On Properties of meromorphic solutions of difference Painlevé I and II equation

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Abstract

In this paper, we investigate some properties of finite order transcendental meromorphic solutions of difference Painlevé I and II equations, and obtain precise estimations of exponents of convergence of poles of difference \( \Delta w(z) = w(z+1) - w(z) \) and divided difference \( \frac{\Delta w(z)}{w(z)} \), and of fixed points of \( w(z+\eta) \) (\( \eta \in \mathbb{C} \setminus \{0\} \)).

Keywords: Difference; Divided Difference; Difference Painlevé Equations; Meromorphic Function

1. Introduction and Results

In this paper, we assume that the reader is familiar with the basic Nevanlinna’s value distribution theory of meromorphic functions (see[2, 10]). In addition, we use the notation \( \sigma(f) \) to denote the order of growth of the meromorphic function \( f(z) \), \( \lambda(f) \) and \( \lambda(\frac{1}{f}) \) to denote, respectively, the exponent of convergence of zeros and poles of \( f(z) \). We also use the notation \( \tau(f) \) to denote the exponent of convergence of fixed points of \( f(z) \) which is defined as

\[
\tau(f) = \limsup_{r \to \infty} \frac{\log N\left(r, \frac{1}{f(z)-\eta}\right)}{\log r}.
\]

We denote by \( S(r,f) \) any quantity satisfying \( S(r,f) = o(T(r,f)) \), as \( r \to \infty \), possibly outside a set with finite measure.

Recently, a number of papers (including [3 – 9, 11 – 14]) have focused on complex difference equations and difference analogues of Nevanlinna’s theory. As the difference analogues of Nevanlinna’s theory are investigated [8, 12], many results on the complex difference equations are rapidly obtained.

Halburd and Korhonen [9] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation

\[
w(z + 1) + w(z - 1) = R(z, w),
\]

where \( R \) is rational in both of its arguments. They proved that if (1.1) has an admissible meromorphic solutions of finite order, then either \( w \) satisfies a difference Riccati equation, or (1.1) may be transformed into some classical difference equations, which include difference Painlevé I equations

\[
w(z + 1) + w(z - 1) = \frac{a z + b}{w(z)} + c,
\]

\[
w(z + 1) + w(z - 1) = \frac{az + b}{w(z)} + c + \frac{e}{w^2(z)},
\]

where \( a, b \) and \( c \) are constants.

In 2010, Chen and Shon [13] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.2) and difference Painlevé II equation (1.5), and proved the following results.

**Theorem A.** (See [13]) Let \( a, b, c \) be constants with \( ac \neq 0 \). If \( w(z) \) is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.2), then:

(i) \( w(z) \) has at most one non-zero finite Borel exceptional value;
(ii) \( \lambda(\frac{1}{w}) = \lambda(w) = \sigma(w) \);
(iii) \( w(z) \) has infinitely many fixed points and satisfies \( \tau(w) = \sigma(w) \).

**Theorem B.** (See [13]) Let \( a, b, c \) be constants with \( a \neq 0 \). If \( w(z) \) is a finite-order transcendental meromorphic solution of the difference Painlevé II equation (1.5), then:

(i) \( w(z) \) has at most one non-zero finite Borel exceptional value;
(ii) \( \lambda(\frac{1}{w}) = \lambda(w) = \sigma(w) \);
(iii) \( w(z) \) has infinitely many fixed points and satisfies \( \tau(w) = \sigma(w) \).

In 2011, Chen [14] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.3) and obtained the following result.

**Theorem C.** (See [14]) Let \( a, b, c \) be constants such that \( ac \neq 0 \). Suppose that \( w(z) \) is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.3), then:

(i) \( w(z) \) has no any Borel exceptional value;
(ii) If \( p(z) \) is a non-constant polynomial, then \( w(z) - p(z) \) has infinitely many zeros and satisfies \( \lambda(w-p) = \sigma(w) \).
In 2012, Chen and Chen [7] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.4) and proved the following result.

**Theorem E.** (See [7]) Let $a, b, c$ be constants such that $|a| + |b| 
eq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.4), then:
(i) $\lambda \left( \frac{1}{w} \right) = \lambda (w) = \sigma (w)$;
(ii) If $p(z)$ is a non-constant polynomial, then $w(z) - p(z)$ has infinitely many zeros and satisfies $\lambda (w - p) = \sigma (w)$.
(iii) If $\lambda \neq 0$, then $w(z)$ has no Borel exceptional value. If $\lambda = 0$, then the Borel exceptional value of $w(z)$ can only come from a set $E = \{ z \in \mathbb{C} \mid \pi^2 - cz - b = 0 \}$.

In this paper, we consider some properties of difference and divided difference of transcendental meromorphic solutions of the difference Painlevé I equations (1.2) – (1.4) and Painlevé II equation (1.5), and obtain the following results.

**Theorem 1.1.** Let $a, b, c$ be constants with $|a| + |b| 
eq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.2), then
(i) $\lambda \left( \frac{1}{w} \right) = \lambda \left( \frac{w}{1} \right) = \sigma (w)$;
(ii) For any $\eta \in \mathbb{C} \setminus \{ 0 \}$, $\tau (w(z + \eta)) = \sigma (w)$.

**Theorem 1.2.** Let $a, b, c$ be constants with $|a| + |b| + |c| 
eq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.3), then
(i) $\lambda \left( \frac{1}{w} \right) = \lambda \left( \frac{w}{1} \right) = \sigma (w)$;
(ii) For any $\eta \in \mathbb{C} \setminus \{ 0 \}$, $\tau (w(z + \eta)) = \sigma (w)$.

**Theorem 1.3.** Let $a, b, c$ be constants with $|a| + |b| + |c| 
eq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.4), then $\lambda \left( \frac{1}{w} \right) = \lambda \left( \frac{w}{1} \right) = \sigma (w)$.

**Theorem 1.4.** Let $a, b, c$ be constants with $|a| + |b| + |c| 
eq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé II equation (1.5), then
(i) $\lambda \left( \frac{1}{w} \right) = \lambda \left( \frac{w}{1} \right) = \sigma (w)$;
(ii) For any $\eta \in \mathbb{C} \setminus \{ 0 \}$, $\tau (w(z + \eta)) = \sigma (w)$.

**Remark 1.1.** From the proofs of Theorems 1.1 – 1.4, we can also obtain that $\lambda \left( \frac{1}{w} \right) = \sigma (w)$ and $\lambda \left( \frac{w}{a} \right) = \sigma (\Delta w)$.

**Remark 1.2.** Generally, $\tau (w(z + \eta)) \neq \tau (w(z))$, where $\eta \in \mathbb{C} \setminus \{ 0 \}$.

**Example 1.1.** The meromorphic function $w(z) = \frac{1}{e^{\pi z} z + 1}$ satisfies the difference Painlevé I equation
$$w(z + 1) + w(z - 1) = \frac{2 w(z)}{1 - w(z)}.$$ with $a = c = 0, b = 2$ satisfying $|a| + |b| = 2 (\neq 0)$. We see that
$$\Delta w(z) = \frac{e^{\pi (z + 1)} - 1}{e^{\pi z} + 1} + \frac{e^{\pi z} - 1}{e^{\pi z} + 1} = \frac{4 e^{\pi z}}{e^{\pi z} + 1},$$
$$\Delta w(z) = \frac{4 e^{\pi z}}{e^{\pi z} + 1} = \frac{4 e^{\pi z}}{e^{\pi z} + 1},$$
$$w(z + 1) - w(z) = \frac{e^{\pi (z + 1)} - 1}{e^{\pi z} + 1} - \frac{e^{\pi z} - 1}{e^{\pi z} + 1} = \frac{4 e^{\pi z}}{e^{\pi z} + 1}.$$

**Example 1.2.** The meromorphic function $w(z) = \frac{1}{e^{\pi z} z + 1}$ satisfies the difference Painlevé II equation
$$w(z + 1) + w(z - 1) = \frac{2 w(z)}{1 - w(z)}.$$ with $a = c = 0, b = 2$ satisfying $|a| + |b| + |c| = 2 (\neq 0)$. We see that
$$\Delta w(z) = \frac{e^{2 \pi z} + 1}{e^{2 \pi z} + 1} = 2 \frac{e^{\pi z}}{e^{\pi z} + 1}.$$
Lemma 2.5. (See [12, Theorem 2.1]) Let f(z) be a meromorphic function with order \( \sigma = \sigma(f) \), \( \sigma < +\infty \), and let \( \eta \) be a fixed non-zero complex number, then for each \( \varepsilon > 0 \), we have

\[
T(r, f(z + \eta)) = T(r, f(z)) + O(r^{\sigma - 1 + \varepsilon}) + O(\log r).
\]

Lemma 2.6. (See [12, Theorem 2.2]) Let f be a meromorphic function with exponent of convergence of poles \( \lambda \frac{1}{f} = \lambda < +\infty, \eta \neq 0 \) be fixed, then for each \( \varepsilon > 0 \),

\[
N(r, f(z + \eta)) = N(r, f(z)) + O(r^{\lambda - 1 + \varepsilon}) + O(\log r).
\]

3. Proof of Theorems

Proof of Theorem 1.1

(i) Firstly, we will prove \( \lambda \left( \frac{1}{f} \right) = \sigma(w) \). By equation (1.2), Lemma 2.1, Lemma 2.5 and \( |a| + |b| 
eq 0 \), we have

\[
2T(r, w(z)) = T\left( r, \frac{az + b + cw(z)}{w(z)} \right) + O(\log r)
\]

\[
= T\left( r, \frac{w(z + 1) + w(z - 1)}{w(z)} \right) + O(\log r)
\]

\[
\leq T\left( r, \frac{w(z + 1)}{w(z)} \right) + T\left( r, \frac{w(z - 1)}{w(z)} \right) + O(\log r)
\]

\[
= 2T\left( r, \frac{w(z + 1)}{w(z)} \right) + S\left( r, w(z) \right) + O(\log r)
\]

\[
= 2T\left( r, \frac{w(z)}{w(z)} \right) + S\left( r, w \right),
\]

that is,

\[
T(r, w) \leq T\left( r, \frac{\Delta w(z)}{w(z)} \right) + S\left( r, w \right).
\]

By equation (1.2), we obtain

\[
w(z)(w(z + 1) + w(z - 1)) = az + b + cw(z).
\]

From (3.5) and Lemma 2.3, we see that for each \( \varepsilon > 0 \), there is a subset \( E_{\varepsilon} \subset (1, \infty) \) having finite logarithmic measure such that for \( |z| = r \notin [0, 1] \cup E_{\varepsilon} \),

\[
m(r, w(z + 1) + w(z - 1)) = O(r^{\sigma(w) - 1 + \varepsilon}) + S(r, w).
\]

It follows from equation (1.2), Lemma 2.1 and \( |a| + |b| \neq 0 \) that

\[
T(r, w(z + 1) + w(z - 1)) = T\left( \frac{az + b}{w(z)} + c \right) = T(r, w) + S(r, w).
\]

From (3.6), (3.7) and Lemma 2.6, we obtain

\[
T(r, w(z)) + S(r, w) = N(r, w(z + 1) + w(z - 1)) \leq N(r, w(z + 1)) + N(r, w(z - 1)) = 2N(r, w(z)) + S(r, w).
\]

It follows from Lemma 2.4 that

\[
m(r, \Delta w(z)) \leq m\left( r, \frac{\Delta w(z)}{w(z)} \right) + m(r, w(z)) = m(r, w(z)) + S(r, w).
\]

From (3.3) – (3.4) and (3.8) – (3.9), we see

\[
The \lambda \left( \frac{1}{f} \right) \geq \sigma(w), \text{that is } \lambda \left( \frac{1}{f} \right) = \sigma(w).
\]

(ii) For any \( \eta \in \mathbb{C} \setminus \{0\} \), substituting \( z + \eta \) into equation (1.2), we obtain

\[
w(z + \eta + 1) + w(z + \eta - 1) = az + b + cw(z + \eta).
\]

Set \( g(z) = w(z + \eta) \). Rewriting equation (3.10) as

\[
g(z)(g(z + 1) + g(z - 1)) = cg(z) + (a(z + \eta) + b).
\]

Denote

\[
P_{1}(z, g) := g(z)(g(z + 1) + g(z - 1)) - cg(z) - (a(z + \eta) + b).
\]

Then, we have

\[
P_{1}(z, g) = 2z^{2} - (a + c)z - (a\eta + b) \neq 0.
\]

From \( P_{1}(z, g) \neq 0 \) and Lemma 2.2, we see

\[
m\left( r, \frac{1}{g(z) - z} \right) = S(r, g).
\]

Thus, by Lemma 2.5, we have

\[
N\left( r, \frac{1}{w(z + \eta) - z} \right) = N\left( r, \frac{1}{g(z) - z} \right) = T(r, g) + S(r, g)
\]

\[
= T\left( r, w(z + \eta) + S(r, w(z + \eta)) \right)
\]

\[
= T\left( r, w(z) + S(r, w) \right).
\]

Hence, for any \( \eta \in \mathbb{C} \setminus \{0\} \), \( \pi(w(z + \eta)) = \sigma(w) \).

This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2**

If \( c = 0 \), equation (1.3) is a special case of equation (1.2). In what
follows, we assume \( c \neq 0 \). (i) Firstly, we will prove \( \frac{1}{g(z)} = \sigma(w) \).

By equation (1.3), Lemma 2.1, Lemma 2.5 and \( c \neq 0 \), we have

\[
3T(r, w(z)) = T\left(r, \frac{(a \zeta + b)w(z) + c}{w^3(z)}\right) + O(\log r)
\]
\[
= T\left(r, \frac{w(z+1) + w(z-1)}{w(z)}\right) + O(\log r)
\]
\[
\leq 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w),
\]

that is,

\[
\frac{3}{2} T(r, w(z)) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w). \tag{3.11}
\]

It follows from (3.11) and Lemma 2.4 that

\[
N\left(r, \frac{\Delta w(z)}{w(z)}\right) = T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right)
\]
\[
\geq \frac{3}{2} T(r, w(z)) + S(r, w).
\]

Thus, \( \lambda \left(\frac{1}{w^3}\right) \geq \sigma(w) \), that is \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \).

Next, we prove \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \). By equation (1.3),

\[
\Delta w(z) - \Delta w(z-1) = \frac{c + (a \zeta + b)w(z) - 2w(z)}{w^3(z)}. \tag{3.12}
\]

From (3.12), Lemma 2.1, Lemma 2.5 and \( c \neq 0 \), we have

\[
3T(r, w(z)) = T\left(r, \frac{c + (a \zeta + b)w(z) - 2w(z)}{w^3(z)}\right) + O(\log r)
\]
\[
= T\left(r, \Delta w(z) - \Delta w(z-1)\right) + O(\log r)
\]
\[
\leq 2T\left(r, \Delta w(z)\right) + S(r, w),
\]

that is,

\[
\frac{3}{2} T(r, w(z)) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w). \tag{3.13}
\]

By Lemma 2.4,

\[
m(r, \Delta w(z)) = m\left(r, \frac{\Delta w(z)}{w(z)}\right) + m(r, w(z)) \leq T\left(r, w(z)\right) + S(r, w).
\]

(3.14)

It follows from (3.13) and (3.14) that

\[
N(r, \Delta w(z)) = T\left(r, \Delta w(z)\right) - m(r, \Delta w(z))
\]
\[
\geq \frac{3}{2} T(r, w(z)) - T\left(r, w(z)\right) + S(r, w)
\]
\[
= \frac{1}{2} T(r, w(z)) + S(r, w).
\]

Thus, \( \lambda \left(\frac{1}{w^3}\right) \geq \sigma(w) \), that is \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \).

(ii) For any \( \eta \in \mathbb{C} \setminus \{0\} \), substituting \( z + \eta \) into equation (1.3), we obtain

\[
w(z + \eta + 1) + w(z + \eta - 1) = \frac{a(z + \eta) + b}{w(z + \eta)} + \frac{c}{w(z + \eta)^2}. \tag{3.15}
\]

Set \( \omega(z) = w(z + \eta) \). Then (3.15) can be rewritten as

\[
g^2(z)(g(z + 1) + g(z - 1)) = g(z)(a(z + \eta) + b) + c.
\]

Denote

\[
P_2(z, g) := g^2(z)(g(z + 1) + g(z - 1)) - g(z)(a(z + \eta) + b) - c = 0.
\]

Then, we have

\[
P_2(z, z) = z^2(z + 1 + z - 1) - z(a(z + \eta) + b) - c \neq 0.
\]

From \( P_2(z, z) \neq 0 \) and Lemma 2.2, we see

\[
m\left(r, \frac{1}{g(z) - z}\right) = S(r, g).
\]

Thus, by Lemma 2.5, we have

\[
N\left(r, \frac{1}{w(z + \eta) - z}\right) = N\left(r, \frac{1}{g(z) - z}\right) = T(r, g) + S(r, g)
\]
\[
= T\left(r, w(z + \eta)\right) + S\left(r, w(z + \eta)\right)
\]
\[
= T\left(r, w(z + \eta)\right) + S\left(r, w(z + \eta)\right)
\]

Hence, for any \( \eta \in \mathbb{C} \setminus \{0\} \), \( \tau\left(w(z + \eta)\right) = \sigma(w) \).

This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3**

Using the same method as the proof of Theorem 1.1, we can easily obtain \( \lambda \left(\frac{1}{w^3}\right) = \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \).

**Proof of Theorem 1.4**

(i) In what follows, we consider three cases: Case 1, \( c = 0 \); Case 2, \( c \neq 0 \), either \( a = 0, b - c = 0 \), or \( a = 0, b + c = 0 \); Case 3, \( c \neq 0 \), either \( a \neq 0, b - c \neq 0 \), or \( b + c = 0 \).

Case 1, \( c = 0 \). Firstly, we prove \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \). By equation (1.5), Lemma 2.1, Lemma 2.5 and \( |a| + |b| \neq 0 \), we have

\[
2T\left(r, w(z)\right) = T\left(r, \frac{az + b - \omega(z)}{1 - w^3(z)}\right) + O(\log r)
\]
\[
= T\left(r, \frac{w(z+1) + w(z-1)}{w(z)}\right) + O(\log r)
\]
\[
\leq 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w),
\]

that is,

\[
T\left(r, w(z)\right) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w). \tag{3.16}
\]

It follows from (3.16) and Lemma 2.4 that

\[
N\left(r, \frac{\Delta w(z)}{w(z)}\right) = T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right)
\]
\[
\geq T\left(r, w(z)\right) + S(r, w).
\]

Thus, \( \lambda \left(\frac{1}{w^3}\right) \geq \sigma(w) \), that is \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \).

Next, we prove \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \). By equation (1.5),

\[
\Delta w(z) - \Delta w(z-1) = \frac{2w^3(z) + (a \zeta + b - 2)w(z)}{1 - w^3(z)}. \tag{3.17}
\]

From (3.17), Lemma 2.1, Lemma 2.5 and \( |a| + |b| \neq 0 \), we have

\[
3T\left(r, w(z)\right) = T\left(r, \frac{2w^3(z) + (a \zeta + b - 2)w(z)}{1 - w^3(z)}\right) + O(\log r)
\]
\[
= T\left(r, \Delta w(z) - \Delta w(z-1)\right) + O(\log r)
\]
\[
\leq 2T\left(r, \Delta w(z)\right) + S(r, w),
\]

that is,

\[
\frac{3}{2} T\left(r, w(z)\right) \leq T\left(r, \Delta w(z)\right) + S(r, w). \tag{3.18}
\]

From (3.14) and (3.18), we have

\[
N\left(r, \Delta w(z)\right) = T\left(r, \Delta w(z)\right) - m(r, \Delta w)
\]
\[
\geq \frac{3}{2} T\left(r, w(z)\right) - T\left(r, w(z)\right) + S(r, w)
\]
\[
= \frac{1}{2} T\left(r, w(z)\right) + S(r, w).
\]

Thus, \( \lambda \left(\frac{1}{w^3}\right) \geq \sigma(w) \), that is \( \lambda \left(\frac{1}{w^3}\right) = \sigma(w) \).

Case 2, \( c \neq 0 \), either \( a = 0, b - c = 0 \), or \( a = 0, b + c = 0 \). We divide
this proof into the following two subcases.

Case 2.1, \(c \neq 0, a = 0, b + c = 0\). Firstly, we prove \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

By equation (1.5), Lemma 2.1, Lemma 2.5 and \(b = c \neq 0\), we have

\[
2T(r, w(z)) = T \left( r, \frac{b}{w(z)(1 - w(z))} \right) + O(1)
\]

\[
= T \left( r, \frac{w^2(z) + w(z) - 1}{w(z)} \right) + O(1)
\]

\[
\leq 2T \left( r, \frac{\Delta w(z)}{w(z)} \right) + S(r, w),
\]

hence,

\[
T(r, w(z)) \leq T \left( r, \frac{\Delta w(z)}{w(z)} \right) + S(r, w). \tag{3.19}
\]

From (3.19) and Lemma 2.4, we see

\[
N \left( r, \frac{\Delta w(z)}{w(z)} \right) = T \left( r, \frac{\Delta w(z)}{w(z)} \right) - m \left( r, \frac{\Delta w(z)}{w(z)} \right)
\]

\[
\geq T(r, w(z)) + S(r, w).
\]

Thus, \(\hat{\lambda} \left( \frac{1}{w(z)} \right) \geq \sigma(w)\), that is \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

Next, we prove \(\lambda \left( \frac{1}{w(z)} \right) = \sigma(w)\). By equation (1.5),

\[
\Delta w(z) - \Delta w(z - 1) = \frac{2w^2(z) - 2w(z) + b}{1 - w(z)}. \tag{3.20}
\]

From (3.20), Lemma 2.1, Lemma 2.5 and \(b = c \neq 0\), we have

\[
2T(r, w(z)) = T \left( r, \frac{2w^2(z) - 2w(z) + b}{1 - w(z)} \right) + O(1)
\]

\[
= T(r, \Delta w(z) - \Delta w(z - 1)) + O(1)
\]

\[
\leq 2T(r, \Delta w(z)) + S(r, w),
\]

that is,

\[
T(r, w(z)) \leq T(r, \Delta w(z)) + S(r, w). \tag{3.21}
\]

By equation (1.5), we obtain

\[
w(z)(w(z + 1) + w(z - 1)) = w(z + 1) + w(z - 1) - b. \tag{3.22}
\]

From (3.22) and Lemma 2.3, we see that for each \(z \in [1, \infty)\) having finite logarithmic measure such that \(|z - r| \notin [0, 1] \cup E_2\),

\[
m(r, w(z + 1) + w(z - 1)) = O(\rho^{\sigma(w)} + r) + S(r, w). \tag{3.23}
\]

By equation (1.5), Lemma 2.1 and \(b = c \neq 0\), we see

\[
T(r, w(z + 1) + w(z - 1)) = T \left( r, \frac{1}{1 - w(z)} \right) = T(r, w(z)) + S(r, w). \tag{3.24}
\]

From (3.23), (3.24) and Lemma 2.6, we obtain

\[
T(r, w(z)) + S(r, w) = N(r, w(z + 1) + w(z - 1)) \leq 2N(r, w(z)) + S(r, w) \tag{3.25}
\]

From (3.4), (3.9) and (3.21) and (3.25), we see

\[
N(r, \Delta w(z)) = T(r, \Delta w(z)) - m(r, \Delta w(z))
\]

\[
\geq T(r, \Delta w(z)) - T(r, w(z)) + \frac{1}{4} T(r, \Delta w(z)) + S(r, w)
\]

\[
= \frac{5}{4} T(r, \Delta w(z)) - T(r, w(z)) + S(r, w)
\]

\[
\geq \frac{1}{4} T(r, w(z)) + S(r, w).
\]

Thus, \(\hat{\lambda} \left( \frac{1}{w(z)} \right) \geq \sigma(w)\), that is \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

Case 2.2, \(c \neq 0, a = 0, b + c = 0\). Using the same method as the proof of subcase 2.1, we can also obtain \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

Case 3, \(c \neq 0, a \neq 0, b - c \neq 0, b + c \neq 0\). Firstly, we prove \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\). By equation (1.5), Lemma 2.1, Lemma 2.5 and \(c \neq 0\), either \(a \neq 0, b - c \neq 0, b + c \neq 0\), or \(b - c = 0, b + c \neq 0\), we have

\[
3T(r, w(z)) = T \left( r, \frac{(az + b)w(z) + c}{w(z)(1 - w(z))} \right) + O(\log r)
\]

\[
= T \left( r, \frac{w(z + 1) + w(z - 1)}{w(z)} \right) + O(\log r)
\]

\[
\leq 2T \left( r, \frac{\Delta w(z)}{w(z)} \right) + S(r, w),
\]

that is,

\[
\frac{3}{2} T(r, w(z)) \leq T \left( r, \frac{\Delta w(z)}{w(z)} \right) + S(r, w). \tag{3.26}
\]

From (3.26) and Lemma 2.4, we see

\[
N \left( r, \frac{\Delta w(z)}{w(z)} \right) = T \left( r, \frac{\Delta w(z)}{w(z)} \right) - m \left( r, \frac{\Delta w(z)}{w(z)} \right)
\]

\[
\geq \frac{3}{2} T(r, w(z)) + S(r, w).
\]

Thus, \(\hat{\lambda} \left( \frac{1}{w(z)} \right) \geq \sigma(w)\), that is \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

Next, we prove \(\lambda \left( \frac{1}{w(z)} \right) = \sigma(w)\). By equation (1.5),

\[
\Delta w(z) - \Delta w(z - 1) = \frac{2w^3(z) + (az + b - 2)b}{1 - w(z)}. \tag{3.27}
\]

From (3.27), Lemma 2.1, Lemma 2.5 and \(c \neq 0\), either \(a \neq 0, b - c \neq 0, b + c \neq 0\), or \(b - c = 0, b + c \neq 0\), we have

\[
3T(r, w) = T \left( r, \frac{2w^3(z) + (az + b - 2)b}{1 - w(z)} \right) + O(\log r)
\]

\[
= T(r, \Delta w(z) - \Delta w(z - 1)) + O(\log r)
\]

\[
\leq 2T(r, \Delta w(z)) + S(r, w),
\]

that is,

\[
\frac{3}{2} T(r, w(z)) \leq T(r, \Delta w(z)) + S(r, w). \tag{3.28}
\]

From (3.14) and (3.28), we have

\[
N(r, \Delta w(z)) = T(r, \Delta w(z)) - m(r, \Delta w(z))
\]

\[
\geq \frac{3}{2} T(r, w(z)) - T(r, w(z)) + S(r, w)
\]

\[
= \frac{1}{2} T(r, w(z)) + S(r, w).
\]

Thus, \(\hat{\lambda} \left( \frac{1}{w(z)} \right) \geq \sigma(w)\), that is \(\hat{\lambda} \left( \frac{1}{w(z)} \right) = \sigma(w)\).

(ii) For any \(\eta \in \mathbb{C} \setminus \{0\}\), substituting \(z + \eta\) into equation (1.5), we obtain

\[
w(z + \eta) + w(z + \eta - 1) = \frac{(a(z + \eta) + b)w(z + \eta) + c}{1 - w(z + \eta)^2}. \tag{3.29}
\]

Set \(g(z) = w(z + \eta)\). Then (3.29) can be rewritten as

\[(1 - g^2(z))(g(z + 1) + g(z - 1)) = g(z)(a(z + \eta) + b) + c. \tag{3.30}\]

Denote

\[P_3(z, \xi) := (1 - g^2(z))(g(z + 1) + g(z - 1)) - g(z)(a(z + \eta) + b) - c = 0. \tag{3.31}\]

Then, we have

\[P_3(z, \xi) = (1 - c^2)(z + 1 + z - 1) - z(a(z + \eta) + b) - c \neq 0. \tag{3.32}\]
From \( P_3(z, z) \neq 0 \) and Lemma 2.2, we see that

\[
m\left( r, \frac{1}{g(z) - z} \right) = S(r, g).
\]

Thus, by Lemma 2.5, we have

\[
N\left( r, \frac{1}{w(z + \eta) - z} \right) = N\left( r, \frac{1}{g(z) - z} \right) = T(r, g) + S(r, g)
\]

\[
= T(r, w(z + \eta)) + S(r, w(z + \eta))
\]

\[
= T(r, w(z)) + S(r, w).
\]

Hence, for any \( \eta \in \mathbb{C} \setminus \{0\} \), \( \tau(w(z + \eta)) = \sigma(w) \).

This completes the proof of Theorem 1.4.

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References


