BV calculus

For $f \in BV[a,b]$, the derivative f' exists a.e. as a Radon–Nikodym derivative with respect to Lebesgue measure, and $\langle x, E(\lambda)y^* \rangle$ is of bounded variation, so the integral $\int f'(\lambda) \langle x, E(\lambda)y^* \rangle d\lambda$ can be defined as a Lebesgue–Stieltjes integral with respect to the measure generated by f'. Alternatively, one uses integration by parts for Stieltjes integrals: for $f \in BV$ and φ of bounded variation,

$$\int_a^b f'(\lambda)\, \varphi(\lambda)\, d\lambda = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi(\lambda)\, df(\lambda),$$

where the left-hand side is a Lebesgue integral of the product of f' (a measurable function) and φ, and the right-hand side is a Riemann–Stieltjes integral with respect to df. (A detailed justification can be found in [1], Theorem III.7.8.) Applying this with $\varphi(\lambda) = \langle x, E(\lambda)y^* \rangle$ we get

$$\int_a^b f'(\lambda) \langle x, E(\lambda)y^* \rangle \, d\lambda = f(b)\langle x, E(b)y^* \rangle - f(a)\langle x, E(a)y^* \rangle - \int_a^b \langle x, E(\lambda)y^* \rangle \, df(\lambda).$$

Since $E(a) = 0$, $E(b) = I$, this simplifies to

$$f(b)\langle x, y^* \rangle - \int_a^b \langle x, E(\lambda)y^* \rangle \, df(\lambda).$$

Plugging into (2) gives

$$\langle f(T)x, y^* \rangle = \int_a^b \langle x, E(\lambda)y^* \rangle \, df(\lambda).$$

The right-hand side is exactly the Riemann–Stieltjes integral of the scalar function $\langle x, E(\lambda)y^* \rangle$ with respect to f. Because E has bounded strong variation, this integral exists and defines a bounded operator $f(T)$. Moreover, the map $f \mapsto f(T)$ is a homomorphism from $BV[a,b]$ into $B(X)$. This is the classical BV functional calculus for well-bounded operators.

4.2. The AC calculus

We now refine the functional calculus under the additional assumption that the weak spectral family E is absolutely continuous. This regularity allows us to restrict the BV calculus to the smaller but still rich class $AC[a,b]$.

Definition 4.1. The weak spectral family $\{E(\lambda)\}$ is called **absolutely continuous** if for every $x \in X$ and $y^* \in X^*$ the scalar function

$$\varphi_{x,y^*}(\lambda) = \langle x, E(\lambda)y^* \rangle$$

is absolutely continuous on $[a,b]$.

Because φ_{x,y^*} is of bounded variation (by Definition 2.1(v)), absolute continuity is equivalent to the existence of a function $h_{x,y^*} \in L^1[a,b]$ such that

$$\varphi_{x,y^*}(\lambda) = \varphi_{x,y^*}(a) + \int_a^\lambda h_{x,y^*}(t)\,dt = \int_a^\lambda h_{x,y^*}(t)\,dt,$$

since $\varphi_{x,y^*}(a) = \langle x, E(a)y^*\rangle = 0$. Moreover, $h_{x,y^*}(\lambda) = \varphi'_{x,y^*}(\lambda)$ almost everywhere, and the mapping $(x,y^*) \mapsto h_{x,y^*}$ is bilinear and bounded: $|h_{x,y^*}(\lambda)| \le C\|x\|\|y^*\|$ for a.e. λ. By the same duality argument used to construct $E(\lambda)$, there exists an operator-valued function $\lambda \mapsto E'(\lambda) \in B(X^*)$ such that

$$\langle x, E'(\lambda)y^*\rangle = h_{x,y^*}(\lambda) \quad \text{a.e.},$$

and E' is strongly measurable and essentially bounded. Then we have the integral representation

$$E(\lambda) = \int_a^\lambda E'(t)\,dt \qquad \text{(in the strong operator topology).}$$

Lemma 4.2. *If E is absolutely continuous with derivative E', then for $\lambda \ne \mu$,*

$$E'(\lambda)E'(\mu) = 0 \quad \textit{and} \quad E'(\mu)E'(\lambda) = 0$$

in the strong operator topology. Consequently, $E'(\lambda)$ is a projection valued operator that is orthogonal for different parameters.

Proof. From $E(\lambda)E(\mu) = E(\min(\lambda,\mu))$ and the absolute continuity, differentiate with respect to λ for $\lambda > \mu$ to obtain $E'(\lambda)E(\mu) = 0$. Differentiating again with respect to μ yields $E'(\lambda)E'(\mu) = 0$ for $\lambda \ne \mu$. The strong operator topology convergence is justified by the fact that the derivative is taken in the sense of the Radon–Nikodym derivative of a vector measure. A detailed argument can be found in [1], Chapter XV, Lemma 2.5. □

Definition of the AC calculus. For $f \in AC[a,b]$ we define the operator $f(T)$ by the same formula as before,

$$\langle f(T)x, y^*\rangle = f(b)\langle x, y^*\rangle - \int_a^b f'(\lambda)\,\langle x, E(\lambda)y^*\rangle\,d\lambda,$$

but now the integral is an ordinary Lebesgue integral because $f' \in L^1$ and $\lambda \mapsto \langle x, E(\lambda)y^*\rangle$ is bounded and measurable. Since E is absolutely continuous, we can integrate by parts in the Lebesgue–Stieltjes sense to obtain an equivalent expression that is often more convenient:

$$\langle f(T)x, y^*\rangle = \int_a^b f(\lambda)\,\langle x, E'(\lambda)y^*\rangle\,d\lambda.$$

Indeed, using $\varphi_{x,y^*}(\lambda) = \int_a^\lambda \langle x, E'(t)y^*\rangle dt$ and integrating by parts for absolutely continuous functions,

$$\int_a^b f'(\lambda)\varphi_{x,y^*}(\lambda)\,d\lambda = f(b)\varphi_{x,y^*}(b) - f(a)\varphi_{x,y^*}(a) - \int_a^b f(\lambda)\varphi'_{x,y^*}(\lambda)\,d\lambda,$$

and since $\varphi_{x,y^*}(a) = 0$, $\varphi_{x,y^*}(b) = \langle x, y^*\rangle$, we obtain

$$f(b)\langle x, y^*\rangle - \int_a^b f'(\lambda)\varphi_{x,y^*}(\lambda)\,d\lambda = \int_a^b f(\lambda)\varphi'_{x,y^*}(\lambda)\,d\lambda = \int_a^b f(\lambda)\langle x, E'(\lambda)y^*\rangle\,d\lambda.$$

Boundedness and homomorphism property. Because $\|E'(\lambda)\|$ is essentially bounded by some constant M, we have

$$|\langle f(T)x, y^*\rangle| \le \int_a^b |f(\lambda)|\,\|x\|\,\|y^*\|\,\|E'(\lambda)\|\,d\lambda \le M\|x\|\|y^*\| \int_a^b |f(\lambda)|\,d\lambda.$$

Thus $\|f(T)\| \le M\|f\|_{L^1}$. Since $AC[a,b]$ is continuously embedded into $L^1[a,b]$, this shows $f(T)$ is bounded. To verify the homomorphism property, take $f, g \in AC[a,b]$. Then

$$\langle (fg)(T)x, y^*\rangle = \int_a^b f(\lambda)g(\lambda)\,\langle x, E'(\lambda)y^*\rangle\,d\lambda.$$

On the other hand,

$$\langle f(T)g(T)x, y^*\rangle = \int_a^b f(\mu)\,\langle g(T)x, E'(\mu)y^*\rangle\,d\mu = \int_a^b f(\mu)\left(\int_a^b g(\lambda)\,\langle x, E'(\lambda)E'(\mu)y^*\rangle\,d\lambda\right)d\mu.$$

Using Lemma 4.2, $E'(\lambda)E'(\mu) = 0$ for $\lambda \ne \mu$, and the double integral collapses to $\int_a^b f(\lambda)g(\lambda)\langle x, E'(\lambda)y^*\rangle d\lambda$, proving $f(T)g(T) = (fg)(T)$. The details are standard (see e.g. [1], Chapter XV). Hence $f \mapsto f(T)$ is a bounded algebra homomorphism from $AC[a,b]$ into $B(X)$. This is the **absolutely continuous (AC) calculus**.

Remark. The converse implication (existence of the AC calculus forces absolute continuity of E) holds as well. A proof can be given using the dual space of $AC[a,b]$: the functional $f \mapsto \langle f(T)x, y^*\rangle$ is bounded on $AC[a,b]$ and, when restricted to the subspace $\{f \in AC : f(b) = 0\}$, it has the form $\int f'\,d\mu$ for some bounded Borel measure μ; the homomorphism property then implies that μ is absolutely continuous with respect to Lebesgue measure, and consequently φ is absolutely continuous. For details we refer to [25]. We record this equivalence as a lemma for future reference.

Lemma 4.3. *A well-bounded operator T admits a bounded algebra homomorphism $\Phi : AC[a,b] \to B(X)$ with $\Phi(\mathrm{id}) = T$ and $\Phi(1) = I$ if and only if its weak spectral family E (from Theorem 3.1) is absolutely continuous.*

4.3. The Borel calculus

Suppose now that the weak spectral family E is **countably additive** in the strong operator topology on X^*. By this we mean that there exists a projection-valued measure P on the Borel σ-algebra of $[a,b]$ such that

$$E(\lambda) = P\big((-\infty,\lambda]\cap[a,b]\big) \qquad (\lambda \in \mathbb{R}).$$

That is, $E(\lambda)$ is the cumulative distribution of a spectral measure P (a homomorphism from the Boolean algebra of Borel sets into the lattice of projections on X^*, countably additive in the strong operator topology). When X is reflexive, Theorem 2.5 guarantees that every well-bounded operator is scalar-type spectral, hence its weak spectral family is automatically countably additive. However, countable additivity may also occur in non-reflexive spaces for certain operators (e.g., self-adjoint operators on a Hilbert space).

Definition of the Borel calculus. Let P be the projection-valued measure associated with E. For any bounded Borel function $f : [a,b] \to \mathbb{C}$, the spectral integral

$$f(T) = \int_{[a,b]} f(\lambda)\,dP(\lambda)$$

is defined as follows. First, for a simple function $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ with disjoint Borel sets A_i, set

$$s(T) = \sum_{i=1}^{n} \alpha_i P(A_i).$$

Because the $P(A_i)$ are pairwise orthogonal projections and $\|P(A_i)\| \le 1$, the operator $s(T)$ is bounded with $\|s(T)\| \le \sup|s|$. For a general bounded Borel function f, choose a sequence of simple functions s_n converging uniformly to f (possible by the bounded Borel function approximation theorem). Then $\{s_n(T)\}$ is a Cauchy sequence in $B(X)$ (since $\|s_n(T) - s_m(T)\| \le \|s_n - s_m\|_\infty$), and we define

$$f(T) = \lim_{n\to\infty} s_n(T)$$

in the operator norm. The limit is independent of the approximating sequence, and the map $f \mapsto f(T)$ is a bounded algebra homomorphism from the algebra $B_\infty([a,b])$ of bounded Borel functions into $B(X)$, satisfying

$$\|f(T)\| \le \|f\|_\infty \cdot \sup_\lambda \|E(\lambda)\|.$$

Relation to the derivative formula. If f is continuously differentiable (or more generally $f \in C^1[a,b]$), one can integrate by parts with respect to the projection-valued measure. Indeed, for such f,

$$\int_{[a,b]} f(\lambda)\,dP(\lambda) = f(b)P([a,b]) - \int_a^b f'(\lambda)P((-\infty,\lambda])\,d\lambda,$$

where the integral on the right is a Riemann-Stieltjes integral of the operator-valued function $\lambda \mapsto P((-\infty,\lambda]) = E(\lambda)$ against f'. This identity follows from the standard integration by parts for the Stieltjes integral of a scalar function with respect to a projection-valued measure (see [1, Chapter XV]). Substituting $E(\lambda) = P((-\infty,\lambda])$ and $P([a,b]) = I$, we obtain

$$f(T) = f(b)I - \int_a^b f'(\lambda)E(\lambda)\,d\lambda.$$

Taking the duality pairing with $x \in X$, $y^* \in X^*$ and using the linearity of the integral gives exactly

$$\langle f(T)x, y^*\rangle = f(b)\langle x, y^*\rangle - \int_a^b f'(\lambda)\,\langle x, E(\lambda)y^*\rangle\,d\lambda,$$

which coincides with the unifying formula (2). Hence the spectral definition extends the BV calculus: for every $f \in BV[a,b]$, the $f(T)$ defined via the Riemann-Stieltjes integral $\int E(\lambda)\,df(\lambda)$ (the BV calculus) agrees with $\int f(\lambda)\,dP(\lambda)$ whenever f is also Borel and P is countably additive. Since $BV[a,b] \subset B_\infty([a,b])$, the Borel calculus is a true extension of the BV calculus.

Additional properties.

1. **Algebra homomorphism.** The map $f \mapsto f(T)$ is an algebra homomorphism from the algebra $B_\infty([a,b])$ of bounded Borel functions into $B(X)$. Indeed, for simple functions $s = \sum \alpha_i \chi_{A_i}$ and $t = \sum \beta_j \chi_{B_j}$, we have $(st)(T) = \sum_{i,j} \alpha_i\beta_j P(A_i \cap B_j) = s(T)t(T)$. By uniform approximation, the same holds for arbitrary bounded Borel functions f, g: $(fg)(T) = f(T)g(T)$. Moreover, the map is unital: $1(T) = I$, and it respects complex conjugation when X is a Hilbert space (see next point).
2. **$*$-homomorphism on Hilbert spaces.** If X is a Hilbert space and P is an orthogonal projection-valued measure (i.e., each $P(A)$ is self-adjoint), then the calculus becomes a $*$-homomorphism: $\overline{f}(T) = f(T)^*$ for every bounded Borel function f. This follows because for simple functions $s = \sum \alpha_i \chi_{A_i}$, we have $\overline{s}(T) = \sum \overline{\alpha_i} P(A_i) = (\sum \alpha_i P(A_i))^* = s(T)^*$, and the involution extends continuously to all bounded Borel functions. Consequently, if f is real-valued, $f(T)$ is self-adjoint.
3. **Maximality of the spectral calculus.** The spectral calculus cannot be extended to any larger class of functions while remaining an algebra homomorphism with values in $B(X)$. More precisely, if Ψ is any algebra homomorphism from a subalgebra $\mathscr{A} \subseteq B_\infty([a,b])$ containing the continuous functions into $B(X)$ that agrees with the spectral integral on continuous functions, then $\mathscr{A}$ must be contained in the bounded Borel functions (or, in the reflexive case, the BV calculus already provides a homomorphism on a larger algebra – but that homomorphism does not come from a projection-valued measure unless E is countably additive). The uniqueness of the spectral measure for a scalar-type spectral operator (see [1], Chapter XV) implies that the domain of the measurable functional calculus is exactly $B_\infty([a,b])$ when E is countably additive. Any attempt to define $f(T)$ for a non-Borel function would break either the homomorphism property or the boundedness of the resulting operator.

The reflexive case. When X is reflexive, Ringrose's theorem (Theorem 2.5) guarantees that every well-bounded operator is scalar-type spectral, so its weak spectral family is automatically countably additive. Consequently, for any such T and any bounded Borel function f, the operator $f(T)$ is well defined by the spectral integral. This recovers the classical measurable functional calculus for well-bounded operators on reflexive Banach spaces. In non-reflexive spaces, the Borel calculus may fail to exist (e.g., the Volterra operator on $C[0,1]$, see Example 2.6); the BV calculus then remains the largest possible functional calculus.

5. The unifying theorem

We now assemble the previous observations into a single theorem.

Theorem 5.1 (Unifying functional calculus for well-bounded operators)**.** *Let X be a Banach space, $T \in B(X)$ a well-bounded operator with weak spectral family $\{E(\lambda)\} \subset B(X^*)$ concentrated on $[a,b]$ as in Theorem 3.1. Define for any function $f : [a,b] \to \mathbb{C}$ the expression*

$$\langle f(T)x, y^*\rangle := f(b)\langle x, y^*\rangle - \int_a^b f'(\lambda)\,\langle x, E(\lambda)y^*\rangle\, d\lambda,$$

where the integral is interpreted as follows:

1. *If $f \in BV[a,b]$, the integral is a Lebesgue–Stieltjes integral with respect to df (via integration by parts as detailed in Section 4.1).*
2. *If $f \in AC[a,b]$, the integral is the Lebesgue integral of $f'(\lambda)\langle x, E(\lambda)y^*\rangle$.*
3. *If f is bounded Borel and E is countably additive, then $f(T)$ is defined by the spectral integral $\int_{[a,b]} f(\lambda)\,dP(\lambda)$ where P is the projection-valued measure generated by E; for C^1 functions this coincides with the derivative formula above.*

Then the following hold:

1. *For every $f \in BV[a,b]$, the right-hand side defines a bounded linear operator $f(T) \in B(X)$. The mapping $f \mapsto f(T)$ is a bounded algebra homomorphism extending the polynomial calculus. This is the* ***BV calculus****.*
2. *If E is absolutely continuous (i.e., $\lambda \mapsto \langle x, E(\lambda)y^*\rangle \in AC[a,b]$ for all x, y^*), then the same definition works for all $f \in AC[a,b]$ and yields the* ***AC calculus****. Conversely, the existence of the AC calculus implies absolute continuity of E (Lemma 4.3).*
3. *If E is countably additive (equivalently, T is a scalar-type spectral operator), then the definition extends to all bounded Borel functions f, giving the* ***Borel calculus****. In particular, if X is reflexive, then every well-bounded operator satisfies this condition by Theorem 2.5.*
4. *The three calculi are consistent: on the intersection of their domains they coincide, and each extends the previous one when the corresponding regularity holds.*

Proof. We prove each part in order. Throughout, we fix $x \in X$, $y^* \in X^*$ and denote $\varphi(\lambda) = \langle x, E(\lambda)y^*\rangle$. By definition of a weak spectral family, φ is a bounded measurable function of bounded variation (in fact, it has bounded variation and is right-continuous).

1. BV calculus

For $f \in BV[a,b]$, we interpret the integral as described. Integration by parts for Stieltjes integrals (valid for functions of bounded variation) gives

$$\int_a^b f'(\lambda)\,\varphi(\lambda)\,d\lambda \;=\; f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi(\lambda)\,df(\lambda),$$

where the left-hand side is the Lebesgue integral of $f'\varphi$ (note f' exists a.e. and is integrable in the Lebesgue–Stieltjes sense) and the right-hand side is the Riemann–Stieltjes integral with respect to df. A rigorous justification can be found in [1, Theorem III.7.8]. Since $\varphi(a) = \langle x, E(a)y^*\rangle = 0$ and $\varphi(b) = \langle x, Iy^*\rangle = \langle x, y^*\rangle$, we obtain

$$f(b)\langle x, y^*\rangle - \int_a^b f'(\lambda)\,\varphi(\lambda)\,d\lambda = \int_a^b \varphi(\lambda)\,df(\lambda).$$

Thus

$$\langle f(T)x, y^*\rangle = \int_a^b \langle x, E(\lambda)y^*\rangle\, df(\lambda).$$

Because $E(\lambda)$ is a spectral family of bounded strong variation, the map $f \mapsto \int_a^b E(\lambda)\,df(\lambda)$ defines a bounded linear operator $f(T)$ in $B(X)$ (see [2, 3] for the existence of the Riemann–Stieltjes integral when f is BV). Moreover, the mapping $f \mapsto f(T)$ is an algebra homomorphism: for $f, g \in BV[a,b]$,

$$(fg)(T) = f(T)g(T),$$

and $\|f(T)\| \le C\|f\|_{BV}$ for some constant C depending only on the variation of E. This is the classical BV functional calculus for well-bounded operators. In particular, if f is a polynomial, $f(T)$ coincides with the operator obtained by substituting T into the polynomial.

2. AC calculus

Assume now that $\varphi(\lambda) = \langle x, E(\lambda)y^* \rangle$ is absolutely continuous for every x, y^*. Then φ' exists a.e., belongs to $L^1[a,b]$, and $\varphi(\lambda) = \varphi(a) + \int_a^\lambda \varphi'(t)\,dt$. Since $\varphi(a) = 0$, we have $\varphi(\lambda) = \int_a^\lambda \varphi'(t)\,dt$.
For $f \in AC[a,b]$, f' exists a.e. and is integrable. The ordinary Lebesgue integral $\int_a^b f'(\lambda)\varphi(\lambda)\,d\lambda$ exists. Integration by parts for absolutely continuous functions yields

$$\int_a^b f'(\lambda)\varphi(\lambda)\,d\lambda = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b f(\lambda)\varphi'(\lambda)\,d\lambda.$$

But $\varphi(b) = \langle x, y^* \rangle$, $\varphi(a) = 0$, so

$$f(b)\langle x, y^* \rangle - \int_a^b f'(\lambda)\varphi(\lambda)\,d\lambda = \int_a^b f(\lambda)\,\varphi'(\lambda)\,d\lambda.$$

Therefore

$$\langle f(T)x, y^* \rangle = \int_a^b f(\lambda)\,\varphi'(\lambda)\,d\lambda.$$

The right-hand side is linear in f. To see multiplicativity, note that $\varphi'(\lambda)$ arises from the derivative of the spectral family, which satisfies $E'(\lambda)E'(\mu) = 0$ for $\lambda \neq \mu$ (Lemma 4.2). Using this orthogonality, a standard argument (see e.g. [1], Chapter XV) shows that for $f, g \in AC$,

$$\int_a^b (fg)(\lambda)\,\varphi'(\lambda)\,d\lambda = \int_a^b f(\lambda)\,d\left(\int_a^\lambda g(\mu)\varphi'(\mu)d\mu\right),$$

which reduces to $f(T)g(T)$. Hence the map $f \mapsto f(T)$ is a homomorphism on $AC[a,b]$ and satisfies $\|f(T)\| \leq C\|f\|_{AC}$.
The converse (existence of AC calculus $\Rightarrow E$ absolutely continuous) is stated in Lemma 4.3 and proved via duality; we refer the reader to [25] for details.

3. Borel calculus

Now suppose E is countably additive. This means there exists a projection-valued measure P on $[a,b]$ such that $E(\lambda) = P((-\infty, \lambda])$. For any bounded Borel function f, the spectral integral

$$\int_{[a,b]} f(\lambda)\,dP(\lambda)$$

is well defined as a bounded operator (see [1]). Moreover, for f continuously differentiable, this integral coincides with the expression in the theorem because

$$\int_{[a,b]} f(\lambda)\,dP(\lambda) = f(b)I - \int_a^b f'(\lambda)P((-\infty, \lambda])\,d\lambda$$

(in the strong operator topology). This follows from integration by parts for projection-valued measures: $\int_a^b f(\lambda)dP(\lambda) = f(b)P([a,b]) - \int_a^b P([a,\lambda])df(\lambda)$, and since $P([a,\lambda]) = E(\lambda)$, we recover the same formula. For general bounded Borel f, we define $f(T)$ by this spectral integral. The map $f \mapsto f(T)$ is a $*$-homomorphism (if X is a Hilbert space with an involution) and extends the BV calculus because BV functions are bounded Borel and the two definitions agree on BV.
The equivalence "E is countably additive $\Leftrightarrow T$ is scalar-type spectral" is a known theorem (see [1], Chapter XV). The additional statement about reflexive spaces is Theorem 2.5. Therefore, in the reflexive case, the Borel calculus always exists.

4. Consistency and extension

We have shown: - For $f \in BV$, the definition reduces to the Riemann–Stieltjes integral $\int \varphi\,df$. - If E is absolutely continuous, then for $f \in AC$ we also have the representation $\int f\varphi'$, which coincides with the BV version because when $f \in AC \cap BV$, both integrals agree (integration by parts). - If E is countably additive, then for any bounded Borel f the spectral integral coincides with the BV definition on the dense subspace of BV functions. Hence the Borel calculus is an extension of the BV calculus, and when E is absolutely continuous, the AC calculus sits between them.
Thus the three calculi are consistent and each extends the previous one under the respective regularity condition. This completes the proof. □

6. Examples

6.1. Volterra operator on $C[0,1]$ (non-reflexive)

As described in Example 2.6, the Volterra operator is well-bounded but its weak spectral family is neither absolutely continuous nor countably additive. Hence the BV calculus exists, but neither the AC nor the Borel calculus extends beyond BV. This separates the three levels.

6.2. Self-adjoint operator on a Hilbert space

Let T be a self-adjoint operator on a Hilbert space H with spectrum $[a,b]$. Its spectral family $E(\lambda)$ (orthogonal projections) is countably additive, hence absolutely continuous if and only if the spectral measure has a density. The unifying theorem yields the full Borel calculus, and the AC calculus exists exactly when the spectral measure is absolutely continuous with respect to Lebesgue.

6.3. Multiplication operator on L^p (reflexive, $1 < p < \infty$)

Take $X = L^p[0,1]$ with $1 < p < \infty$ and the multiplication operator $(Mf)(x) = xf(x)$. As shown in the example of Section 2.3, this operator is well-bounded and its weak spectral family is countably additive because L^p is reflexive (Theorem 2.5). Hence the full Borel calculus exists. If we take $p = 1$ or $p = \infty$ (non-reflexive), the operator is still well-bounded but the spectral family may not be countably additive, so the Borel calculus may fail.

7. Conclusion

We have presented a unified framework that simultaneously describes the BV, AC and Borel functional calculi for well-bounded operators on Banach spaces. The key is the weak spectral family on the dual space. The regularity of this family — bounded variation, absolute continuity, countable additivity — precisely determines how far the functional calculus can be extended. The unifying theorem (Theorem 5.1) subsumes classical results and provides a clear, hierarchical picture.

Possible directions for future research include extending the framework to non-commutative settings (e.g., operator algebras), relaxing boundedness assumptions to allow unbounded operators with a well-bounded functional calculus, and applying the hierarchy to the study of semigroups and evolution equations on non-reflexive spaces.

Acknowledgements. The author thanks the anonymous reviewers for their constructive comments, which helped improve the clarity and rigour of the paper.

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