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Some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions on the co-ordinates

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Abstract

In this paper, the authors introduce a new concept "(s_1,m_1)-(s_2,m_2)-convex function on co-ordinates" and establish some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions of 2-variables on the co-ordinates.

Keywords: *co-ordinated convex function;* (s_1,m_1)-(s_2,m_2)*-convex function; inequality.*

1 Introduction

The following inequality, named after Simpson, is one of the best known results in the literature.

Theorem A. *Let* $f:I\subset\mathbb{R}_0=[0,\infty)\to\mathbb{R}=(-\infty,\infty)$ *be a four times continuously differentiable mapping on* $[a,b]$ *and* $\|f^{(4)}\|_\infty=\sup_{x\in[a,b]}|f^{(4)}(x)|<\infty$. *Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty. \tag{1.1}$$

For more information and recent development on Simpson's type inequalities, please refer to [1, 2, 3] and closely related references therein.

In [4], J. Park obtained the following generalized identity for some partial differentiable mappings on a bi-dimensional interval.

Lemma 1. *Let* $f:\Delta\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. *If* $\frac{\partial^2 f}{\partial t\,\partial\lambda}\in L(\Delta)$, *then for* $r_1,r_2\ge2$ *and* $h_1,h_2\in(0,1)$ *with* $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1}$, $\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, *we have*

$$\begin{aligned}
I(f)(h_1,h_2,r_1,r_2)&=\left[\frac{(r_1-2)(r_2-2)}{r_1r_2}\right]f(h_1a+(1-h_1)b,h_2c+(1-h_2)d)\\
&+\left[\frac{r_1-2}{r_1r_2}\right]\left[f(h_1a+(1-h_1)b,c)+f(h_1a+(1-h_1)b,d)\right]+\left[\frac{r_2-2}{r_1r_2}\right]\left[f(a,h_2c+(1-h_2)d)+f(b,h_2c+(1-h_2)d)\right]\\
&+\frac{1}{r_1r_2}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]-\frac{1}{r_2(b-a)}\int_a^b\left[f(x,c)+(r_2-2)f(x,h_2c+(1-h_2)d)+f(x,d)\right]\mathrm{d}x\\
&-\frac{1}{r_1(d-c)}\int_c^d\left[f(a,y)+(r_1-2)f(h_1a+(1-h_1)b,y)+f(b,y)\right]\mathrm{d}y+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&=(b-a)(d-c)\int_0^1\int_0^1 p(h_1,r_1,t)q(h_2,r_2,\lambda)\frac{\partial^2 f}{\partial t\,\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}t\,\mathrm{d}\lambda,
\end{aligned}$$

where

$$p(h_1,r_1,t)==\begin{cases} t-\dfrac{1}{r_1}, & t\in[0,h_1] \\ t-\dfrac{r_1-1}{r_1}, & t\in[h_1,1] \end{cases} \quad \text{and } q(h_2,r_2,t)=\begin{cases} \lambda-\dfrac{1}{r_2}, & \lambda\in[0,h_2], \\ \lambda-\dfrac{r_2-1}{r_2}, & \lambda\in[h_2,1]. \end{cases}$$

Moreover, J. Park established some Simpson-like type inequalities for co-ordinated s -convex mappings in the second sense.

The main purpose of this paper is to introduce the new concept "co-ordinated (s_1,m_1) - (s_2,m_2) -convex function" and establish some new inequalities for such kind of functions on the co-ordinates.

2 Co-ordinated (s_1,m_1)-(s_2,m_2)-convex functions

In [2, 4-10], Dragomir et al. and Park considered among others the class of mappings which are convex and s -convex on the co-ordinates.

Let us now consider a bi-dimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $b>a$ and $d>c$.

Definition 1. ([8]) *A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality*

$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le\alpha f(x,y)+(1-\alpha)f(z,w)$

holds for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$. If the inequality is reversed, then f is said to be concave on Δ.

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir in [6] as follows.

Definition 2. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbf{R},f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbf{R},f_x(v)=f(x,v)$ are convex for all $x\in[a,b]$ and $y\in[c,d]$.*

A formal definition for co-ordinated convex functions may be stated as follows.

Definition 3. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\le t\lambda f(x,y)+t(1-\lambda)f(x,w)+\lambda(1-t)f(z,y)+(1-t)(1-\lambda)f(z,w)$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ and $t,\lambda\in[0,1]$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex functions which are not convex. See [9].

Definition 4 ([4, 8, 10]) *For the bi-dimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}_0^2=[0,\infty)^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be s -convex on Δ if the inequality*

$f\left(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w\right)\le\alpha^s f(x,y)+(1-\alpha)^s f(z,w)$

holds for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$ and for some fixed $s\in(0,1]$.

Definition 5. ([4, 8]) *A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is called* S*-convex on co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R},f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are s -convex for all $x\in[a,b]$, $y\in[c,d]$ and $s\in(0,1]$, i.e., f_y and f_x are s -convex for some fixed $s\in(0,1]$.*

Theorem B. ([8]) *Every s -convex mapping $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ is s -convex on the co-ordinates. But, the converse is not true in general.*

A formal definition for co-ordinated s -convex mappings may be stated as follow.

Definition 6. ([4,8]) A *mapping $f:\Delta\to\mathbb{R}$ is called* S*-convex on the co-ordinates on Δ if the inequality*

$$f\left(tx+(1-t)z,\lambda y+(1-\lambda)w\right)\le t^s\lambda^s f\left(x,y\right)+t^s\left(1-\lambda\right)^s f\left(x,w\right)+\lambda^s\left(1-t\right)^s f\left(z,y\right)+\left(1-t\right)^s\left(1-\lambda\right)^s f\left(z,w\right)$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ *and* $t,\lambda\in[0,1]$ *and* $s\in(0,1]$.

J. Park gave the notions of (s,m)-convexity and (s,m)–convexity on the co-ordinates for the function f on a rectangle from the plane $\mathbb{R}^2$ as follows.

Definition 7 ([11]). *A mapping* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is* (s,m)*-convex on* Δ *if the inequality*

$$f\left(\alpha x+(1-\alpha)z,\alpha y+m(1-\alpha)w\right)\le\alpha^s f\left(x,y\right)+m\left(1-\alpha^s\right)f\left(z,w\right)$$

holds for all $(x,y),(z,w)\in\Delta$ *and* $\alpha\in[0,1]$ *and for some fixed* $s,m\in(0,1]$.

Definition 8 ([11]). *A function* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is called* (s,m)*- convex on co-ordinates on* Δ *if the partial mappings* $f_y:[0,b]\to\mathbb{R},f_y(u)=f(u,y)$ *and* $f_x:[0,d]\to\mathbb{R},\ f_x(v)=f(x,v)$ *are* (s,m)*-convex for all* $x\in[0,b],y\in[0,d]$ *and* $s,m\in(0,1]$, *i.e.,* f_y *and* f_x *are* (s,m)*-convex with some fixed* $s,m\in(0,1]$.

Motivated by the above definitions, we now introduce the (s_1,m_1)-(s_2,m_2)-convexity on the co-ordinates for a function $f(x)$ on a rectangle from the plane $\mathbb{R}^2$ as follows:

Definition 9. *A function* $f:\Delta_0=[0,b]\times[0,d]\subset\mathbb{R}_0^2\to\mathbb{R}$ *is called* (s_1,m_1)-(s_2,m_2) *convex on co-ordinates on* Δ *if the inequality*

$$f\left(tx+m_1(1-t)z,\lambda y+m_2(1-\lambda)w\right)$$
$$\le t^{s_1}\lambda^{s_2}f\left(x,y\right)+m_1t^{s_1}\left(1-\lambda^{s_2}\right)f\left(x,w\right)+m_2\lambda^{s_2}\left(1-t^{s_1}\right)f\left(z,y\right)+m_1m_2\left(1-t^{s_1}\right)\left(1-\lambda^{s_2}\right)f\left(z,w\right)$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ with $t,\lambda\in[0,1]$ and $s_1,s_2,m_1,m_2\in(0,1]$.

3 Some inequalities for (s_1,m_1)-(s_2,m_2)-convex functions

Theorem 1 *Let* $f:\mathbb{R}_0^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m=[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b>a>0$ *and* $d>c>0$. *If* $\frac{\partial^2 f}{\partial t\,\partial\lambda}\in L(\Delta_m)$ *is a co-ordinated* (s_1,m_1)-(s_2,m_2)*-convex mapping on* Δ_m *for* $s_1,s_2,m_1,m_2\in(0,1]$, *then for* $r_1,r_2\ge2$ *and* $h_1,h_2\in(0,1)$ *with* $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1},\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, *the inequality*

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|\le\mu(h_1,r_1,s_1)\left\{\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|+m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|\right\}$$
$$+\nu(h_1,r_1,s_1)\left\{m_1\mu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|+m_1m_2\nu(h_2,r_2,s_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\right\},\quad(3.1)$$

where

$$\mu(h,r,s)=M(r,s)+N(h,s),\ \nu(h,r,s)=W(h,r)-M(r,s)-N(1-h,s),$$

and

$$M(r,s)=\frac{2+2(r-1)^{s+2}+r^{s+1}(s-r+2)}{(s+1)(s+2)r^{s+2}},\ N(h,s)=\frac{h^{s+1}((2h-1)s+2(h-1))}{(s+1)(s+2)},$$
$$W(h,r)=\frac{1}{2}-h+h^2+\frac{(2-r)}{r^2}.$$

Proof. From Lemma 1 and the co-ordinated (s_1,m_1)-(s_2,m_2)-convexity of $\frac{\partial^2 f}{\partial t\,\partial\lambda}$, we have

$$
\begin{aligned}
&\frac{1}{(\boldsymbol{b}-\boldsymbol{a})(\boldsymbol{d}-\boldsymbol{c})}\left|\boldsymbol{I}(\boldsymbol{f})(\boldsymbol{h}_1,\boldsymbol{h}_2,\boldsymbol{r}_1,\boldsymbol{r}_2)\right| \\
&\le \int_0^1\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right| \left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\boldsymbol{t}\boldsymbol{a}+\boldsymbol{m}_1(1-\boldsymbol{t})\frac{\boldsymbol{b}}{\boldsymbol{m}_1}, \boldsymbol{\lambda}\boldsymbol{c}+\boldsymbol{m}_2(1-\boldsymbol{\lambda})\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right| \mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{\lambda} \\
&\le \int_0^1\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right| \left\{\boldsymbol{t}^{s_1}\boldsymbol{\lambda}^{s_2}\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{c})\right| + \boldsymbol{m}_2\boldsymbol{t}^{s_1}(1-\boldsymbol{\lambda}^{s_2})\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\boldsymbol{a},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|\right. \\
&\quad \left. + \boldsymbol{m}_1(1-\boldsymbol{t}^{s_1})\boldsymbol{\lambda}^{s_2}\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\boldsymbol{c}\right)\right| + \boldsymbol{m}_1\boldsymbol{m}_2(1-\boldsymbol{t}^{s_1})(1-\boldsymbol{\lambda}^{s_2})\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|\right\} \mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{\lambda} \\
&= \left(\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|\boldsymbol{t}^{s_1}\mathrm{d}\boldsymbol{t}\right)\left\{\left(\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|\boldsymbol{\lambda}^{s_2}\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{c})\right|\right. \\
&\quad \left. + \boldsymbol{m}_2\left(\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|(1-\boldsymbol{\lambda}^{s_2})\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\boldsymbol{a},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|\right\} \\
&\quad + \left(\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|(1-\boldsymbol{t}^{s_1})\mathrm{d}\boldsymbol{t}\right)\left\{\boldsymbol{m}_1\left(\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|\boldsymbol{\lambda}^{s_2}\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\boldsymbol{c}\right)\right|\right. \\
&\qquad \left. + \boldsymbol{m}_1\boldsymbol{m}_2\left(\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|(1-\boldsymbol{\lambda}^{s_2})\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|\right\}. \qquad (3.2)
\end{aligned}
$$

Note that

$$
\begin{gathered}
\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|\boldsymbol{t}^{s_1}\mathrm{d}\boldsymbol{t} = \boldsymbol{\mu}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{s}_1);\ \int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|\boldsymbol{\lambda}^{s_2}\mathrm{d}\boldsymbol{t} = \boldsymbol{\mu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2); \\
\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|\mathrm{d}\boldsymbol{t} = \frac{1}{2}-\boldsymbol{h}_1+\boldsymbol{h}_1^2+\frac{(2-\boldsymbol{r}_1)}{\boldsymbol{r}_1^2} = \boldsymbol{W}(\boldsymbol{h}_1,\boldsymbol{r}_1); \\
\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{t})\right|\mathrm{d}\boldsymbol{t} = \frac{1}{2}-\boldsymbol{h}_2+\boldsymbol{h}_2^2+\frac{(2-\boldsymbol{r}_2)}{\boldsymbol{r}_2^2} = \boldsymbol{W}(\boldsymbol{h}_2,\boldsymbol{r}_2); \\
\int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|(1-\boldsymbol{t}^{s_1})\mathrm{d}\boldsymbol{t} = \int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|\mathrm{d}\boldsymbol{t} - \int_0^1 \left|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})\right|\boldsymbol{t}^{s_1}\mathrm{d}\boldsymbol{t} = \boldsymbol{W}(\boldsymbol{h}_1,\boldsymbol{r}_1)-\boldsymbol{\mu}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{s}_1); \\
\int_0^1 \left|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})\right|(1-\boldsymbol{\lambda}^{s_2})\mathrm{d}\boldsymbol{t} = \boldsymbol{W}(\boldsymbol{h}_2,\boldsymbol{r}_2)-\boldsymbol{\mu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2).
\end{gathered}
$$

Substituting the above equations into (3.2) and rearranging leads to (3.1). This completes the proof.

Corollary 1. *Under the conditions of Theorem 1,*

(i) *if we choose* $\boldsymbol{h}_1=\boldsymbol{h}_2=\frac{1}{2}$, $\boldsymbol{r}_1=\boldsymbol{r}_2=6$, *and* $\boldsymbol{s}_1=\boldsymbol{s}_2=\boldsymbol{m}_1=\boldsymbol{m}_2=1$ *in* (3.1), *then*

$$\left|\boldsymbol{I}(\boldsymbol{f})\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \le \left(\frac{5}{72}\right)^2 \boldsymbol{M}(\boldsymbol{b}-\boldsymbol{a})(\boldsymbol{d}-\boldsymbol{c}).$$

(ii) *if we choose* $\boldsymbol{h}_1=\boldsymbol{h}_2=\frac{1}{2}$, $\boldsymbol{r}_1=\boldsymbol{r}_2=2$, *and* $\boldsymbol{s}_1=\boldsymbol{s}_2=\boldsymbol{m}_1=\boldsymbol{m}_2=1$ *in* (3.1), *then*

$$\left|\boldsymbol{I}(\boldsymbol{f})\left(\frac{1}{2},\frac{1}{2},2,2\right)\right| \le \left(\frac{1}{8}\right)^2 \boldsymbol{M}(\boldsymbol{b}-\boldsymbol{a})(\boldsymbol{d}-\boldsymbol{c}),$$

where

$$\boldsymbol{M} = \left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{c})\right| + \left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{d})\right| + \left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{b},\boldsymbol{c})\right| + \left|\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{t}\,\partial \boldsymbol{\lambda}}(\boldsymbol{b},\boldsymbol{d})\right|.$$

Theorem 2 *Let* $f:\mathbb{R}_0^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m=[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b>a>0$ *and* $d>c>0$. *If* $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q\in L(\Delta_m)$ *is a co-ordinated* (s_1,m_1) - (s_2,m_2) *-convex mapping, where* $s_1,s_2,m_1,m_2\in(0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *for* $q>1$, *then for* $r_1,r_2\ge 2$ *and* $h_1,h_2\in(0,1)$ *with* $\frac{1}{r_1}\le h_1\le\frac{r_1-1}{r_1},\frac{1}{r_2}\le h_2\le\frac{r_2-1}{r_2}$, *the inequality*

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|\le\mu_3^{\frac{1}{p}}\nu_3^{\frac{1}{p}}\left\{\frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q\right.\right.$$
$$\left.\left.+m_1s_1\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1s_1m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}\quad(3.3)$$

is true, where

$$\mu_3=\frac{2+(r_1-r_1h_1-1)^{p+1}+(r_1h_1-1)^{p+1}}{r_1^{p+1}(p+1)}\ \text{ and }\ \nu_3=\frac{2+(r_2-r_2h_2-1)^{p+1}+(r_2h_2-1)^{p+1}}{r_2^{p+1}(p+1)}.$$

Proof. From Lemma 1, we have

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|$$
$$\le\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(ta+m_1(1-t)\frac{b}{m_1},\lambda c+m_2(1-\lambda)\frac{d}{m_2}\right)\right|\mathrm{d}t\mathrm{d}\lambda$$
$$\le\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{p}}\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(ta+m_1(1-t)\frac{b}{m_1},\lambda c+m_2(1-\lambda)\frac{d}{m_2}\right)\right|^q\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{q}}.\quad(3.4)$$

Hence, by the inequality (3.4) and the co-ordinated (s_1,m_1) - (s_2,m_2) -convexity of $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q$, it follows that

$$\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right|\le\left\{\int_0^1\int_0^1\left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{p}}\left\{\frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q\right.\right.$$
$$\left.\left.+m_2s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{m_2}\right)\right|^q+m_1s_1\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2s_1s_2\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}.$$

Note that

$$\int_0^1\left|p(h_1,r_1,t)\right|^p\mathrm{d}t=\frac{2+(r_1-r_1h_1-1)^{p+1}+(r_1h_1-1)^{p+1}}{r_1^{p+1}(p+1)}$$

and

$$\int_0^1\left|q(h_2,r_2,\lambda)\right|^p\mathrm{d}t=\frac{2+(r_2-r_2h_2-1)^{p+1}+(r_2h_2-1)^{p+1}}{r_2^{p+1}(p+1)}.$$

Substituting the above equations into (3.4) and simplifying results in (3.3). The proof is complete.

Corollary 2 *Under the conditions of Theorem 2,*

(i) *if we choose* $h_1=h_2=\frac{1}{2}$, $r_1=r_2=6$, *and* $s_1=s_2=m_1=m_2=1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right|\le\left[\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right]^{\frac{2}{p}}(b-a)(d-c)M_q^{\frac{1}{q}}.$$

(ii) *if we choose* $\eta_1=\eta_2=\frac{1}{2},\rho_1=\rho_2=2$, *and* $\sigma_1=\sigma_2=\mu_1=\mu_2=1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},2,2\right)\right|\le\left[\frac{1}{2^p(p+1)}\right]^{\frac{2}{p}}(b-a)(d-c)M_q^{\frac{1}{q}}.$$

(iii) *if we choose* $\eta_1=\eta_2=\frac{1}{2},\rho_1=\rho_2=6$, $\sigma_1=\sigma_2=\mu_1=\mu_2=1$, *and* $p=1$, *then*

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right|\le\left(\frac{5}{36}\right)^2(b-a)(d-c)M_q^{\frac{1}{q}},$$

where

$$M_q=\frac{1}{4}\left(\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(b,d)\right|^q\right).$$

Theorem 3 *Let* $f:\mathbb{R}_0^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta_m=[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ *for* $b>a>0$ *and* $d>c>0$. *If* $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q\in L(\Delta_m)$ *is a co-ordinated* (σ_1,μ_1) *-* (σ_2,μ_2) *- convex mapping for* $\sigma_1,\sigma_2,\mu_1,\mu_2\in(0,1]$, $q>1$, *and* $\frac{1}{p}+\frac{1}{q}=1$, *then, for* $\rho_1,\rho_2\ge 2$ *and* $\eta_1,\eta_2\in(0,1)$ *with* $\frac{1}{\rho_1}\le\eta_1\le\frac{\rho_1-1}{\rho_1},\frac{1}{\rho_2}\le\eta_2\le\frac{\rho_2-1}{\rho_2}$, *the inequality*

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\left|I(f)(\eta_1,\eta_2,\rho_1,\rho_2)\right|&\le\left\{\left(\frac{1}{2}-\eta_1+\eta_1^2+\frac{(2-\rho_1)}{\rho_1^2}\right)\left(\frac{1}{2}-\eta_2+\eta_2^2+\frac{(2-\rho_2)}{\rho_2^2}\right)\right\}^{1-\frac{1}{q}}\\&\times\left\{\mu(\eta_1,\rho_1,\sigma_1)\left[\mu(\eta_2,\rho_2,\sigma_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+\mu_2\nu(\eta_2,\rho_2,\sigma_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{\mu_2}\right)\right|^q\right]\right.\\&\left.+\nu(\eta_1,\rho_1,\sigma_1)\left[\mu_1\mu(\eta_2,\rho_2,\sigma_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{\mu_1},c\right)\right|^q+\mu_1\mu_2\nu(\eta_2,\rho_2,\sigma_2)\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(\frac{b}{\mu_1},\frac{d}{\mu_2}\right)\right|^q\right]\right\}^{\frac{1}{q}}.\end{aligned}\quad(3.5)$$

holds true, where μ *and* ν *are given in Theorem 2.1.*

Proof. From Lemma 1, it follows that

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\left|I(f)(\eta_1,\eta_2,\rho_1,\rho_2)\right|&\le\int_0^1\int_0^1\left|p(\eta_1,\rho_1,t)q(\eta_2,\rho_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\mathrm{d}t\mathrm{d}\lambda\\&\le\left\{\int_0^1\int_0^1\left|p(\eta_1,\rho_1,t)q(\eta_2,\rho_2,\lambda)\right|\mathrm{d}t\mathrm{d}\lambda\right\}^{1-\frac{1}{q}}\left\{\int_0^1\int_0^1\left|p(\eta_1,\rho_1,t)q(\eta_2,\rho_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}t\mathrm{d}\lambda\right\}^{\frac{1}{q}}.\end{aligned}\quad(3.6)$$

It is straightforward that

$$\int_0^1\left|p(\eta_1,\rho_1,t)\right|\mathrm{d}t=\frac{1}{2}-\eta_1+\eta_1^2+\frac{(2-\rho_1)}{\rho_1^2},\ \int_0^1\left|q(\eta_2,\rho_2,\lambda)\right|\mathrm{d}\lambda=\frac{1}{2}-\eta_2+\eta_2^2+\frac{(2-\rho_2)}{\rho_2^2}.$$

Since $\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\right|^q$ is a co-ordinated (σ_1,μ_1) *-* (σ_2,μ_2) *-*convex mapping on Δ, we have

$$\begin{aligned}&\int_0^1\int_0^1\left|p(\eta_1,\rho_1,t)q(\eta_2,\rho_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}t\mathrm{d}\lambda\\&\le\int_0^1\int_0^1\left|p(\eta_1,\rho_1,t)q(\eta_2,\rho_2,\lambda)\right|\left\{t^{\sigma_1}\lambda^{\sigma_2}\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}(a,c)\right|^q+\mu_2 t^{\sigma_1}(1-\lambda^{\sigma_2})\left|\frac{\partial^2 f}{\partial t\,\partial\lambda}\left(a,\frac{d}{\mu_2}\right)\right|^q\right.\end{aligned}$$

$$+\boldsymbol{m}_1(1-\boldsymbol{t}^{s_1})\boldsymbol{\lambda}^{s_2}\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\boldsymbol{c}\right)\right|^{\boldsymbol{q}}+\boldsymbol{m}_1\boldsymbol{m}_2(1-\boldsymbol{t}^{s_1})(1-\boldsymbol{\lambda}^{s_2})\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|^{\boldsymbol{q}}\Bigg\}\mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{\lambda}$$

$$=\left(\int_0^1|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})|\boldsymbol{t}^{s_1}\mathrm{d}\boldsymbol{t}\right)\left\{\left(\int_0^1|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})|\boldsymbol{\lambda}^{s_2}\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{c})\right|^{\boldsymbol{q}}\right.$$

$$\left.+\boldsymbol{m}_2\left(\int_0^1|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})|(1-\boldsymbol{\lambda}^{s_2})\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\boldsymbol{a},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|^{\boldsymbol{q}}\right\}$$

$$+\left(\int_0^1|\boldsymbol{p}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{t})|(1-\boldsymbol{t}^{s_1})\mathrm{d}\boldsymbol{t}\right)\left\{\boldsymbol{m}_1\left(\int_0^1|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})|\boldsymbol{\lambda}^{s_2}\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\boldsymbol{c}\right)\right|^{\boldsymbol{q}}\right.$$

$$\left.+\boldsymbol{m}_1\boldsymbol{m}_2\left(\int_0^1|\boldsymbol{q}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{\lambda})|(1-\boldsymbol{\lambda}^{s_2})\mathrm{d}\boldsymbol{\lambda}\right)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|^{\boldsymbol{q}}\right\}$$

$$=\boldsymbol{\mu}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{s}_1)\left\{\boldsymbol{\mu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}(\boldsymbol{a},\boldsymbol{c})\right|^{\boldsymbol{q}}+\boldsymbol{m}_2\boldsymbol{\nu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\boldsymbol{a},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|^{\boldsymbol{q}}\right\}$$

$$+\boldsymbol{\nu}(\boldsymbol{h}_1,\boldsymbol{r}_1,\boldsymbol{s}_1)\left\{\boldsymbol{m}_1\boldsymbol{\mu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\boldsymbol{c}\right)\right|^{\boldsymbol{q}}+\boldsymbol{m}_1\boldsymbol{m}_2\boldsymbol{\nu}(\boldsymbol{h}_2,\boldsymbol{r}_2,\boldsymbol{s}_2)\left|\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{t}\,\partial\boldsymbol{\lambda}}\left(\frac{\boldsymbol{b}}{\boldsymbol{m}_1},\frac{\boldsymbol{d}}{\boldsymbol{m}_2}\right)\right|^{\boldsymbol{q}}\right\}.$$

Combining the above equations and inequalities reveals the assertion (3.5). The proof is completed.

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References

[1] M. Alomari, M. Darus, and S.S. Dragomir, *New inequalities of Simpson's type for* $\boldsymbol{s}$ *-convex functions with applications*, RGMIA Res. Rep. Coll. Vol.12, No.4, (2009), Available online at http://rgmia.org/v12n4.php.

[2] S.S. Dragomir, R.P. Agarwal, and P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl., 5(2000), pp.533-579.

[3] Z. Liu, *An inequality of Simpson type*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. Vol.461, No.2059**,** (2005), pp.2155-2158.

[4] J. Park, *Generalizations of Simpson-like type inequality for co-ordinated* $\boldsymbol{s}$ *-convex mappings in the second sense*, International Journal of Mathematics and Mathematical Sciences, **2012** (2012), Article ID 715751, 16 pages; doi:10.1155/2012/715751.

[5] M. Alomari and M. Darus, *Co-ordinated* $\boldsymbol{s}$ *-convex function in the first sense with some Hadamard-type inequality*, Int. J. Contemp. Math. Sci. **3** Vol.3, No.29-32, (2008), pp.1557-1567.

[6] S.S. Dragomir, *On Hadamard's inequality for co-ordinated convex function in a rectangle from the plane*, J. Inequal. Appl. Vol.5, No.5**,**(2011), pp.775-788.

[7] J. Park, *Generalizations of Simpson's type inequality for twice differentiable convex mapping*, Far East J. Math. Sci. Vol.52, No.1, (2011), pp.43-55.

[8] M.A. Latif and M. Alomari, *On Hadmard-type inequalities for* $\boldsymbol{h}$ *-convex functions on the co-ordinates*. Int. J. Math. Anal. Vol.33, No.3, (2009), pp.1645--1656.

[9] S.S. Dragomir, *Selected Topics on Hermite-Hadamard Inequality and Applications*, RGMIA Monographs, Victoria University, 2000; Available online at http://rgmia.org/monographs/hermite_hadamard.html.

[10] M. Alomari and M. Darus, *The Hadamard's inequality for* $\boldsymbol{s}$ *-convex function of 2-variables on the co-ordinates*, Int. J. Math. Anal. Vol.13, No.2, (2008), pp. 629--638.

[11] J. Park, *Some Hadamard's type inequality for co-ordinated* $(\boldsymbol{s},\boldsymbol{m})$ *-convex mapping in the second sense*. Far East J. Math. Sci. Vol.51, No.2, (2011), pp.205--216.