



Existence Results for Sequential Fractional Differential Equations with Advanced Arguments

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Abstract

The existence and uniqueness results for the sequential fractional differential equations involving Caputo differential operator with advanced arguments and boundary conditions are investigated in this paper. The uniqueness of the solution is established using the classical contraction mapping principle and the existence results are derived via D.O'Regan's fixed point theorem. An example is presented to illustrate our findings.

Keywords: Advanced arguments; boundary value problem; contraction mapping; fixed point theorem; sequential fractional differential equations

1. Introduction

Recently, the fractional order differential equations have emerged as the foundational resources for researchers to model the complicated problems, which are of significant scientific interest. In particular, the fractional differential equations with advanced argument have been a topic of active research with applications in the fields of physics, astronomy, engineering, etc. For relevant research on this results, we refer the readers to see [1, 2, 3, 4, 5, 6, 7, 8]. Significantly, sequential fractional differential equations is more effective in the field of modelling, therefore researches have shown increasing interest in this domain see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. In [23] authors Rizqan and Dhaigude investigated existence results for nonlinear fractional differential equations with advanced argument and integral boundary conditions.

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t), x(\varphi(t))), \\ x(0) = \lambda \int_0^T x(s) ds + r \end{cases}$$

here the authors adopted the monotone iterative technique to obtain the existence and uniqueness results and Banach fixed point theorem to prove the uniqueness results.

Further in [24] Murad et al. discussed the existence and uniqueness using Banach fixed point theorem and Schauder fixed point theorem for the given boundary value problem

$$\begin{cases} {}^C D^{\alpha} \rho(s) = \phi(s, \rho(s), \int_0^s \kappa(s, r, \rho(r)) dr, s \in \mathbb{J} = [0, S], 2 < \alpha \leq 3, \\ \beta \rho(0) + \gamma \rho(S) = \delta, \\ \lambda \rho'(0) + \nu \rho'(S) = \mu, \\ i \rho''(0) + j \rho''(S) = \ell, \end{cases}$$

which is the Caputo fractional derivative involved in the fractional integro-differential equations with some boundary conditions.

In this paper, we investigate the following boundary value problem for the sequential fractional differential equations with advanced argument:

$$\begin{cases} {}^C D_{1-}^{\gamma} \left\{ {}^C D_{0+}^{\delta} [g(z) - y(z, g(z))] \right\} = f(z, g(z), g(\theta(z))), z \in J = [0, 1], \\ a_1 g(0) + a_2 g(1) = a_3 \\ b_1 g'(0) + b_2 g'(1) = b_3 \\ \alpha_1 g''(0) + \alpha_2 g''(1) = \alpha_3 \end{cases} \quad (1)$$

where ${}^C D^\gamma, {}^C D^\delta$ denote the Caputo fractional derivative of order γ and δ respectively, $0 < \delta \leq 1, 2 < \gamma \leq 3, a_1, a_2, a_3, b_1, b_2, b_3, \alpha_1, \alpha_2$ and α_3 are constants and $f(z, g(z), g(\theta(z))) \in C(J \times \mathbb{R}^2, \mathbb{R}),$

$\theta \in C(J, J), z \leq \theta(z), y : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ are given continuous functions with $y(0, g(0)) = 0$ and $\Lambda = \{(z, w) : 0 \leq w \leq z \leq 1\}.$

The sequential fractional differential equations with advanced arguments studied in this paper have meaningful applications in several scientific disciplines. In control theory, systems with predictive feedback naturally give rise to advanced arguments, where the control input depends on anticipated future states. In biological modeling, anticipatory behavior of species or neurons can be described through advanced functional dependencies. In viscoelastic materials, the sequential Caputo operator captures layered memory effects that single-operator models fail to represent accurately. In economics and finance, sequential fractional models with advanced arguments describe the behavior of agents who make decisions based on future expectations. The existence and uniqueness results established in this paper provide a rigorous mathematical foundation for these applied models.

To the best of our knowledge, the existence and uniqueness of solutions for sequential fractional differential equations involving the Caputo differential operator with advanced arguments under boundary conditions has not yet been studied in the literature. Motivated by this observation, in the present paper we investigate this class of problems. The uniqueness of the solution is established by using the classical Banach contraction mapping principle, while the existence result is derived by applying D. O'Regan's fixed point theorem. The advanced argument $z \leq \theta(z)$ introduces anticipatory dependence into the sequential Caputo framework. Unlike single-operator models, the sequential Caputo framework captures layered memory effects while admitting standard integer-order boundary conditions. This combination makes the analysis more challenging and the results more general compared to existing works in the literature.

The remainder of this paper is organized as follows. Section 2 reviews the fundamental definitions and theorems necessary for developing the main results. Section 3 presents the existence results concerning the sequential fractional differential equations. In Section 4, we provide a concrete example that demonstrates the practical significance of our findings.

2. Preliminaries

This section recalls some basic definitions and an auxiliary lemma from fractional calculus that are needed in the sequel.

The space $(G, \|\cdot\|)$ is a Banach space if $G = \{g(v) \mid g(v) \in C([0, 1], \mathbb{R})\}$ is the space with the norm $\|g\| = \sup\{|g(v)|, v \in [0, 1]\}.$

Definition 1. [25] *The right and left Riemann-Liouville fractional integrals of order ρ for a continuous function $h \in L_1[0, 1]$ are respectively, defined as*

$$I_{1-}^\rho h(v) = \frac{1}{\Gamma(\rho)} \int_t^1 \frac{h(u)}{(u-v)^{1-\rho}} du,$$

$$I_{0+}^\rho h(v) = \frac{1}{\Gamma(\rho)} \int_0^t \frac{h(u)}{(v-u)^{1-\rho}} du,$$

where $\rho > 0, \Gamma(\rho)$ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $\mathbb{R}^+.$

Definition 2. [25] *The right and the left Caputo fractional derivative of order ρ of a function $h \in AC^n[0, 1]$ are respectively, defined as*

$${}^C D_{1-}^\rho h(v) = (-1)^n I_{1-}^{n-\rho} h^{(n)}(v),$$

$${}^C D_{0+}^\rho h(v) = \frac{1}{\Gamma(n-\rho)} \int_0^v \frac{h^{(n)}(u)}{(v-u)^{\rho+1-n}} du = I_{0+}^{n-\rho} h^{(n)}(v),$$

where $n = [\rho] + 1, v > 0, n - 1 < \rho < n, [\rho]$ represents the integer part of the real number $\rho.$

Definition 3. [25] *If $\rho > 0,$ let $h, {}^C D_{0+}^\rho h(t), {}^C D_{1-}^\rho h(t) \in L^1[0, 1],$ then*

$$I_{0+}^\rho {}^C D_{0+}^\rho h(v) = h(v) + c_0 + c_1 v + c_2 v^2 + \dots + c_{n-1} v^{n-1},$$

$$I_{1-}^\rho {}^C D_{1-}^\rho h(v) = h(v) + d_0 + d_1(1-v) + d_2(1-v)^2 + \dots + d_{n-1}(1-v)^{n-1},$$

where $c_i, d_i \in \mathbb{R}, i = 0, 1, \dots, n - 1 (n = [\rho] + 1).$

Lemma 1. *Let $Y \in C([0, 1], \mathbb{R}), \mu \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $Y(0) = 0,$ then the linear fractional differential system's solution enhanced with boundary conditions*

$$\begin{cases} {}^C D_{1-}^\gamma \{ {}^C D_{0+}^\delta [g(z) - Y(z)] \} = \mu(z), z \in [0, 1], 0 < \delta \leq 1, 2 < \gamma \leq 3, \\ a_1 g(0) + a_2 g(1) = a_3, \\ b_1 g'(0) + b_2 g'(1) = b_3, \\ \alpha_1 g''(0) + \alpha_2 g''(1) = \alpha_3, \end{cases}$$

is equivalent to the fractional integral equation

$$\begin{aligned}
 g(z) &= \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma} \mu(w) dw \\
 &\quad - \frac{z^{\delta}}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma} \mu(1) dw \right. \\
 &\quad \left. + \delta \Lambda_1 + Y''(1) - \frac{\alpha_3}{\alpha_2} \right] + \frac{z^{\delta+1}}{\delta+1} \Lambda_1 \\
 &\quad - \frac{1}{a_1+a_2} \left\{ \frac{a_2}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma} \mu(1) dw + a_2 Y(1) \right. \\
 &\quad \left. - a_3 - \frac{a_2}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma} \mu(1) dw \right. \right. \\
 &\quad \left. \left. + \delta \Lambda_1 + Y''(1) - \frac{\alpha_3}{\alpha_2} \right] + \frac{a_2}{(\delta+1)} \Lambda_1 \right\} + Y(z)
 \end{aligned} \tag{2}$$

Proof. Applying the fractional integral operator I_{0+}^{δ} to both sides and using Definition 3, we obtain

$$\begin{aligned}
 g(z) &= I_{0+}^{\delta} (I_{1-}^{\gamma} \mu(z) + c_0 + c_1 z) + c_2 + Y(z) \\
 &= I_{0+}^{\delta} I_{1-}^{\gamma} \mu(z) + \frac{z^{\delta}}{\Gamma(\delta+1)} c_0 + \frac{z^{\delta+1}}{\Gamma(\delta+2)} c_1 + c_2 + Y(z).
 \end{aligned} \tag{3}$$

Using the conditions $a_1 g(0) + a_2 g(1) = a_3$, we get

$$a_2 I_{0+}^{\delta} I_{1-}^{\gamma} \mu(1) + \frac{a_2 c_0}{\Gamma(\delta+1)} + \frac{a_2 c_1}{\Gamma(\delta+2)} + (a_1 + a_2) c_2 + a_2 Y(1) = a_3. \tag{4}$$

Applying the boundary conditions $b_1 g'(0) + b_2 g'(1) = b_3$, we obtain

$$b_2 \left[I_{0+}^{\delta-1} I_{1-}^{\gamma} \mu(1) + \frac{\delta}{\Gamma(\delta+1)} c_0 + \frac{(\delta+1)}{\Gamma(\delta+2)} c_1 + Y'(1) \right] = b_3. \tag{5}$$

Making use of the boundary condition $\alpha_1 g''(0) + \alpha_2 g''(1) = \alpha_3$, this yields

$$\alpha_2 \left[I_{0+}^{\delta-2} I_{1-}^{\gamma} \mu(1) + \frac{\delta(\delta-1)}{\Gamma(\delta+1)} c_0 + \frac{(\delta+1)\delta}{\Gamma(\delta+2)} c_1 + Y''(1) \right] = \alpha_3. \tag{6}$$

Solving the equation (5) and (6), it follows that

$$c_1 = \frac{\Gamma(\delta+2)}{(\delta+1)} \left\{ (\delta-1) \left[I_{0+}^{\delta-1} I_{1-}^{\gamma} \mu(1) + Y'(1) - \frac{b_3}{b_2} \right] - I_{0+}^{\delta-2} I_{1-}^{\gamma} \mu(1) - Y''(1) + \frac{\alpha_3}{\alpha_2} \right\}$$

Now substituting the value of c_1 in (6), we get

$$c_0 = \frac{\Gamma(\delta+1)}{\delta(\delta-1)} \left[-\delta \Lambda_1 - I_{0+}^{\delta-2} I_{1-}^{\gamma} \mu(1) - Y''(1) + \frac{\alpha_3}{\alpha_2} \right]$$

Finally applying c_0 and c_1 values in (4), we deduce that

$$\begin{aligned}
 c_2 &= \frac{1}{a_1+a_2} \left\{ a_3 - a_2 I_{0+}^{\delta} I_{1-}^{\gamma} \mu(1) - a_2 Y(1) \right. \\
 &\quad \left. - \frac{a_2}{\delta(\delta-1)} \left[-\delta \Lambda_1 - I_{0+}^{\delta-2} I_{1-}^{\gamma} \mu(1) - Y''(1) + \frac{\alpha_3}{\alpha_2} \right] - \frac{a_2}{(\delta+1)} \Lambda_1 \right\}
 \end{aligned}$$

$$\text{where } \Lambda_1 = (\delta-1) \left[I_{0+}^{\delta-1} I_{1-}^{\gamma} \mu(1) + Y'(1) - \frac{b_3}{b_2} \right] - I_{0+}^{\delta-2} I_{1-}^{\gamma} \mu(1) - Y''(1) + \frac{\alpha_3}{\alpha_2}.$$

Using c_0, c_1 and c_2 values in (3)

$$\begin{aligned}
 g(z) &= \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma} \mu(w) dw \\
 &\quad - \frac{z^{\delta}}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma} \mu(1) dw \right. \\
 &\quad \left. + \delta \Lambda_1 + Y''(1) - \frac{\alpha_3}{\alpha_2} \right] + \frac{z^{\delta+1}}{\delta+1} \Lambda_1 \\
 &\quad - \frac{1}{a_1+a_2} \left\{ \frac{a_2}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma} \mu(1) dw + a_2 Y(1) \right. \\
 &\quad \left. - a_3 - \frac{a_2}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma} \mu(1) dw \right. \right. \\
 &\quad \left. \left. + \delta \Lambda_1 + Y''(1) - \frac{\alpha_3}{\alpha_2} \right] + \frac{a_2}{(\delta+1)} \Lambda_1 \right\} + Y(z)
 \end{aligned}$$

The converse follows by direct verification and is omitted. \square

Lemma 2. [26] Let G be an open set in a closed, convex set C of a Banach space E . Assume $0 \in G$. Also assume that $T(\bar{G})$ is bounded and that $T : \bar{G} \rightarrow C$ is given by $T = T_1 + T_2$, in which $T_1 : \bar{G} \rightarrow E$ is continuous and completely continuous and $T_2 : \bar{G} \rightarrow E$ is a nonlinear contraction.

Then, either

(C1) T has a fixed point $g \in \bar{G}$; or

(C2) there exist a point $g \in \partial G$ and $\varepsilon \in (0, 1)$ with $g = \varepsilon T(g)$, where \bar{G} and ∂G respectively, represent the closure and boundary of G .

Consider $\lambda_r = \{g \in C([0, 1], \mathbb{R}) : \|g\| < r\}$, and indicate the highest number by

$$B_r = \max\{|f(z, g, g(\theta))| : (z, g, g(\theta)) \in [0, 1] \times [-r, r] \times [-r, r]\}.$$

Following Lemma 1, we express problem (2) as a fixed point problem $g = Fg$, where $F : X \rightarrow X$ is determined by

$$\begin{aligned} Fg(z) = & \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma f(w, g(w), g(\theta(w))) dw + y(z, g(z)) \\ & - \frac{z^\delta}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right. \\ & + \delta\Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \left. \right] + \frac{z^{\delta+1}}{(\delta+1)} \Lambda_1 \\ & - \frac{a_2}{(a_1+a_2)} \left\{ \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right. \\ & + y(1, g(1)) - \frac{a_3}{a_2} \\ & - \frac{1}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right. \\ & \left. \left. + \delta\Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \right] + \frac{\Lambda_1}{(\delta+1)} \right\} \end{aligned} \tag{7}$$

Let us define $F_i, i = 1, 2 : X \rightarrow X$ by

$$\begin{aligned} F_1g(z) = & \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma f(w, g(w), g(\theta(w))) dw \\ & - \frac{z^\delta}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right] \\ & - \frac{a_2}{(a_1+a_2)} \left\{ \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right. \\ & \left. - \frac{1}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma f(1, g(1), g(\theta(1))) dw \right] \right\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} F_2g(z) = & y(z, g(z)) - \frac{z^\delta}{\delta(\delta-1)} \left[\delta\Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \right] \\ & + \frac{z^{\delta+1}}{(\delta+1)} \Lambda_1 - \frac{a_2}{(a_1+a_2)} \left\{ \left[y(1, g(1)) - \frac{a_3}{a_2} \right] \right. \\ & \left. - \frac{1}{\delta(\delta-1)} \left[\delta\Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \right] + \frac{\Lambda_1}{(\delta+1)} \right\} \end{aligned} \tag{9}$$

Clearly,

$$Fg(z) = F_1g(z) + F_2g(z), \quad z \in [0, 1]. \tag{10}$$

3. Existence Results

The main existence results are established under the following assumptions.

(A1) $|f(z, c(z), c(\theta(z))) - f(z, d(z), d(\theta(z)))| \leq M|c(z) - d(z)| + N|c(\theta(z)) - d(\theta(z))|, \forall z \in J, c, d \in \mathbb{R}$;

where M and N are nonnegative constants.

(A2) y and y'' achieving the condition

$$|y(c) - y(d)| \leq \hbar_1 |c - d|, \quad \hbar_1 < \lambda_2^{-1}, \quad c, d \in C([0, 1], \mathbb{R}), \quad \hbar_1 > 0,$$

$$|y''(c) - y''(d)| \leq \hbar_2 |c - d|, \quad y \in C^3([0, 1]), \quad \hbar_2 > 0.$$

(A3) $\omega := (\lambda_1(M+N) + \lambda_2\hbar_1 + \lambda_3\hbar_2) < 1$.

(A4) there exists a nonnegative function $k \in C([0, 1], \mathbb{R})$ and a nondecreasing function $\varphi : [0, \infty) \rightarrow (0, \infty)$

$$|f(z, g, g(\theta))| \leq k(z)\varphi(\|g\|) \text{ for any } (z, x, x(\theta)) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$$

(A5) $\sup_{r \in (0, \infty)} \frac{r}{\lambda_1 \|k\| \varphi(r)} > \frac{1}{1 - \hbar_1 \lambda_2 - \hbar_2 \lambda_3}$, where

$$\begin{aligned} \lambda_1 &= \left(\frac{a_1}{a_1 + a_2} \right) \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad + \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] \int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ \lambda_2 &= \left(\frac{a_1}{a_1 + a_2} \right) \text{ and } \lambda_3 = \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] \end{aligned}$$

Theorem 1. Let $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $y : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ represents a continuous function. Suppose that (A1)-(A3) holds, then there exists a unique solution for the boundary value problem (1).

Proof. For $c, d \in X$ and for all $z \in [0, 1]$, based on the assumptions (A1) and (A2), it follows that

$$\begin{aligned} \|Fc - Fd\| &\leq \sup_{z \in [0, 1]} \left\{ \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma |f(w, c(w), c(\theta(w))) - f(w, d(w), d(\theta(w)))| dw \right. \\ &\quad + |y(z, c(z)) - y(z, d(z))| + \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] \\ &\quad \frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma |f(1, c(1), c(\theta(1))) - f(1, d(1), d(\theta(1)))| dw \\ &\quad + \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] |y''(1, c(1)) - y''(1, d(1))| \\ &\quad - \frac{a_2}{(a_1+a_2)\Gamma(\delta)} \int_0^z I_{1-}^\gamma |f(1, c(1), c(\theta(1))) - f(1, d(1), d(\theta(1)))| dw \\ &\quad \left. - \frac{a_2}{(a_1+a_2)} |y(1, c(1)) - y(1, d(1))| \right\} \\ &\leq (M+N) \|c-d\| \left(\frac{a_1}{a_1+a_2} \right) \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad + \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] \int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad + \hbar_1 \|c-d\| \left(\frac{a_1}{a_1+a_2} \right) + \hbar_2 \|c-d\| \left[\frac{a_2}{\delta(\delta-1)(a_1+a_2)} - \frac{z^\delta}{\delta(\delta-1)} \right] \end{aligned}$$

Recognizing the integral expressions as $\lambda_1, \lambda_2, \lambda_3$ defined in (A5), we get

$$\begin{aligned} \|Fc - Fd\| &\leq (\lambda_1(M+N) + \lambda_2 \hbar_1 + \lambda_3 \hbar_2) \|c-d\| \\ &\leq \omega \|c-d\|. \end{aligned}$$

Hence, $\|Fc - Fd\| \leq \omega \|c-d\|$.

By (A3) we have $\omega < 1$, $F : X \rightarrow X$ is a contraction. Therefore, the contraction mapping principle leads to the theorem's conclusion. \square

Theorem 2. Let $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ denotes a continuous function. Assume (A1)-(A5) holds then there exists at least one solution on $[0, 1]$ for the boundary value problem (1).

Proof. (A5) gives that, there exists a number $r_0 > 0$ such that

$$\sup_{r_0 \in (0, \infty)} \frac{r_0}{\lambda_1 \|k\| \varphi(r_0)} > \frac{1}{1 - \hbar_1 \lambda_2 - \hbar_2 \lambda_3}.$$

The proof is carried out in several steps, showing that F_1 and F_2 satisfy all the conditions of Lemma 2.

Step 1. F_1 is completely continuous and continuous.

Initially, we show that $F_1(\bar{\lambda}_{r_0})$ is bounded. For given any $t \in \bar{\lambda}_{r_0}$, we obtain

$$\begin{aligned} \|F_1 g\| &= \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma |f(w, g(w), g(\theta(w)))| dw \\ &\quad - \frac{z^\delta}{\delta(\delta-1)\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \\ &\quad - \frac{a_2}{(a_1+a_2)} \left[\frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right. \\ &\quad \left. - \frac{1}{\delta(\delta-1)\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right] \\ &\leq B_r \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad - \frac{z^\delta}{\delta(\delta-1)} \int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad - \frac{a_2}{(a_1+a_2)} \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\quad + \frac{a_2}{(a_1+a_2)\delta(\delta-1)} \int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} dm dw \\ &\leq B_r \lambda_1. \end{aligned}$$

Therefore, $F_1(\bar{\lambda}_{r_0})$ is uniformly bounded.

For any $t_1, t_2 \in [0, 1], t_1 < t_2$, we get

$$\begin{aligned} &F_1 g(t_2) - F_1 g(t_1) \\ &= \frac{1}{\Gamma(\delta)} \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-1} - (t_1-w)^{\delta-1} \right] I_{1-}^\gamma |f(w, g(w), g(\theta(w)))| dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-1} I_{1-}^\gamma |f(w, g(w), g(\theta(w)))| dw \right\} - \frac{1}{\delta(\delta-1)\Gamma(\delta-2)} \\ &\quad \left\{ (t_2^\delta - t_1^\delta) \int_0^{t_1} \left[(t_2-w)^{\delta-3} - (t_1-w)^{\delta-3} \right] I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right. \\ &\quad \left. + t_2^\delta \int_{t_1}^{t_2} (t_2-w)^{\delta-3} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right\} \\ &\quad - \frac{a_2}{(a_1+a_2)\Gamma(\delta)} \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-1} - (t_1-w)^{\delta-1} \right] I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-1} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right\} + \frac{a_2}{(a_1+a_2)\delta(\delta-1)\Gamma(\delta-2)} \\ &\quad \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-3} - (t_1-w)^{\delta-3} \right] I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-3} I_{1-}^\gamma |f(1, g(1), g(\theta(1)))| dw \right\} \\ &\leq \frac{B_r}{\Gamma(\delta)\Gamma(\gamma+1)} \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-1} - (t_1-w)^{\delta-1} \right] dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-1} dw \right\} \\ &\quad - \frac{B_r}{\Gamma(\gamma+1)\Gamma(\delta-2)\delta(\delta-1)} \left\{ (t_2^\delta - t_1^\delta) \int_0^{t_1} \left[(t_2-w)^{\delta-3} - (t_1-w)^{\delta-3} \right] dw \right. \\ &\quad \left. + t_2^\delta \int_{t_1}^{t_2} (t_2-w)^{\delta-3} dw \right\} \\ &\quad - \frac{B_r a_2}{\Gamma(\gamma+1)\Gamma(\delta)(a_1+a_2)} \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-1} - (t_1-w)^{\delta-1} \right] dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-1} dw \right\} \\ &\quad - \frac{B_r a_2}{\Gamma(\gamma+1)\Gamma(\delta-2)(a_1+a_2)\delta(\delta-1)} \left\{ \int_0^{t_1} \left[(t_2-w)^{\delta-3} - (t_1-w)^{\delta-3} \right] dw \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-w)^{\delta-3} dw \right\} \end{aligned}$$

$|F_1 g(t_2) - F_1 g(t_1)| \rightarrow 0$, as $t_2 - t_1 \rightarrow 0$ and is independent of g .

Therefore, F_1 is equicontinuous.

By Arzel-Ascoli Theorem, the set $F_1(\bar{\lambda}_{r_0})$ is relatively compact. Now, assume $g_n \subset \bar{\lambda}_{r_0}$ with $\|g_n - g\| \rightarrow 0$. Consequently, the convergence $g_n \rightarrow g$ is uniform on $[0, 1]$. As a result of uniform continuity of $f(z, q, q\theta)$ on the compact set $[0, 1] \times [-r_0, r_0] \times [-r_0, r_0]$, $\|f(z, g_n(z), g_n(\theta(z))) - f(z, g(z), g(\theta(z)))\| \rightarrow 0$ uniformly on $[0, 1]$. Therefore $\|F_1 g_n - F_1 g\| \rightarrow 0$ as $n \rightarrow \infty$, establishing the continuity of F_1 .

Step 2. Consider $F_2 : \bar{\lambda}_{r_0} \rightarrow C([0, 1], \mathbb{R})$ is a contraction.

According to (A2), for $g_1, g_2 \in C([0, 1], \mathbb{R})$, we obtain

$$\begin{aligned} |F_2 g_1(z) - F_2 g_2(z)| &= |y(z, g_1(z)) - y(z, g_2(z))| \\ &\quad + \left(\frac{a_2 - (a_1 + a_2) z^\delta}{(a_1 + a_2) \delta (\delta - 1)} \right) |y''(1, g_1(1)) - y''(1, g_2(1))| \\ &\quad - \frac{a_2}{(a_1 + a_2)} |y(1, g_1(1)) - y(1, g_2(1))| \\ &\leq \hbar_1 \|g_1 - g_2\| \left(\frac{a_1}{a_1 + a_2} \right) + \hbar_2 \|g_1 - g_2\| \left(\frac{a_2 - (a_1 + a_2) z^\delta}{(a_1 + a_2) \delta (\delta - 1)} \right) \\ &\leq (\hbar_1 \lambda_2 + \hbar_2 \lambda_3) \|g_1 - g_2\| \end{aligned}$$

when supremum is taken over $z \in [0, 1]$, it produces

$$|F_2 g_1(z) - F_2 g_2(z)| \leq H_0 \|g_1 - g_2\|, \quad H_0 = \hbar_1 \lambda_2 + \hbar_2 \lambda_3 < 1.$$

Since $H_0 < 1$, F_2 is a contraction.

Step 3. We will show that $F(\bar{\lambda}_{r_0})$ is a bounded set.

Using (A3)

$$F_2(\bar{\lambda}_{r_0}) \leq f \lambda_2 r_0$$

for any $t \in \bar{\lambda}_{r_0}$. Thus, the fact that $F_1(\bar{\lambda}_{r_0})$ is bounded confirms that $F(\bar{\lambda}_{r_0})$ is a bounded set.

Step 4. To show that the case (C2) is not satisfied in Lemma 2.

Suppose, for contradiction, that (C2) holds. Then there exists $\varepsilon \in (0, 1)$ also $g \in \partial \lambda_{r_0}$ in which $g = \varepsilon Fg$. Thus, we obtain $\|g\| = r_0$ and

$$\begin{aligned} g(z) &= \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} f(m, g(m), g(\theta(m))) dm dw + y(z, g(z)) \\ &\quad - \frac{z^\delta}{\delta(\delta-1)} \left[\int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} f(1, g(1), g(\theta(1))) dm dw \right. \\ &\quad \left. + \delta \Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \right] + \frac{z^{\delta+1}}{(\delta+1)} \Lambda_1 \\ &\quad - \frac{a_2}{(a_1 + a_2)} \left\{ \int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} f(1, g(1), g(\theta(1))) dm dw \right. \\ &\quad \left. + y(1, g(1)) - \frac{a_3}{a_2} \right. \\ &\quad \left. - \frac{1}{\delta(\delta-1)} \left[\int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} f(1, g(1), g(\theta(1))) dm dw \right. \right. \\ &\quad \left. \left. + \delta \Lambda_1 + y''(1, g(1)) - \frac{\alpha_3}{\alpha_2} \right] + \frac{\Lambda_1}{(\delta+1)} \right\}, \quad z \in [0, 1]. \end{aligned}$$

Based on assumptions (A4) - (A5), we obtain

$$r_0 \leq \lambda_1 \|k\| \varphi(r_0) + \hbar_1 \lambda_2 r_0 + \hbar_2 \lambda_3 r_0$$

Thus, we get

$$\frac{r_0}{\lambda_1 \|k\| \varphi(r_0)} \leq \frac{1}{1 - \hbar_1 \lambda_2 - \hbar_2 \lambda_3}$$

This contradicts (A5), hence (C2) does not hold. Thus the operators F_1 and F_2 fulfill all the criteria of Lemma 2. Therefore, there exists at least one fixed point $g \in \bar{\lambda}_{r_0}$ for the operator F , which corresponds to a solution of problem (1). This completes the proof. \square

4. Example

Consider the boundary value problem

$$\begin{cases} {}^C D_{1-}^{\frac{5}{2}} \left\{ {}^C D_{0+}^{\frac{1}{4}} [g(z) - y(z, g(z))] \right\} = f(z, g(z), g(\theta(z))), \quad z \in [0, 1], \\ \frac{2}{5} g(0) + \frac{3}{8} g(1) = \frac{5}{7} \\ \frac{4}{7} g'(0) + \frac{1}{7} g'(1) = \frac{5}{9} \\ \frac{3}{7} g''(0) + \frac{5}{6} g''(1) = \frac{1}{5} \end{cases} \quad (11)$$

where $\gamma = \frac{5}{2}$, $\delta = \frac{1}{4}$, $a_1 = \frac{2}{5}$, $a_2 = \frac{3}{8}$, $\theta(z) = z^\eta$, $0 < \eta < 1$

$$f(z, g(z), g(\theta(z))) = \frac{z+1}{55} g(z) + \frac{z^2+1}{80} g(z^\eta) \quad (12)$$

One can verify that f satisfies assumption (A1) with $M = \frac{1}{55}$ and $N = \frac{1}{80}$.

Here, $y(z, g(z)) = \frac{2}{7} + \frac{|g(z)|}{45(2+|g(z)|)}$.

It can be verified that,

$$|y(z, c) - y(z, d)| \leq h_1 |c - d|$$

with $h_1 = \frac{1}{45}$ and $h_2 = \frac{2}{45}$.

Further, $\lambda_1 = 14.26756$, $\lambda_2 = 0.516129$, $\lambda_3 = 1.904135$. Then

$$\omega := \lambda_1(M + N) + \lambda_2 h_1 + \lambda_3 h_2 \approx 0.533852.$$

Thus, all the conditions of Theorem 1 with $f(z, c, c(\theta))$ defined by (12) are satisfied and as a result of this, Theorem 1 concludes that there is a unique solution for the problem (11).

5. Conclusion

In this paper, we have investigated the existence and uniqueness of solutions for sequential fractional differential equations involving the Caputo differential operator with advanced arguments under nonlocal boundary conditions. The uniqueness result established in Theorem 1 guarantees the mathematical well-posedness of the proposed model, ensuring a single predictable solution under the Lipschitz conditions (A1)–(A3), which is physically meaningful in control systems and biological models with anticipatory behavior.

The existence result of Theorem 2, derived via D. O'Regan's fixed point theorem, is significant in that it requires only mild growth conditions on the nonlinearity f , making it applicable to a broader class of problems compared to results based on Krasnosel'skii or Leray–Schauder alternatives. The decomposition $F = F_1 + F_2$, where F_1 is completely continuous and F_2 is a contraction, provides a flexible analytical tool that can be adapted to related problems in fractional analysis.

The results of this paper open several directions for future research, including stability analysis, numerical approximation, extension to generalized fractional operators such as Hilfer and ψ -Caputo derivatives, and coupled systems of sequential fractional differential equations with advanced arguments.

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