Existence of the solutions to convolution equations with distributional kernels

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Abstract

It is proved that a class of convolution integral equations of the Volterra type has a global solution, that is, solutions defined for all \( t \geq 0 \). Smoothness of the solution is studied.

Keywords: Volterra equations; distributional kernels

1. Introduction

Consider the equation:

\[
u(t) = \int_0^t \frac{(t-s)^{a}}{\Gamma(a+1)} u(s) ds + f(t) := Vu + f, \quad (1)
\]

where \( t \geq 0; \ a = \text{const} \neq -1, -2, \ldots, \) (2)

and

\[Vu := V_a * u, \quad V_a := \frac{t^a}{\Gamma(a+1)}. \quad (3)
\]

Here \( \Gamma(\cdot) \) is the Gamma function, and (3) is a convolution with the distribution \( V_a \), see [1]. Thus, equation (1) is a Volterra equation with kernel that is not absolutely integrable for \( a < -1 \). There is a large literature on integral equations, [2], but the usual methods to study such equations are based on the assumption that the kernel of the operator \( V \) belongs to \( L^p \) with \( p \geq 1 \).

The goal of this paper is to develop a method to study (1) with a distributional kernel \( \frac{t^a}{\Gamma(a+1)} \). The basic known result (see [1]) is the property of convolution

\[V_a * V_b = V_b * V_a = V_{a+b}. \quad (4)
\]

Our result is formulated in Theorem 1.

Theorem 1. Equation (1) with \( a < -1 \) is uniquely solvable. Its solution \( u \) exists for all \( t \geq 0 \). It belongs to the space of functions which is of the same smoothness as \( V_{-a}f \).

In the next section a proof is given.

2. Proof

Proof of Theorem 1. The idea of the proof is to apply \( V_{-a} \) to equation (1) and use the formula

\[V_a * V_{-a} = I, \quad (5)
\]

where \( I \) is the identity operator whose kernel is the delta-function. Applying \( V_{-a} \) to (1) one gets

\[V_{-a}u = u + V_{-a}f, \quad (6)
\]

or

\[u = -V_{-a}u + V_{-a}f. \quad (7)
\]

Suppose that \( a < -1 \). Then \( -a > 1 \) and the operator \( V_{-a} \) is a convolution with a continuous kernel. Therefore equation (7) is a Volterra integral equation with a continuous kernel. Consequently, this equation has a unique solution \( u \) for all \( t \geq 0 \). This solution can be calculated by iterations. Equations (1) and (7) are equivalent because \( V_a * V_{-a} = I \). Therefore, Theorem 1 is proved.

2 References