Common fixed point theorems for weakly compatible non-self mappings in metric spaces of hyperbolic type

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Abstract

In this paper, we establish common fixed point theorems for a pair of weakly compatible nonself mappings satisfying generalized contractive conditions in metric space of hyperbolic type. The results generalize and extend some results in literature.

Keywords: common fixed points, generalized contractive mapping, metric space of hyperbolic type, nonself mappings, weakly compatible mappings.

1. Introduction

In literature, fixed point theory has diverse results on fixed point theorems for self-mappings in metric and Banach spaces. However, an area that seems not broadly investigated is the fixed point theorems for non-self mappings. Kirk [1] extended the metric space to metric space of hyperbolic type by replacing Krasnoselski’s result with the framework of convex metric space. The study of fixed point theorems for multivalued non-self mappings in a metric space \((X, d)\) was initiated by Assad [2] and Assad and Kirk [3]. Many authors have studied the existence and uniqueness of fixed and common fixed points result for nonself contraction mappings in cone metric spaces [see: 4, 5, 6, 7]. Some authors studied common fixed point theorems for non-self mappings in metric spaces of hyperbolic type [See: 8, 9]. Motivated by Jankovic et al. [7], we prove some common fixed point theorems for a pair of weakly compatible non-self mappings satisfying a generalized contraction condition in the setting of metric space of hyperbolic type. Throughout our consideration, we suppose that \((X, d)\) is a metric space which contains a family \(L\) of metric segments (isometric images of real line segment) such that

a) each two points \(x,y \in X\) are endpoints of exactly one number \(seg[x,y]\) of \(L\), and
b) If \(u,x,y \in X\) and if \(z \in seg[x,y]\) satisfies \(d(x,z) = \lambda d(x,y)\) for \(\lambda \in [0,1]\) then

\[
    d(u,z) \leq (1-\lambda) d(u,x) + \lambda d(u,y)
\]

A space of this type is called metric space of hyperbolic type.

The following definition was introduced by Jungck et al. [4] in the setting of cone metric spaces.

Definition 1.1 Let \((X, d)\) be a complete cone metric space, let \(C\) be a non empty closed subset of \(X\), and let \(f, g : C \to X\) be non-self mappings. Denote for \(x, y \in C\)

\[
    M^f_\delta = \{ d(gx,gy),d(fx,gx),d(fy,gy), \frac{d(fx,gy)+d(fy,gx)}{2} \}\quad(1.2)
\]

Then \(f\) is called a generalized \(\delta M\)-contractive mapping in \(C\) into \(X\) if, for some \(\lambda \in (0, \sqrt{2}-1)\), there exists \(U(x,y) \in M^f_\delta\) such that for all \(x,y \in C\),

\[
    d(fx, fy) \leq \lambda U(x,y)
\]

2. Main results

Jankovic et al. [7] proved the following fixed point theorem for a pair of non-self mappings defined on a nonempty closed subset of complete metrically convex cone metric spaces with new contractive conditions.

Theorem 2.1: Let \((X, d)\) be a complete cone metric space, let \(K\) be a non empty closed subset of \(X\) such that for each \(x \in C\) and \(y \notin C\) there exists a point \(z \in \delta K\) (the boundary of \(K\)) such that \(d(x,z) + d(z,y) = d(x,y)\). Suppose that \(f, g : C \to X\) are such that \(f\) is a generalized \(\delta M\)-contractive mapping of \(C\) into \(X\) and

(i) \(\delta C \subseteq gC, fC \cap C \subseteq gC,\)
(ii) \(gx \in \delta C \Longrightarrow fx \in C,\)

(iii) \(gC\) is closed in \(X\).

Then the pair \((f,g)\) has a coincidence point. Moreover, if \((f,g)\) are coincidentally commuting, then \((f,g)\) have a unique common fixed point.

In this paper, we extend the above theorem to fixed point theorem of weakly compatible non- self mappings in metric space of hyperbolic type.

We state and prove our main result as follows.
Theorem 2.2: Let X be a metric space of hyperbolic type. K a non-empty closed subset of X and δK the boundary of K. Let δK be nonempty and let T : K → X and f : fK ∩ (K) → X be two non-self-mappings satisfying the following conditions:
\[ d(f(x), y) \leq \lambda d(x, y), \]
where
\[ \mu \in \{ d(Tx, Ty), d(Tx, fx), d(Ty, fy), d(Tx, fy)+d(fy, fx) / 2 \} \]
(2.1)
for all x, y ∈ C, 0 < \lambda < 1. If
(i) δK ∩ TK, fK ∩ K ∩ TK,
(ii) Tx ∈ δK ⇒ fx ∈ K,
(iii) fK ∩ K is complete.
Then f and T have a coincidence point z in X. Moreover, if f and T are weakly compatible, then z is the unique common fixed point of f and T.

Proof: Let x ∈ δK be arbitrary. We construct three sequences \{xn\} and \{zn\} in K and a sequence \{yn\} in fK ⊂ X as follows. Choose z0 = x. Since z0 ∈ δK then there exists x0 ∈ K such that
\[ z_0 = T x_0 \in \delta K. \]
By (iii) f x0 ∈ K. Now choose y1 = f x0 with y1 ∈ fK ∩ K. This implies that f y0 ∈ fK ∩ K ∩ TK. Set y1 = f x0, we choose x1 ∈ K such that T x1 = y1. Hence z1 = T x1 = f x0 = y1. This gives y2 = f y1.
Since y2 ∈ fK ∩ K then y2 ∈ TK by (ii). Let x1 ∈ K with z1 = T x1 ∈ δK such that z2 = T x2 = f y1 = y2. If y1 = y2 ∉ K, then there exists z2 ∈ δK(y2 ∉ K) such that z2 ∈ seg[y1, y2]. Since y2 ∈ K, then by (i) we have T y2 = z2. Hence z2 ∈ δK ∩ seg[y1, y2].
We can choose y3 ∈ fK ∩ K and, by (ii), y3 ∈ TK and let x2 ∈ K such that T x2 = y3 = f x2. Continuing in the process, we construct three sequences \{xn\} ⊂ K, \{zn\} ⊂ K and \{yn\} ⊂ fK ⊂ X such that
(a) yn = f xn−1
(b) zn = T xn
(c) zn = yn if and only if yn ∈ K.
(d) zn = yn whenever yn ∉ K and zn ∈ δK such that zn ∈ δK ∩ seg[f xn−2, f xn−1].

Now, we show that \{xn\} ∩ K and \{yn\} ∩ K = \{zn\} ∩ K = \{zn\} for all n. From (a), (b), (c) and (d) we can establish three possibilities.
(1) zn = yn ∈ K and zn+1 = yn+1
(2) zn = yn ∈ K but zn+1 = zn
(3) zn ≠ yn ∈ K in which case zn ∈ δK ∩ seg[f xn−2, f xn−1].

Case (1)
Let zn = yn ∈ K and zn+1 = yn+1. Using (2.1) we obtain
\[ d(zn, zn+1) ≤ \lambda d(yn, yn+1) \]
where
\[ \mu_n \in \{ d(T x_n−1, x_n), d(T x_n−1, x_n+1), d(T x_n, x_n+1), d(T x_n, x_n+1) + d(f x_n, x_n+1) / 2 \} \]
(2.1)
\[ = \{ d(z_n−1, z_n), d(z_n−1, y_n), d(z_n, y_{n+1}), d(z_n, y_n) + d(z_{n+1}, y_n) / 2 \} \]
\[ = \{ d(z_n−1, z_n), d(z_n−1, z_n), d(z_n, z_{n+1}), d(z_n, y_{n+1}) + d(z_{n+1}, y_n) / 2 \} \]
\[ = \{ d(z_n−1, z_n), d(z_n−1, z_n), d(z_n, z_{n+1}), d(z_n, y_{n+1}) + d(z_{n+1}, y_{n+1}) / 2 \} \]

Obviously, there are infinite many n such that at least one of the following cases holds:
I: d(zn, zn+1) ≤ \lambda d(zn−1, zn)
II: d(zn, zn+1) ≤ \lambda d(zn−1, zn)

III: d(zn, zn+1) ≤ \lambda d(zn, zn+1 + d(zn, zn+1))
A contradiction.

IV: d(zn, zn+1) ≤ \lambda (d(zn, zn+1) + d(x_n, x_{n+1}))
\[ \leq \lambda d(z_{n+1}, z_n) \]

From I, II, III, IV it follows that
\[ d(z_{n+1}, zn+1) ≤ \lambda d(z_{n+1}, zn) \]
(2.2)

Therefore
\[ d(x_{n+1}, y_{n+1}) ≤ d(x_n, y_n) + (1 - \lambda) d(x_n, y_n) = d(x_n, y_n) \]

Hence
\[ z \in \text{seg}[x_n, y_n] \]

Since zn+1 ∈ \text{seg}[y_{n+1}, y_{n+1}]. Hence
\[ d(z_{n+1}, zn+1) = d(y_{n+1}, zn+1) = d(y_n, zn+1) - d(z_{n+1}, zn+1) \]

In view of case (1), we obtain
\[ d(y_{n+1}, zn+1) ≤ \lambda d(z_{n+1}, zn) \]

This implies that
\[ d(z_{n+1}, zn+1) ≤ \lambda d(z_{n+1}, zn). \]

Case (3)
z ≠ y. Then z ∈ δK ∩ \text{seg}[f xn−2, f xn−1]. i.e. \[ z_0 \in δK ∩ \text{seg}[y_{n+1}, y_{n+1}]. \]
By remark (2.3) we have zn+1 = y_{n+1} and zn+1 = y_{n+1}. This implies that
\[ d(z_{n+1}, zn+1) = d(z_{n+1}, y_{n+1}) \]
\[ ≤ d(y_{n+1}, y_{n+1}) \]
\[ = d(y_{n+1}, zn+1) - d(z_{n+1}, zn+1) + d(y_{n+1}, zn+1) \]
\[ = d(y_{n+1}, zn+1) - d(z_{n+1}, zn+1) + d(y_{n+1}, zn+1) \]
(2.3)

We shall find \[ d(y_{n+1}, zn+1) \] and \[ d(y_{n+1}, zn+1) \]. Since \[ zn+1 = y_{n+1} \] then we can conclude that
\[ d(y_{n+1}, zn+1) ≤ \lambda d(z_{n+1−2}, zn−1) \]
(2.4)

Now
\[ d(y_{n+1}, zn+1) = d(f xn−1, f xn) ≤ \lambda \mu_n \]

where
\[ \mu_n \in \{ d(T x_{n−1}, x_n), d(T x_{n−1}, x_{n−1}), d(T x_n, x_{n−1}), d(T x_n, f x_{n−1}), d(T x_{n−1}, f x_{n−1}) / 2 \} \]
(2.1)
\[ = \{ d(z_{n−1}, zn), d(z_{n−1}, zn), d(z_{n−1}, y_{n−1}), d(z_{n−1}, y_{n−1}) + d(z_{n−2}, y_{n−1}) / 2 \} \]
\[ = \{ d(z_{n−1}, zn), d(z_{n−1}, zn), d(z_{n−1}, zn), d(z_{n−1}, zn) + d(z_{n−2}, y_{n−1}) / 2 \} \]
\[ = \{ d(z_{n−1}, zn), d(z_{n−1}, zn), d(z_{n−1}, zn), d(z_{n−1}, zn) + d(z_{n−2}, y_{n−1}) / 2 \} \]

Clearly, there are infinite many n such that at least one of the following cases holds:
Following the procedure of Assad and Kirk [3], it can be easily verify by induction that for $n > 1$
\[ d(z_n, z_{n+1}) \leq k d(z_{n-1}, z_n) \]  \hspace{1cm} (2.5)
where $k = \max\{\lambda, \lambda + \beta, \frac{1}{1 - \beta}, \frac{23}{2} \}$

Combining Cases 1, 2, 3 we get
\[ d(z_n, z_{n+1}) \leq k \omega_n \]
where $\omega_n \in \{d(z_{n-1}, z_n), d(z_{n-1}, z_n)\}$ and
\[
\omega = \max\{\lambda, \lambda + \beta, \frac{1}{1 - \beta}, \frac{23}{2} \}
\]

Following the procedure of Assad and Kirk [3], it can be easily verify by induction that for $n > 1$
\[ d(z_n, z_{n+1}) \leq k \omega_n \]
where $\omega_2 \in \{d(z_0, z_1), d(z_1, z_2)\}$.

For $n > m$ and using (2.5) and the triangle inequality we have
\[ d(z_n, z_m) \leq d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \cdots + d(z_{m+1}, z_m) \leq (k^2 + k + \cdots + k^{m-1}) \omega_2 \leq \frac{k^m - 1}{k - 1} \omega_2 \rightarrow 0, \quad \text{as} \quad m \rightarrow \infty. \]

The sequence is Cauchy. Since $z_n = f x_{n-1} \in f K \cap K$ is complete, there is some $z \in f K \cap K$ such that $z_n \rightarrow z$. Let $w$ in $K$ be such that $Tw = z$. By the construction of $\{z_n\}$, there is a subsequence $\{z_{n_k}\}$ such that $z_{n_k} = y_{n_k} = f x_{n_k-1}$ and $f x_{n_k-1} \rightarrow z$. We show that $f w = z$.
\[ d(fw, z) \leq d(fw, f x_{n_k-1}) + d(f x_{n_k-1}, z) \leq \lambda \mu_n + d(f x_{n_k-1}, z) \]
where
\[ \mu_n = \{d(Tw, T x_{n_k-1}), d(T x_{n_k-1}, f x_{n_k-1})\}, d(Tw, f w), \]
\[ \frac{d(Tw, f x_{n_k-1}) + d(T x_{n_k-1}, f w)}{2} \]
Taking $z_{n_k} = y_{n_k} = f x_{n_k-1} \rightarrow z$ as $n \rightarrow \infty$ yields
\[ \mu_n \in \{0, d(fw, f w), \frac{d(fw, f w)}{2} \} \]
Thus, we have
\[ d(fw, z) \leq \lambda d(fw, z) + d(f x_{n_k-1}, z) \leq d(fw, z) \leq \lambda d(fw, z) \]
Since $\lambda < 1$ then $d(fw, z) = 0$. This implies $z = fw$.
\[ d(fw, z) \leq \frac{1}{2} d(fw, z) \]
Since $\lambda < 1$ then $d(fw, z) = 0$. Hence $z =fw$. In all cases we have $z =fw$.
Suppose that $T$ and $f$ are weakly compatible, then we have $z = fw = Tw = fTw = Tw = Tz$.
Next we prove that $z = f z = T z$. Suppose $z \neq f z$ then using 2.1 we obtain
\[ d(f z, z) = d(fz, fw) \leq \lambda \mu \]
where
\[ \mu \in \{d(Tz, Tw), d(Tz, f z), d(Tw, f w), d(Tw, f w), \frac{d(fw, f w)}{2} \} \]
\[ \leq \{d(z, f z), d(z, f z), \frac{d(fz, f z)}{2} \} \]
Case (i)
\[ d(fz, z) \leq \lambda d(fz, z) \] is a contradiction. Hence $z = f z$
Case (ii)
\[ d(fz, z) \leq \frac{1}{2} d(fz, z) \]
It is also a contradiction. This implies that $z = f z$. Therefore we obtain $z = f z = T z$. Thus $T$ and $f$ have a common fixed point. The uniqueness of the common fixed point follows easily from (2.1).

**Remark 2.4**: Theorem 2.2 is an extension of the result of jankovic [7].
Setting $T = I_k$, the identity mapping of $X$ in Theorem 2.2, we obtain the following result.

**Corollary 2.5**: Let $(X,d)$ be metric space of hyperbolic type, $K$ a non-empty closed subset of $X$ and $\delta K$ the boundary of $K$. Let $\delta K$ be nonempty such that $f : K \rightarrow K$ satisfies the condition
\[ d(fx, fy) \leq \lambda \mu \]
where
\[ \mu \in \{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \} \]
for all $x,y \in k$, $0 < \lambda < 1$ and $f$ has the additional property that for each $x \in \delta K$ and $f x \in K$. Then $f$ has a unique fixed point.

**Corollary 2.6**: Let $X$ be a metric space of hyperbolic type, $K$ a non-empty closed subset of $X$ and $\delta K$ the boundary of $K$. Let $\delta K$ be nonempty and let $T : K \rightarrow X$ and $f : K \cap T(K) \rightarrow X$ be two non-self- mappings satisfying the following conditions:
\[ d(fx, fy) \leq \lambda (d(Tx, fx) + d(Ty, fy)) \]
for all $x,y \in C$, $0 < \lambda < \frac{1}{2}$. If
(i) $\delta K \subset TK$, $fK \cap K \subset TK$,
(ii) $Tx \in \delta K \implies fx \in K$,
(iii) $fK \cap K$ is complete.

Then $f$ and $T$ have a coincidence point $z$ in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $z$ is the unique common fixed point of $f$ and $T$.

**Example 2.7**: Let $X$ be the set of real numbers with the usual metric, $K = [0, +\infty)$ and let $T : K \to X$ and $f : K \cap T(K) \to X$ be two non-self mappings defined by $Tx = 4x$ and $fx = \frac{4x}{\lambda}$ for all $x \in K$.

Taking $x = \frac{1}{2}$ and $y = \frac{1}{4}$ we obtain $\lambda = \frac{1}{4}$. Thus $T$ and $f$ satisfied (2. 1) and all the hypotheses in Theorem 2.2 are satisfied. $T$ and $f$ have a unique common fixed point $z = 0$.

3. Conclusion

In this section, we proved that in a metric space of hyperbolic type, two non-self mappings $f$ and $T$ satisfying certain contractive conditions have a coincidence point. Moreover, if the maps are weakly compatible then $f$ and $T$ have a unique common fixed point. We gave an example to validate our results.

References