Global Journal of Mathematical Analysis, 5 (1) (2017) 19-20

SPC

Global Journal of Mathematical Analysis

Website: www.sciencepubco.com/index.php/GJMA

doi: 10.14419/gjma.v5i1.7306

Research paper

Global existence and estimates of the solutions to nonlinear integral equations

Alexander G. Ramm[1*]

[1] *Department of Mathematics, Kansas State University, , Manhattan, KS 66506, USA, http://www.math.ksu.edu/~ramm*
[*] *Corresponding author E-mail: ramm@math.ksu.edu*

Abstract

It is proved that a class of nonlinear integral equations of the Volterra-Hammerstein type has a global solution, that is, solutions defined for all $t \geq 0$, and estimates of these solutions as $t \to \infty$ are obtained. The argument uses a nonlinear differential inequality which was proved by the author and has broad applications.

Keywords: *nonlinear integral equations*

1. Introduction

Consider the equation:

$$u(t) = \int_0^t e^{-a(t-s)} h(u(s))ds + f(t) := T(u), \quad t \geq 0; \quad a = const > 0. \tag{1}$$

that is, Volterra-Hammerstein equation. There is a large literature on nonlinear integral equations,[1], [6]. The usual methods to study such equations include fixed-point theorems such as contraction mapping principle and degree theory, (Schauder and Leray-Schauder theorems). The goal of this paper is to give a new approach to a study of equation (1). We give sufficient conditions for the global existence of solutions to (1) and their estimates as $t \to \infty$.
Denote $f' := \frac{df}{dt}$. By $c > 0$ various constants will be denoted.
Let us formulate our assumptions:

$$|h(u)| \leq c|u|^b, \quad |h'(u)| \leq c|u|^{b-1}, \quad b \geq 2, \tag{2}$$

$$|f(t)| + a|f'(t)| \leq ce^{-a_1 t}, \qquad a_1 = const > 0. \tag{3}$$

Our approach is based on the author's results on the nonlinear differential inequality formulated in Theorem 1 (see [2]–[5]). These results have been used by the author in a study of stability of solutions to abstract nonlinear evolution problems ([5]).
Denote $\mathbb{R}_+ = [0, \infty)$.
Theorem 1. *Let $g \geq 0$ solve the inequality*

$$g'(t) \leq -ag(t) + \alpha(t, g) + \beta(t), \quad t \geq 0, \quad a = const > 0, \tag{4}$$

where $\alpha(t, g) \geq 0$ and $\beta(t) \geq 0$ are continuous functions of t, $t \in \mathbb{R}_+$ and $\alpha(t, g)$ is locally Lipschitz with respect to g. If there exists a function $\mu(t) > 0$, defined on $\mathbb{R}_+$, $\mu \in C^1(\mathbb{R}_+)$, such that

$$\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{1}{\mu(t)}\Big(a - \frac{\mu'(t)}{\mu(t)}\Big), \quad \forall t \geq 0, \tag{5}$$

and

$$g(0)\mu(0) \leq 1, \tag{6}$$

then g exists on $\mathbb{R}_+$ and

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0. \tag{7}$$

A proof of Theorem 1 can be found in [5]. Its idea is described in Section 2.
The result of this paper is formulated in Theorem 2.
Theorem 2. *Assume that (2) and (3) hold, $a \geq 2$, $b \geq 2$, $c \in (0, 0.75)$, $p \in (0, \min(0.75a, a_1))$, $R = (b-1)^{1/b}$. Then any solution to (1) exists on $\mathbb{R}_+$ and satisfies the estimate*

$$|u(t)| \leq R^{-1} e^{-pt}, \qquad \forall t \geq 0, \qquad p \in (0, min(0.25a_1, a)). \tag{8}$$

In Section 2 Theorem 2 is proved.

2. Proof of Theorem 2

Let us reduce equation (1) to the form suitable for an application of Theorem 1. Differentiate (1) and get

$$u' = f' - a\int_0^t e^{-a(t-s)} h(u(s))ds + h(u(t)). \tag{9}$$

Let $g(t) := |u(t)|$ and take into account that $|F(t)| \leq ce^{a_1 t}$, $F := f' + af$.
From (1) one gets $\int_0^t e^{-a(t-s)} h(u(s))ds = u - f$. This and equation (9) imply

$$u' = f' - a(u - f) + h(u(t)). \tag{10}$$

Therefore, one gets

$$u' = -au + h(u) + F, \quad F := f' + af. \tag{11}$$

Multiply ([11]) by $\bar{u}$, where $\bar{u}$ stands for complex conjugate of u, and get

$$u'\bar{u} = -ag^2 + h(u)\bar{u} + F\bar{u}. \tag{12}$$

One has

$$u'\bar{u} + u(\bar{u})' = \frac{dg^2}{dt} = 2gg'. \tag{13}$$

We define the derivative as $g' = \lim_{h\to+0}\frac{g(t+h)-g(t)}{h}$. With this definition, $g(t)$ is differentiable at every point if $u(t)$ is continuously differentiable for all $t\geq 0$. Any solution $u(t)$ to (1) is continuously differentiable under our assumptions. Take complex conjugate of (12), add the resulting equation to (12) and take into account (13). This yields

$$2gg' = -2ag^2 + 2Re(h(u)\bar{u}) + 2Re(F\bar{u}). \tag{14}$$

Since $g\geq 0$, one derives from (14), using assumptions (2) and (3), that

$$g'(t) \leq -ag(t) + cg^b + ce^{-a_1 t}. \tag{15}$$

Let

$$\mu(t) = Re^{pt}, \quad R = const > 0, \quad p\in(0,\min(0.25a,a_1)). \tag{16}$$

Condition (5) can be written as

$$\frac{c}{R^b e^{bpt}} + ce^{-a_1 t} \leq \frac{1}{Re^{pt}}(a-p), \quad t\in\mathbb{R}_+. \tag{17}$$

This inequality holds if

$$\frac{c}{R^{b-1}e^{(b-1)pt}} + cRe^{-(a_1-p)t} \leq \frac{3a}{4}, \quad t\in\mathbb{R}_+. \tag{18}$$

Inequality (18) holds if $b\geq 1$ and

$$\frac{1}{R^{b-1}} + R \leq \frac{3a}{4c}. \tag{19}$$

The minimum of the left side of (19) is attained at $R=(b-1)^{1/b}$ and is equal to $\frac{b}{(b-1)^{(b-1)/b}}$. Thus, (19) holds if

$$\frac{b}{(b-1)^{(b-1)/b}} \leq \frac{3a}{4c}. \tag{20}$$

For example, assume that

$$a\geq 2, \qquad c\leq 0.75.$$

Then (20) holds if $b\leq 2(b-1)^{(b-1)/b}$, that is, if

$$b^b \leq 2^b(b-1)^{b-1}. \tag{21}$$

Inequality (21) holds if $b\geq 2$. Thus, by Theorem1, any solution $u(t)$ of (1) exists globally and

$$|u(t)| \leq \frac{e^{-pt}}{R}, \tag{22}$$

provided that

$$|u(0)|R\leq 1, \quad R=(b-1)^{1/b}, \quad a\geq 2, \quad b\geq 2, \quad c=0.75, \quad p\in(0,\min(0.25a,a_1)). \tag{23}$$

Inequality $|u(0)|R\leq 1$ holds if $|f(0)|R\leq 1$. By assumption (3) this inequality holds if $c\leq\frac{1}{R}$. Theorem 2 is proved. □

Let us prove existence of a solution to (1) using the contraction mapping principle and Theorem 2.

By estimate (22) one has $|u(t)|\leq\frac{1}{R}$ for all $t\geq 0$. Therefore, using assumptions (2) and (3), one gets

$$|Tu| \leq c + \frac{c}{aR^b} \leq \frac{1}{R}, \tag{24}$$

provided that $cR\leq\frac{1}{1+\frac{1}{aR^b}}$. For $R=(b-1)^{1/b}$ this inequality holds if c is sufficiently small. If (24) holds, then T maps the ball $B_R := \{u : ||u||\leq\frac{1}{R}\}$ into itself. Here $||u|| = max_{t\geq 0}|u(t)|$.

On the ball B_R the operator T is a contraction:

$$||Tu-Tv|| \leq ||\int_0^t e^{-a(t-s)}c|\eta^{b-1}|ds||||u-v|| \leq \frac{c}{R^{b-1}a}||u-v||, \tag{25}$$

where the assumption (2) was used, and η is the "middle" element between u and v, $||\eta||\leq\frac{1}{R}$. The integral in (25) is estimated as follows:

$$||\int_0^t e^{-a(t-s)}c|\eta^{b-1}|ds|| \leq \frac{c}{R^{b-1}}max_{t\geq 0}\int_0^t e^{-a(t-s)}ds \leq \frac{c}{R^{b-1}a}. \tag{26}$$

If

$$\frac{c}{R^{b-1}a} < 1, \tag{27}$$

then T is a contraction on B_R. Condition (27) holds if c is sufficiently small. Thus, if condition (27) and the assumptions of Theorem 2 hold, then, by the contraction mapping principle, there exists a unique solution to (1) in the ball B_R. □

For convenience of the reader we sketch the idea of the proof of Theorem 1 following [2]—[5].

Inequality (5) can be written for the function $w=\frac{1}{\mu}$ as follows:

$$-aw + \alpha(t,w) + \beta(t) \leq w'. \tag{28}$$

From (4) and (28) by a comparison lemma for ordinary differential equations it follows that

$$0\leq g(t)\leq\frac{1}{\mu(t)}, \tag{29}$$

provided that $g(0)\leq w(0)=\frac{1}{\mu(0)}$. The last inequality is the assumption (6). Since $\mu(t)>0$ and is assumed to be defined for all $t\geq 0$, the function $w=\frac{1}{\mu}$ is defined for all $t\geq 0$. Since $0\leq g(t)\leq\frac{1}{\mu(t)}$, and $g(t) := |u(t)|$, the function u is defined for all $t\geq 0$.

If $\lim_{t\to\infty}\mu(t)=\infty$, then $\lim_{t\to\infty}|u(t)|=0$ by estimate (29).

References

[1] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

[2] A.G.Ramm, Asymptotic stability of solutions to abstract differential equations, Journ. of Abstract Diff. Equations and Applications (JADEA), 1, N1, (2010), 27-34.

[3] A.G.Ramm, A nonlinear inequality and evolution problems, Journ, Ineq. and Special Funct., (JIASF), 1, N1, (2010), 1-9.

[4] A.G.Ramm, Stability of solutions to some evolution problems, Chaotic Modeling and Simulation (CMSIM), 1, (2011), 17-27.

[5] A.G.Ramm, Large-time behavior of solutions to evolution equations, in Handbook of Applications of Chaos Theory, Chapman and Hall/CRC, (ed. C.Skiadas), pp. 183-200.

[6] P. Zabreiko et al, *Integral equations: a reference text*, Leyden, Noordhoff International Pub., 1975.